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Probability and Pascal's Triangle
DIY (Do-It-Yourself) Math Workshop

Dr. Ralph deLaubenfels

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Teacher-Scholar Institute

Columbus, Ohio

2020

DIY (Do-It YOURSELF)

PROBABILITY and PASCAL'S TRIANGLE

WORKSHOP

As with all DIY Workshops,

Writing/drawings in red — are written on a chalkboard & possibly spoken;

Writing in quotes in black "______" is said out loud to students & not written;

Writing not in quotes in black ______ is suggested & not spoken or written

TABLE OF CONTENTS

I. PREREQUISITES

For non-optional material:
arithmetic, as in Saxon 87.

For optional material:
1 year of high-school algebra,
as in Saxon Algebra I.

II. MATERIALS NEEDED

Two or more chalkboards, that we will call Board 1, Board 2, etc.

For each participant, including yourself, need a pen, a ruler, & a copy of Pascal's triangle with the left column blank, as you will find at the end of this exposition.

III. PROBABILITY

"Consider the following two almost-identical questions:"

Board 1

QUESTION 1: Should I wear a safety helmet at a construction site?

QUESTION 2: Should I wear a safety helmet when I'm asleep in my bed?

"In both scenarios, I could sustain head injuries if I don't wear a safety helmet; for example, when I'm asleep in my bed, I might dream so dramatically that I hurl myself onto the floor head first.

Yet the answers to these two questions might be different."

Ask students what their answers to QUESTIONS 1 & 2 are; eventually, hopefully, the answers

"YES to QUESTION 1,"

"NO to QUESTION 2"

will arise

You could write those answers on Board 1 next to the questions:

Board 1 supplement

YES QUESTION 1 ...

NO QUESTION 2 ...

Ask students:

"Why different answers to QUESTIONS 1 & 2?"

"How are the scenarios different?"

"The difference is an example of PROBABILITY;

head injuries are less likely in the second scenario"

Board 2

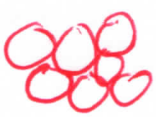
A less likely (lower probability) event affects our actions less.

"We prefer a number for probability. Let's denote"

Board 2 continued

P(A) [reads "P of A"] for probability of the event A.

new Board 1

Example A bag contains 20 M + Ms  ...

3 of which are green.

Board 1 continued

If I choose a random M&M from the bag, what is the probability it will be green?

Ask students; hopefully will eventually get

Board 1 continued

$\frac{3}{20} = 15\% = P(\text{green})$

"In general, probability is relative frequency"

new Board 2

$$P(A) = \frac{(\text{number of outcomes in } A)}{(\text{number of possible outcomes})}$$

if all outcomes are equally likely.

"In this setting, probability means counting; once for the numerator, once for the denominator."

IV. COIN FLIPPING

new Board 1

A coin is fair if, on each flip,

$P(H) = P(\text{Heads}) = P(\text{Tails}) = P(T)$

= (ask students)

$= \frac{1}{2}$ ← desired outcome

↖ what you could get

"We'd like to worry about the number of Heads, when flipping a fair coin. For example, if we get $5 for each Head, the number of Heads is of interest.

Many things can be modelled as number of Heads when flipping a fair coin."

new Board 2

Examples

1. Number of female offspring when reproducing

2. Number of correct answers, when guessing on each problem in a true/false quiz

NOTE: In Example 1, NEED female & male offspring equally likely, on each birth.

