TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI

# Pythagorean Theorem
# DIY (Do-It-Yourself) Math Workshop

Dr. Ralph deLaubenfels

TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI

Teacher-Scholar Institute

Columbus, Ohio

2020

# DIY (Do-It-Yourself)

# PYTHAGOREAN THEOREM WORKSHOP

As with all DIY Workshops,

Writing / drawings in red —
are written on chalkboard
& possibly spoken;

Writing in quotes in black
" ——— " is said out loud to
students, not written;

Writing not in quotes in black
——— is (suggested), not
spoken or written

# TABLE OF CONTENTS

# IX (Optional) Proof of Converse of Pythagorean Theorem

p. 65

# I. PREREQUISITES

For full generality of results, the use of letters for general numbers will be needed.

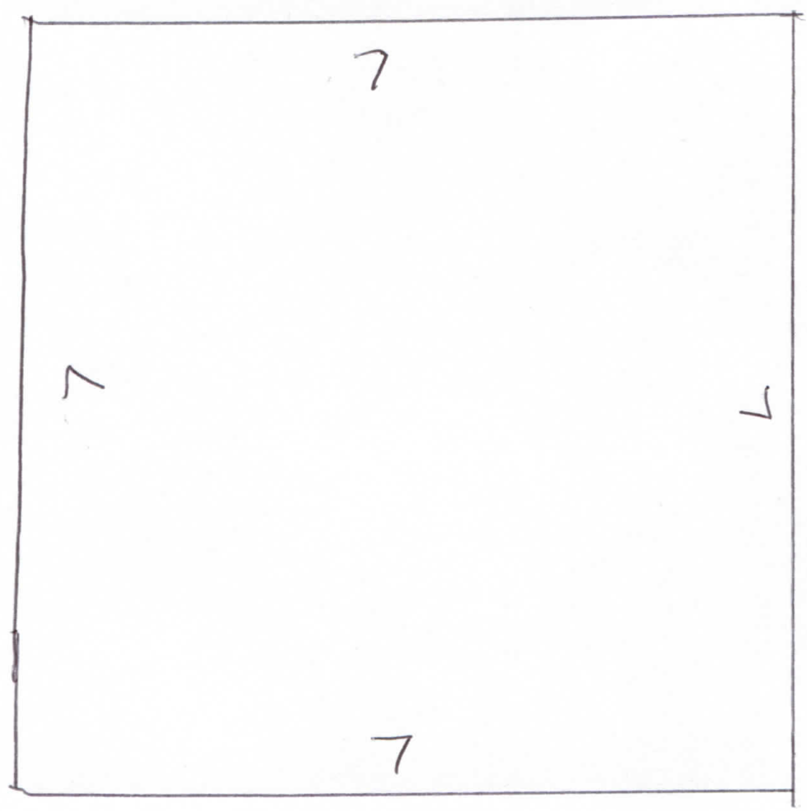
The following topics will be briefly reviewed:

squares & squaring; square roots; areas of rectangles; angles; right angle; right triangle