"We want probabilities of getting specified numbers of Heads, when flipping a fair coin."

new Board 1

$$\left.\begin{aligned} P(0\ Hs)\ (= P(T)) &= \tfrac{1}{2} \\ P(1\ H) &= \tfrac{1}{2} \end{aligned}\right\}\ 1\text{ flip}$$

2 flips: make tree diagram

"with 1st flip on the left, 2nd flip to the right of the 1st flip, followed by possible strings of Hs (Heads) & Ts (Tails)." (point)

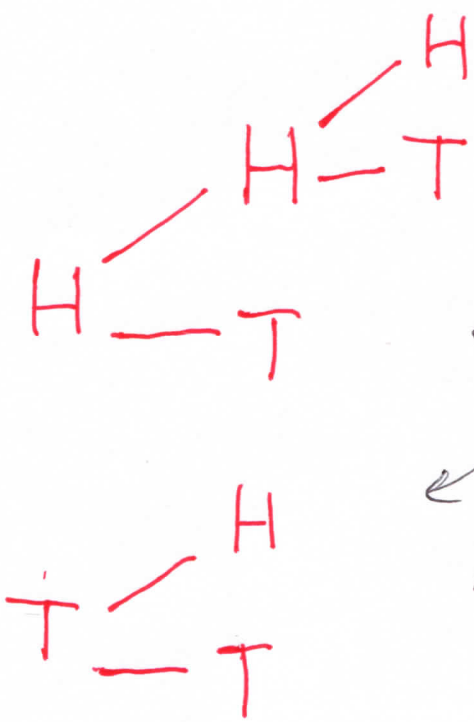
new Board 2

H	H	HH	2 Hs
	T	HT	1 H
T	H	TH	1 H
	T	TT	0 Hs

# of Hs	0	1	2	
# of ways	1	2	1	sum = 4
prob.	$1/4 = 1/2^2$	$2/4 = 2/2^2$	$1/4 = 1/2^2$	

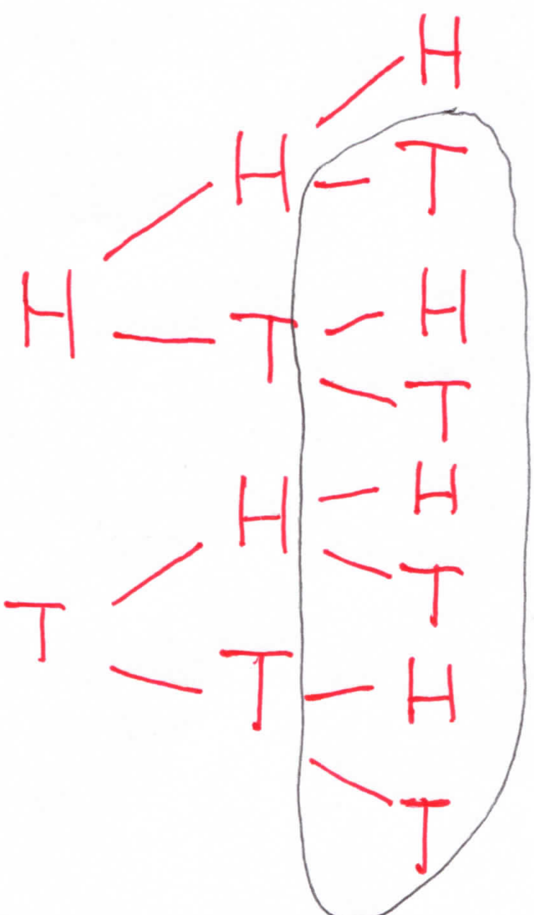
new Board 1

3 flips: "Expand the tree diagram for 2 flips"