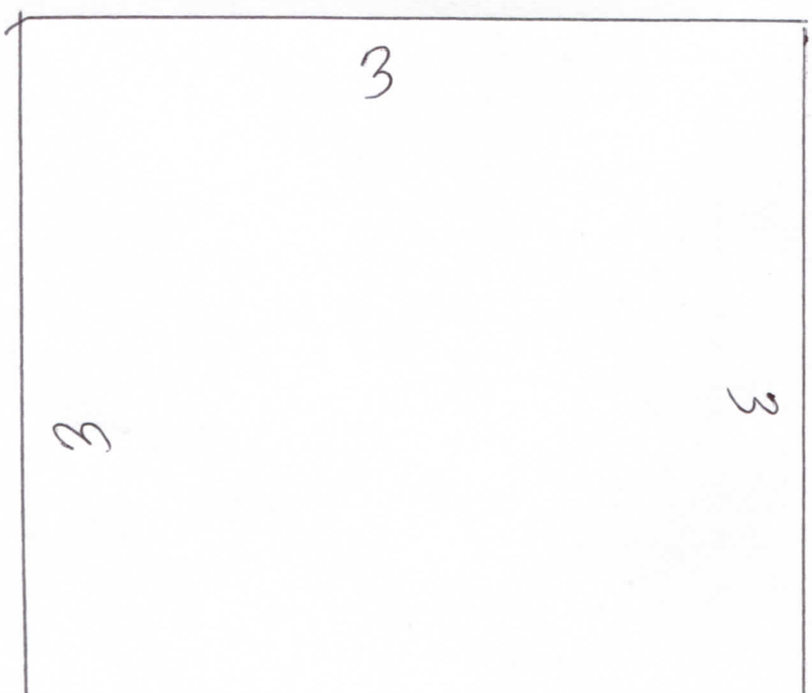
# II. MATERIALS NEEDED FOR PART 1

Two chalkboards, that we will call Board 1 and Board 2, and, for each participant (1)-(5) not drawn to scale):

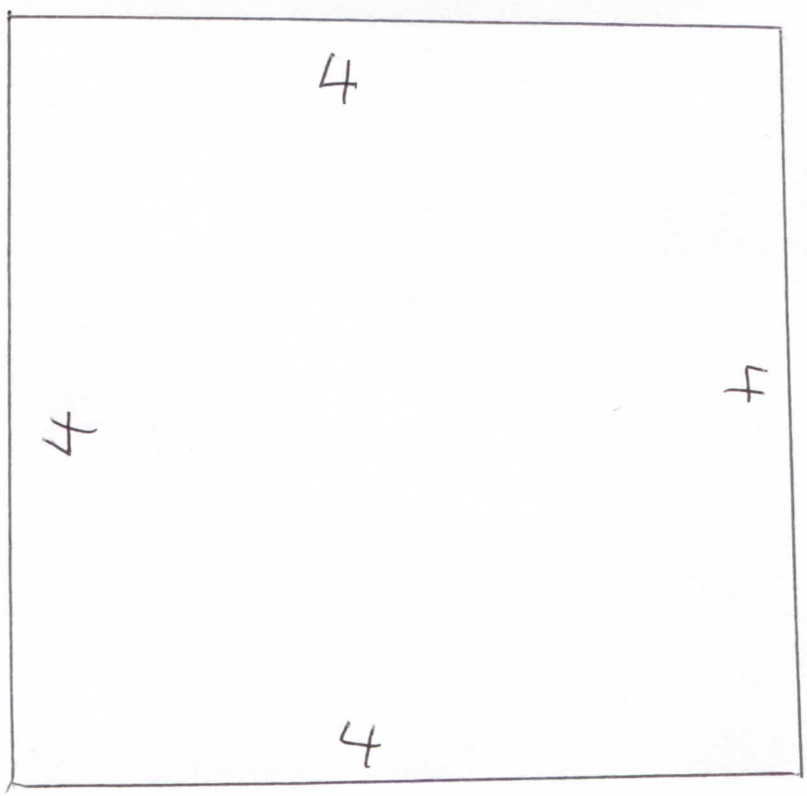
(1) 2 cardstock squares, each of side $8\frac{1}{2}$ inches, with sides labelled 7



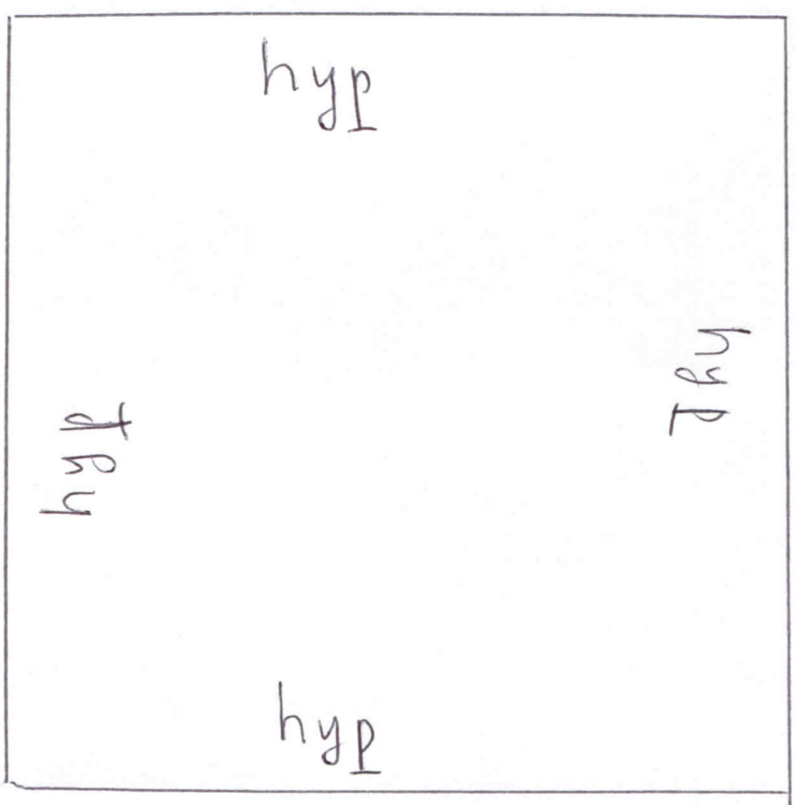
(2) 1 cardstock square of side $3\,5/8$ inches, with sides labeled 3



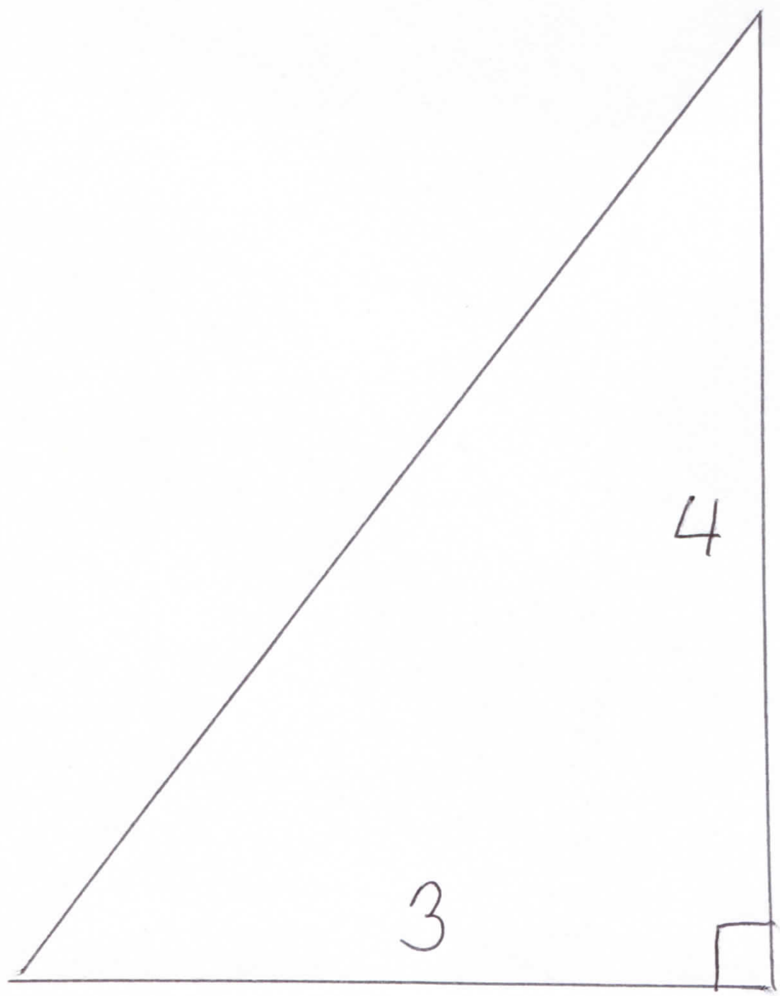
(3) 1 cardstock square of side $4\frac{7}{8}$ inches, with sides labeled 4