Have a student fill in $<^{H}_{T}$ s, on Board 1

new Board 1



HHH 3

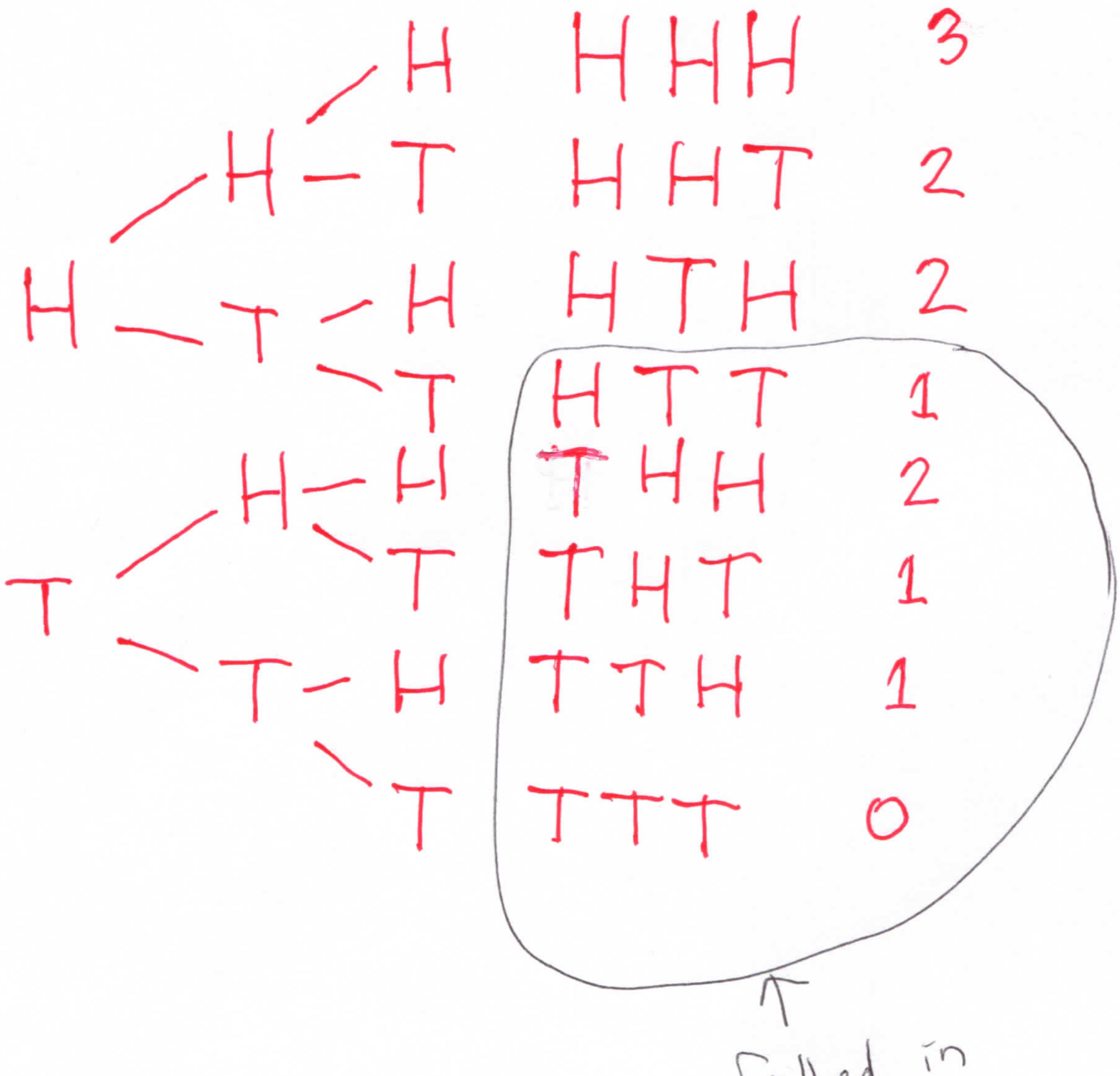
HHT 2

HTH 2

filled in by student

have another student fill in

new Board 1



filled in
by student

new Board 2