(4) 1 cardstock square of side $6\frac{1}{16}$ inches, with sides labeled hyp (short for hypotenuse)



(5) 8 right triangles, each with a leg of length $3\frac{5}{8}$ inches (labeled 3) and a leg of length $4\frac{7}{8}$ inches (labeled 4)



RATIONALE

for the materials needed:

If a quasi-cubit (q.c.)

is defined to be $(8.5 \times \frac{1}{7})$

inches, then the squares in

(1) each have side ~ 7 q.c.,

the square in (2) has side

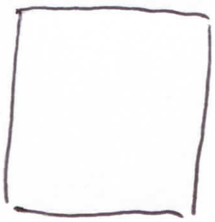
~ 3 q.c., the square in (3) has

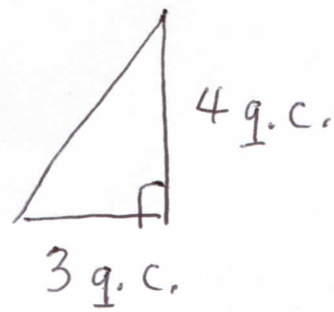
side ~ 4 q.c., the square in (4)

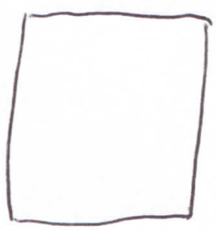
has side ~ 5 q.c. (don't tell

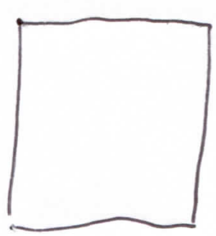
the participants this), and

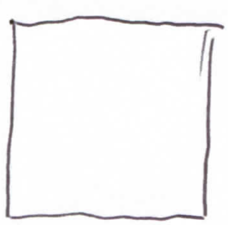
each right triangle
in (5) has legs of length
3 q.c. and 4 q.c.

(1)  7 q.c.

(5) 


(2)  3 q.c.

(3)  4 q.c.

(4)  5 q.c. (don't tell participants)

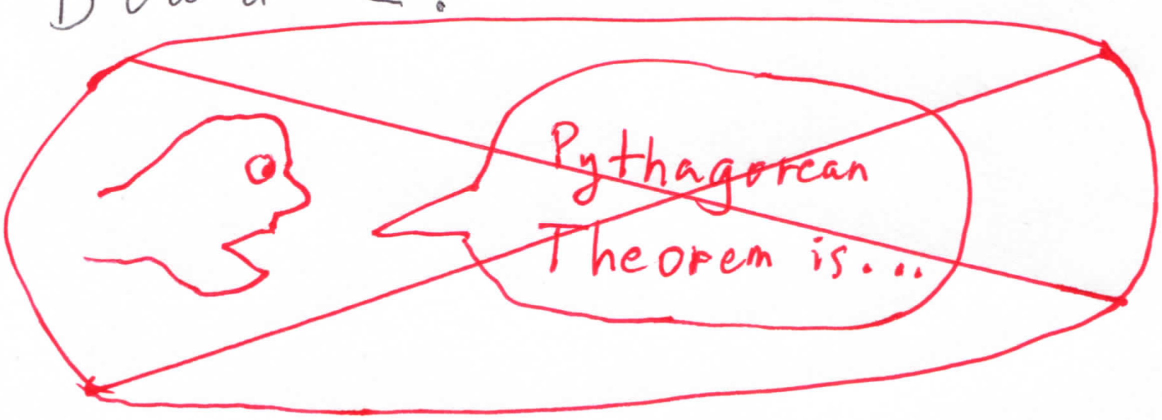
Board 1:

# PYTHAGOREAN THEOREM

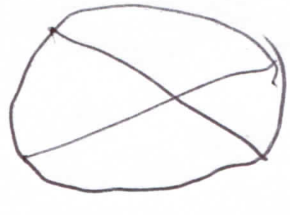
# III. PRELIMINARIES

"Rules of engagement: If you already know the Pythagorean Theorem, please don't say it."

Board 2:




"Note the Math Busters symbol"

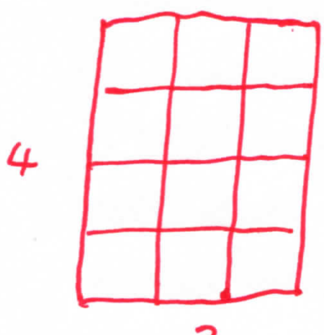


"We will discover the Pythagorean Theorem, without any prior knowledge of said theorem."

new Board 1:

# PRELIMINARIES

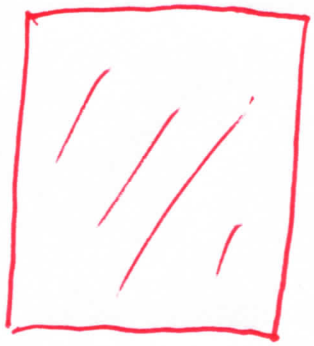
## Area of rectangle




$12 = 3 \times 4 = $ area

("Count the number of little squares")

In general:



height

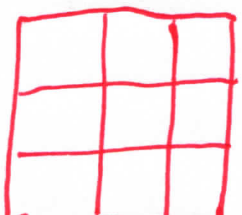
base

area =

(height) × (base)

new Board 2:

3

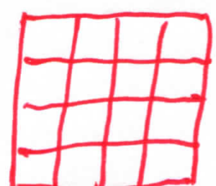


3

area = (ask students)

$9 = 3 \times 3 = 3^2$ ("3 squared")

Area of 4



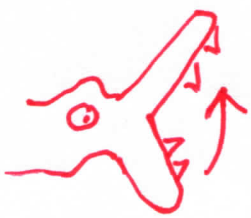
4

(ask students)

$= 16 = 4 \times 4 = 4^2$ ("4 squared")

new Board 1:

<u>Angle</u> between two lines is like a door or crocodile jaws opening:



large angle measure

small angle measure

zero angle measure

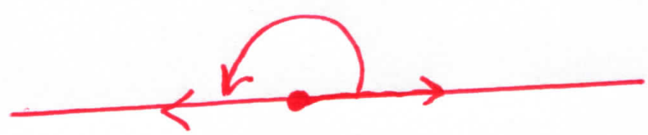
If door is available to open, show zero (closed door), small, and large angle measures.

new Board 2:



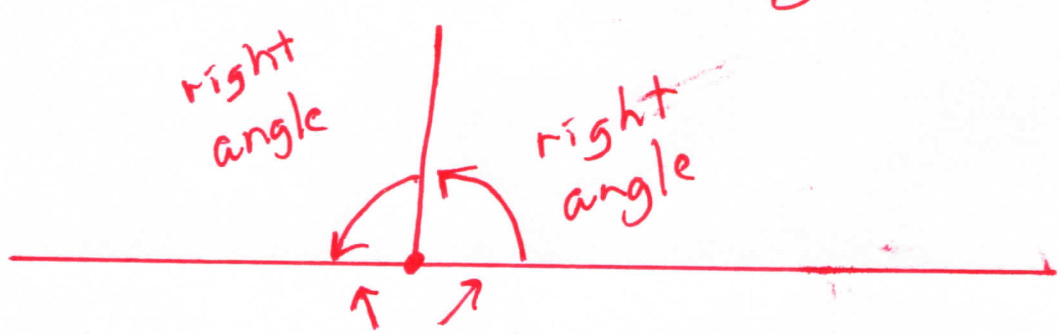
## Straight Angle:

Reverse direction on straight line



## Right Angle:

half a straight angle




equal angles

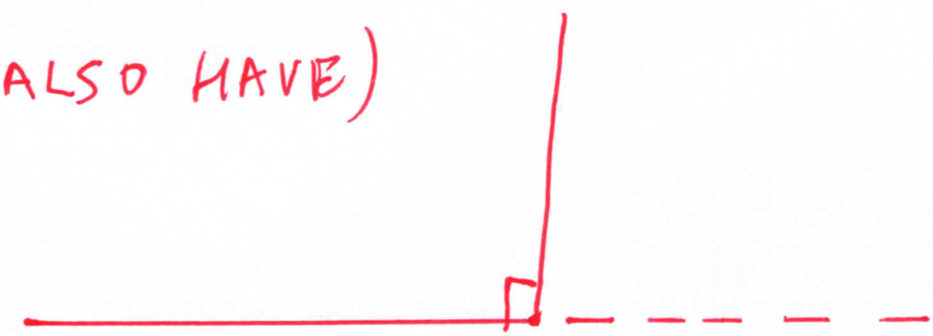
("Euclid's definition")

new Board 1:



TRADITIONAL SYMBOL for right angle:

(ALSO HAVE)



new Board 2:

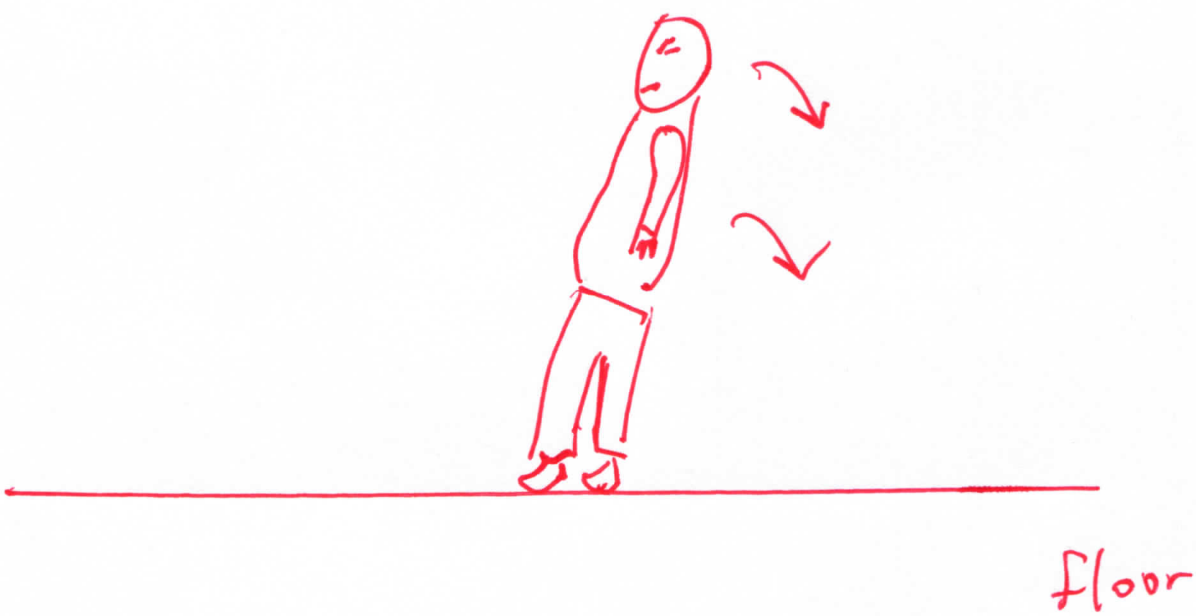
"Consider the following building"


"Is anything wrong?"

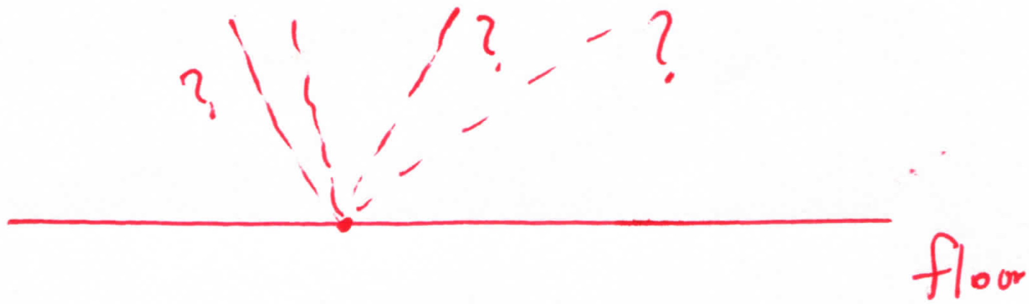
p. 18

"Try standing like that wall" (You should try to lean over as far as possible)

new Board 1:

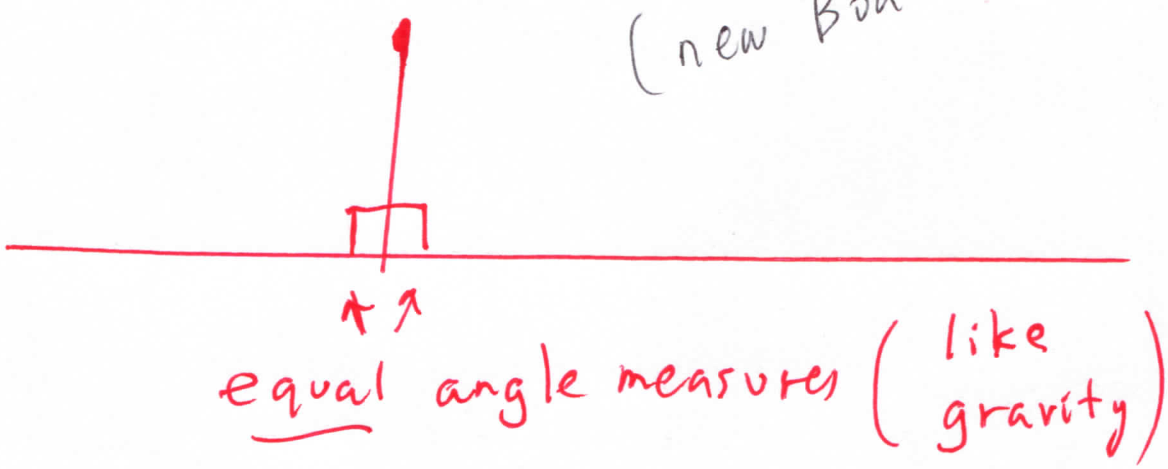


"How would you fix the wall?" (in Board 2)




Student come up & draw on Board 2?

(new Board 2)




"NOTE that we need a right angle."

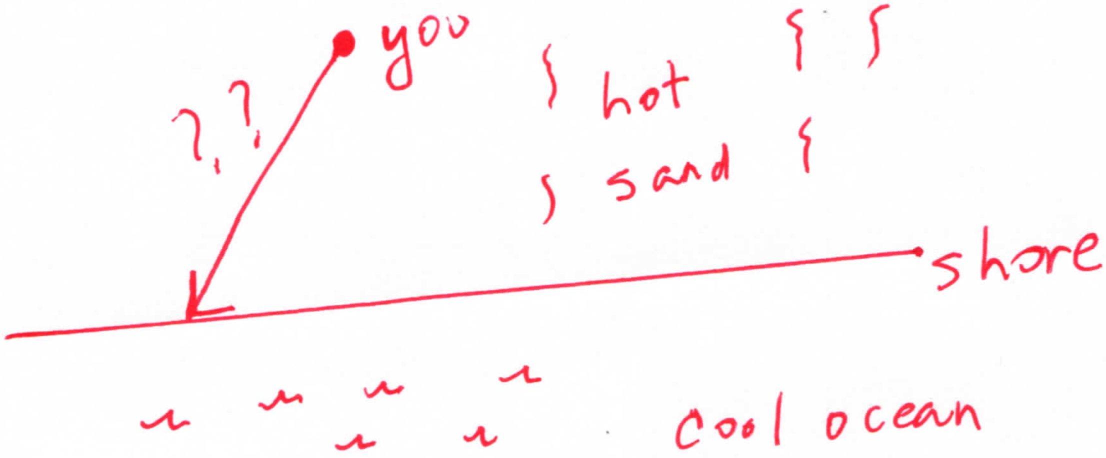
new Board 1:



Two lines are orthogonal or perpendicular if they form a right angle.