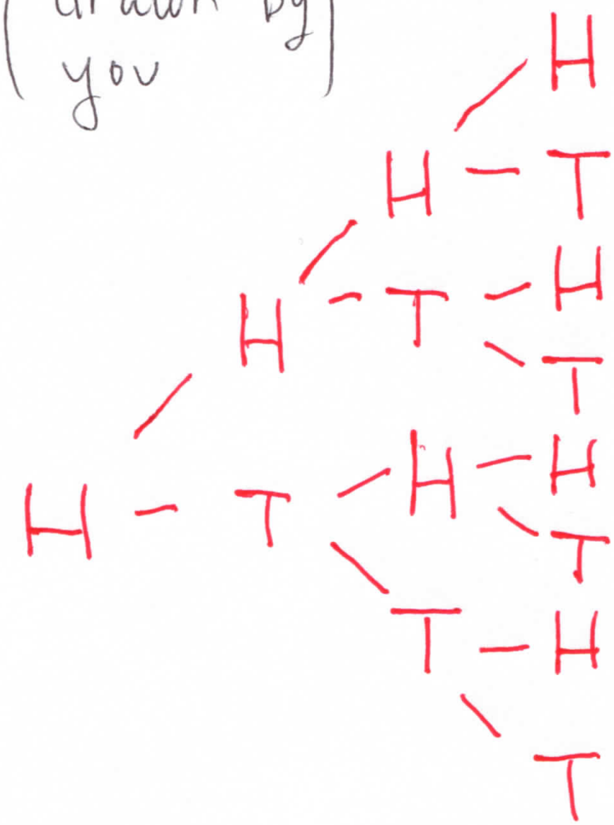
3 flips:

# of Hs	0	1	2	3
# of ways (ask students)	1	3	3	1 (sum = 8)
prob.	$1/8 = 1/2^3$	$3/8 = 3/2^3$	$3/8 = 3/2^3$	$1/8 = 1/2^3$

"4 flips?"

new Board 1

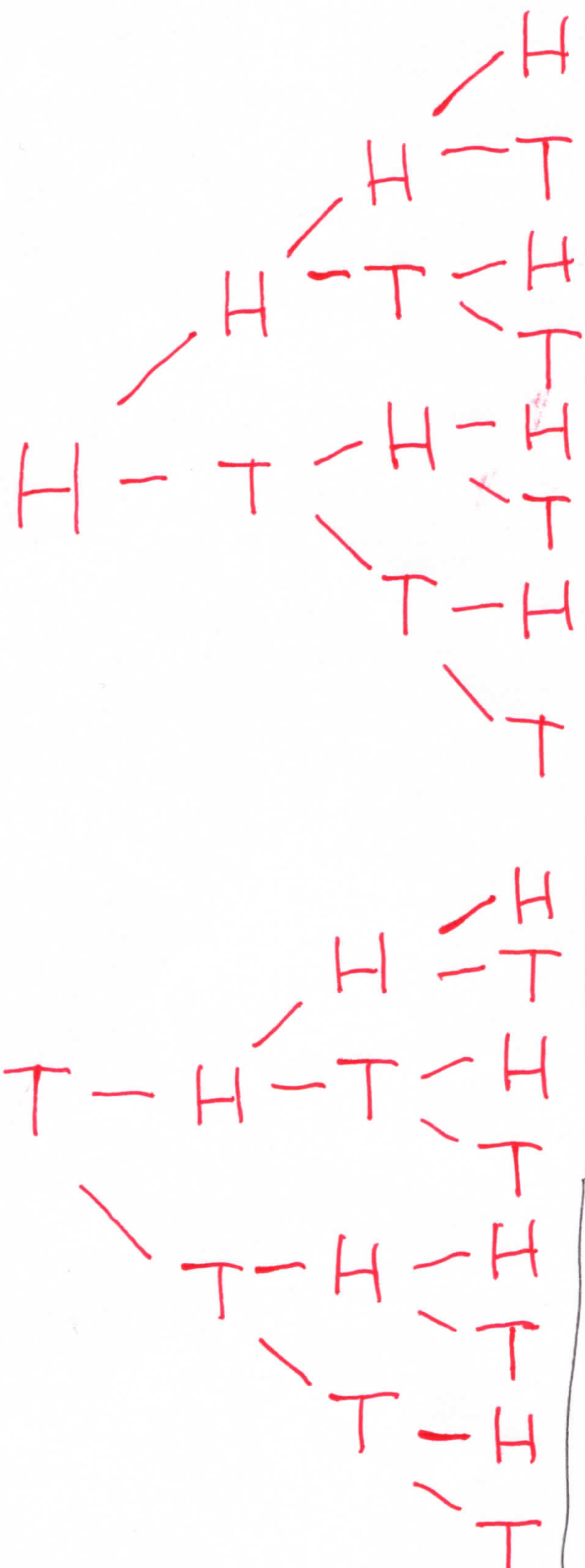
(drawn by you)



have a student or students fill in strings of H_s & T_s,

& count # of H_s in each string, as with 3 flips

new Board 1 completion of previous page



HHHH	4
HHHT	3
HHTH	3
HHTT	2
HTHH	3
HTHT	2
HTTH	2
HTTT	1
THHH	3
THHT	2
THTH	2
THTT	1
TTHH	2
TTHT	1
TTTH	1
TTTT	0

← filled in by students

new Board 2

4 flips

# of Hs	0	1	2	3	4	
# of ways	1	4	6	4	1	(sum = 16)
prob.	$\frac{1}{16} = \frac{1}{2^4}$	$\frac{4}{16} = \frac{4}{2^4}$	$\frac{6}{16} = \frac{6}{2^4}$	$\frac{4}{16} = \frac{4}{2^4}$	$\frac{1}{16} = \frac{1}{2^4}$	

drawn by you

filled in by students

"These trees are getting traumatic.

Here's a different picture, called Pascal's triangle."

V. PASCAL'S TRIANGLE

HAND OUT, to each participant, Pascal's triangle, with left column empty (as drawn in this document at the end of the exposition).

ASK STUDENTS: "What pattern do you see, going down Pascal's triangle; e.g., look at how the row beginning 1 8 28... is constructed from the row above."

new Board 1

1 7 21 35 21 7 1

1 8 . . . 28 8 1

Each entry is sum of two entries directly above.

"Pascal's triangle continues indefinitely; for example, below the row beginning

new Board 2

1 12 66 . . . we have

1 13 78 . . .

"Let's organize our prior results about coin flipping."

new Board 1

2 flips:

# of H's	0	1	2
# of ways	1	2	1
prob.	1/4	2/4	1/4

3 flips:

# of H's	0	1	2	3
# of ways	1	3	3	1
prob.	1/8	3/8	3/8	1/8

4 flips:

# of H's	0	1	2	3	4
# of ways	1	4	6	4	1
prob.	1/16	4/16	6/16	4/16	1/16

"Where do the numbers under 3 flips" (point to # of ways row under 3 flips) "appear in Pascal's triangle?"

new Board 2

								sum
			1					1
		1		1				2
	1		2		1			4
1		3		3		1		8

"In the probabilities for 3 flips, the numerators are in the row 1 3 3 1" (point to that row of Pascal's triangle);

"The denominators are all 8, appearing under sum, to the right of the row 1 3 3 1" (point to the number 8 on Pascal's triangle).

"For 4 flips, look at the row on Pascal's triangle beginning 1 4:"

new Board 2

					Sum
1	4	6	4	1	
0 Hs	1 H	2 Hs	3 Hs	4 Hs	16

"For example,"

new Board 2

$P(2 \text{ Hs in } 4 \text{ flips}) = \frac{6}{16}$

1 4 6 4 1 16

numerator denominator

TELL STUDENTS: "Fill in the left column of your Pascal's triangle with the number of flips:"

new Board 2

flips								sum
0				1				1
1			1		1			2
2		1		2		1		4
3	1		3		3		1	8
⋮	⋰			⋮			⋱	⋮

See next page; check that students filled in their left column correctly.

Pascal's Triangle

flips		sum
0	1	1
1	1 1	2
2	1 2 1	4
3	1 3 3 1	8
4	1 4 6 4 1	16
5	1 5 10 10 5 1	32
6	1 6 15 20 15 6 1	64
7	1 7 21 35 35 21 7 1	128
8	1 8 28 56 70 56 28 8 1	256
9	1 9 36 84 126 126 84 36 9 1	512
10	1 10 45 120 210 252 210 120 45 10 1	1,024
11	1 11 55 165 330 462 462 330 165 55 11 1	2,048
12	1 12 66 220 495 792 924 792 495 220 66 12 1	4,096

p. 32

"For example, let's say we want to write down all probabilities associated with 5 flips of a fair coin. Go to Pascal's triangle, the row for 5 flips:"

new Board 2

flips		sum
5	1 5 10 10 5 1	32 $(=2^5)$
	numerator	denominator

Board 2 continued

5 flips:

# of Hs	0	1	2	3	4	5
# of ways	1	5	10	10	5	1
prob.	$1/32$	$5/32$	$\frac{10}{32}$	$\frac{10}{32}$	$5/32$	$1/32$

e.g., $P(3\text{ Hs in }5\text{ flips}) = \frac{10}{32}$.

"Use Pascal's triangle to get the following probabilities, where all coins are fair."

new Board 1

1. P(2 Hs in 7 flips) = ?

"Go to the 7 flips row, that is, the row beginning 1 7 ...

0 Hs	1 H	2 Hs	...		
1	7	21	35	35	...

$\rightarrow \frac{21}{2^7} = \frac{21}{128}$

2. P(6 Hs in 10 flips) = ?

1	10	45	120	210	252	210	120	45	10	1
0 H	1 H	2 Hs	3 Hs	4 Hs	5 Hs	6 Hs				

(You should count a point: 0 Hs, 1 H, 2 Hs, etc.)

$\frac{210}{2^{10}} = \frac{210}{1,024}$

new Board 2

3. What is the probability of getting 60% on a 10-problem true/false quiz, if you guess on each problem?

SAME as Example 2: $\boxed{\dfrac{210}{1{,}024}}$

4. P(4 Hs in 12 flips) = ?

1	12	66	220	495	. . .
0 Hs	1 H	2 Hs	3 Hs	4 Hs	

$$\rightarrow \frac{495}{2^{12}} = \boxed{\frac{495}{4{,}096}}$$

5. If 12 offspring are produced, what is the probability that exactly 4 of them are girls, assuming boys & girls are equally likely, with each birth?

SAME as Example 4: $\boxed{\dfrac{495}{4{,}096}}$

MORE?? P(6 Hs in 8 flips) = ?

(ANSWER: $\frac{28}{256}$)

P(3 Hs in 6 flips) = ?

(ANSWER: $\frac{20}{64}$)

VI. MORE PASCAL'S TRIANGLE (OPTIONAL)

"It is often the case that subsets of a specified size must be chosen from a set. For example, a social club wishes to choose a pair from its ranks to compete in a dance contest."

new Board 1

TERMINOLOGY: For $k = 0, 1, 2, \ldots$ & $n \geq k$, $C_{n,k}$ is the number of subsets of size k that may be chosen from a set of size n.

"In our dance contest,"

new Board 2

$C_{n,2}$ equals the number of pairs we could choose from n people.

Board 1 continued

$C_{n,k}$ is called combinations of n things taken k at a time; it is also called "n choose k" & denoted $\binom{n}{k}$.

"Let's suppose our social club has 5 people."

new Board 2

$C_{5,2}$ equals the number of pairs (subsets of size 2) that may be chosen from the 5 people in the club.

Labelling the club members

$\{A, B, C, D, E\}$,

"let's write down all such pairs!"

Board 2 continued

$$\left.\begin{array}{llll} AB & AC & AD & AE \\ & BC & BD & BE \\ & & CD & CE \\ & & & DE \end{array}\right\}$$

10 pairs; that is

$C_{5,2} = 10$

"Please note that $C_{5,2} = 10$ appears in Pascal's triangle:"

new Board 1

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & & & = C_{5,2} & & & & & &
\end{array}
$$

In the language of coin flipping, $C_{5,2} = 10$ = the number of ways to get 2 Hs in 5 flips.

"In general, it may be shown that"

new Board 2

$C_{n,k}$ = # of ways of getting k Hs in n flips

"d may be found on Pascal's triangle, just as we did with coin flipping."

"For example, if our social club had 9 members & we wanted a double date (4 people) from our club, the number of double dates that could be chosen (denoted $C_{9,4}$) may be found by looking at the row of Pascal's triangle that begins 1 9 ...

new Board 1

0 Hs	1 H	2 Hs	3 Hs	4 Hs	
1	9	36	84	126	...

$\uparrow$
$C_{9,4}$

We see 126 possible double dates chosen from 9 people; that is,

$$C_{9,4} = 126$$

new Board 2

Example 1 A bag contains 12 M&Ms, 3 of which are green. If I choose 8 M&Ms at random from the bag, what is the probability that none are green?

new Board 1

RECALL $P(A) = \dfrac{(\text{\# of outcomes in } A)}{(\text{\# of possible outcomes})}$

Board 2 continued

OUTCOMES here are 8 M&Ms chosen from 12; there are $C_{12,8}$ ways to do that.

Our desired outcome (denoted A) is 8 M&Ms chosen from the non-green M&Ms. "How many M&Ms are not green?"

$C_{9,8}$ ways to get 8 non-green M&Ms.

Board 1 continued

Our probability is $\frac{C_{9,8}}{C_{12,8}}$

= (from Pascal's triangle) $\frac{9}{495}$

new Board 2

Example 2 (ask students)

Choose 6 M&Ms at random from a bag of 10, 2 of which are green. What is the probability none are green?

(after students work)

$\frac{C_{8,6}}{C_{10,6}}$ = (Pascal) $\frac{28}{210}$

new Board 1

Example 3 (ask students)

A box contains 11 cookies, 5 of which are chocolate. If I choose 3 at random, what is the probability none are chocolate?

(after students work)

$$\frac{C_{6,3}}{C_{11,3}} = (\text{Pascal}) \; \frac{20}{165}$$

new Board 2

Example 4 (ask students)

A plumber's guild contains 9 people, including the 3 Stooges, Curly, Moe, & Larry. If I choose 2 people at random from the guild, what is the probability neither is a Stooge?

(after students work)

$$\frac{C_{6,2}}{C_{9,2}} = (\text{Pascal}) \ \frac{15}{36}$$

new Board 1

Example 5 (ask students)

SAME as Example 4, except choose 4.

(after students work)

$$\frac{C_{6,4}}{C_{9,4}} = (\text{Pascal}) \ \frac{15}{126}$$

"Our last application of Pascal's triangle is not directly about probability."

Ask students:

"Does "

new Board 1

$$(a+b)^2 \stackrel{?}{=} (a^2+b^2)\ ?$$

"No; the consequences would be disconcerting, as we will now demonstrate."

"Suppose, hypothetically, that"

new Board 2

$(a+b)^2$ does equal (a^2+b^2). (?)

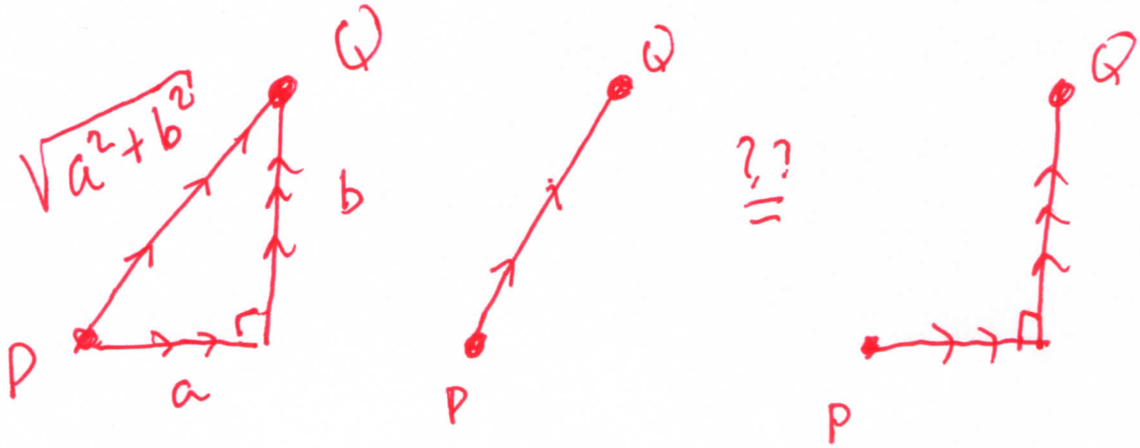
"Assuming a & b are positive, we then have"

Board 2 continued

$$(a+b) = \sqrt{a^2+b^2}$$

"By the Pythagorean theorem, we conclude that, in a right triangle, the length of the hypotenuse equals the sum of the lengths of the legs."

new Board 1



"That equality of lengths is not believable."

"The truth is more complicated:"

new Board 2

$$(a+b)^2 = (a+b)(a+b) =$$
$$a(a+b) + b(a+b) = a^2 + ab + ba + b^2$$
$$= (a^2 + 2ab + b^2).$$

"We used the Distributive Law, twice."

"Third powers are worse"

Board 2 continued

$$(a+b)^3 = (a+b)(a+b)^2 =$$

$$(a+b)(a^2+2ab+b^2) =$$

$$a(a^2+2ab+b^2) + b(a^2+2ab+b^2)$$

$$= \ldots = (a^3+3a^2b+3ab^2+b^3).$$

Similarly, $(a+b)^4 =$

$$(a^4+4a^3b+6a^2b^2+4ab^3+b^4).$$

"This process is tedious; but compare our coefficients to the rows of Pascal's triangle!"

new Board 1

$$(a+b)^1 = 1\cdot a + 1\cdot b$$

$$(a+b)^2 = 1\cdot(a^2) + 2\cdot(ab) + 1\cdot(b^2)$$

$$(a+b)^3 = 1\cdot(a^3) + 3\cdot(a^2b) + 3\cdot(ab^2) + 1\cdot(b^3)$$

$$(a+b)^4 = 1\cdot(a^4) + 4\cdot(a^3b) + 6\cdot(a^2b^2)$$
$$+ 4\cdot(ab^3) + 1\cdot(b^4)$$

"Suppose we wanted $(a+b)^{10}$.
Go to the row in Pascal's
that begins 1 10 . . . : "

new Board 2

1 10 45 120 210 252 210 120 45 10 1

"Use those numbers for coefficients of"

$a^{10}, a^9b, a^8b^2, \ldots, ab^9, b^{10}$:

$$(a+b)^{10} = 1\cdot(a^{10}) + 10\cdot(a^9b) + 45\cdot(a^8b^2)$$
$$+ 120\cdot(a^7b^3) + 210\cdot(a^6b^4) + 252\cdot(a^5b^5)$$
$$+ 210\cdot(a^4b^6) + 120\cdot(a^3b^7) + 45\cdot(a^2b^8)$$
$$+ 10\cdot(ab^9) + 1\cdot(b^{10}).$$

"Note that the exponents add up to 10, in each term."

Ask students (have them come to board?):

new Board 1

Use Pascal's triangle to expand $(a+b)^8$

Should eventually get

~~new~~ Board 1 continued

$a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3$
$+ 70a^4b^4 + 56a^3b^5 + 28a^2b^6$
$+ 8ab^7 + b^8$

"For more about

probability, see"

new Board 2

https://teacherscholarinstitute.com

MATH MAGNIFICATIONS,

especially

Probability and Counting &

Probability Introduction

Pascal's Triangle

	Sum
1	1
1 1	2
1 2 1	4
1 3 3 1	8
1 4 6 4 1	16
1 5 10 10 5 1	32
1 6 15 20 15 6 1	64
1 7 21 35 35 21 7 1	128
1 8 28 56 70 56 28 8 1	256
1 9 36 84 126 126 84 36 9 1	512
1 10 45 120 210 252 210 120 45 10 1	1,024
1 11 55 165 330 462 462 330 165 55 11 1	2,048
1 12 66 220 495 792 924 792 495 220 66 12 1	4,096