"Here is another example of how important orthogonality is"

new Board 2:

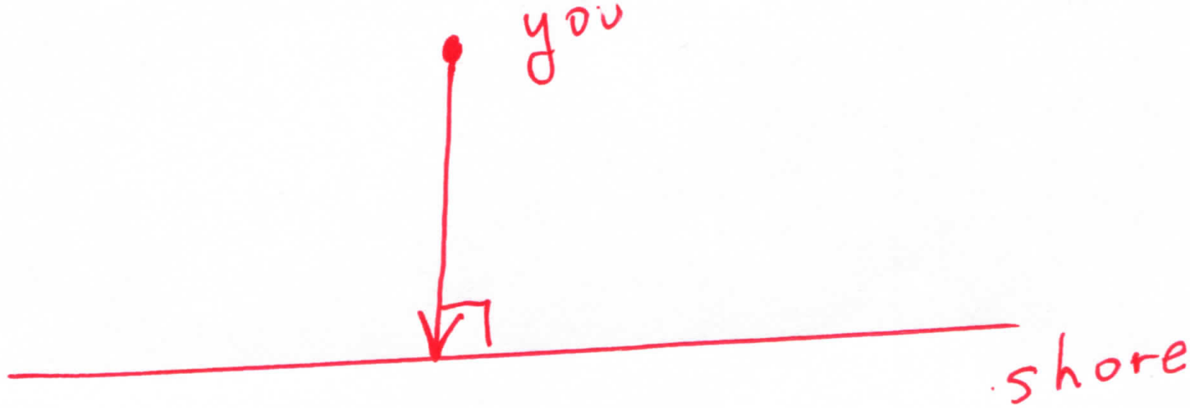



"Imagine burning your feet on hot sand at the beach"

"What's wrong with my picture of travelling to the ocean to cool off?"

new Board 1

Shortest route to the ocean:

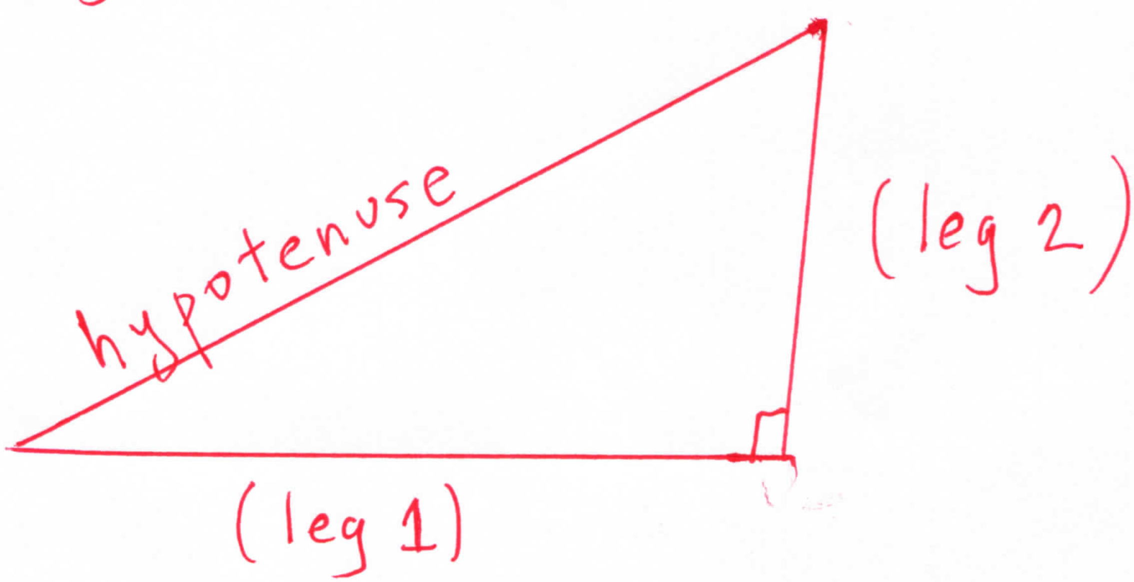



"You want your path to the ocean to be orthogonal or perpendicular to the shore."

# IV. PENGUINS

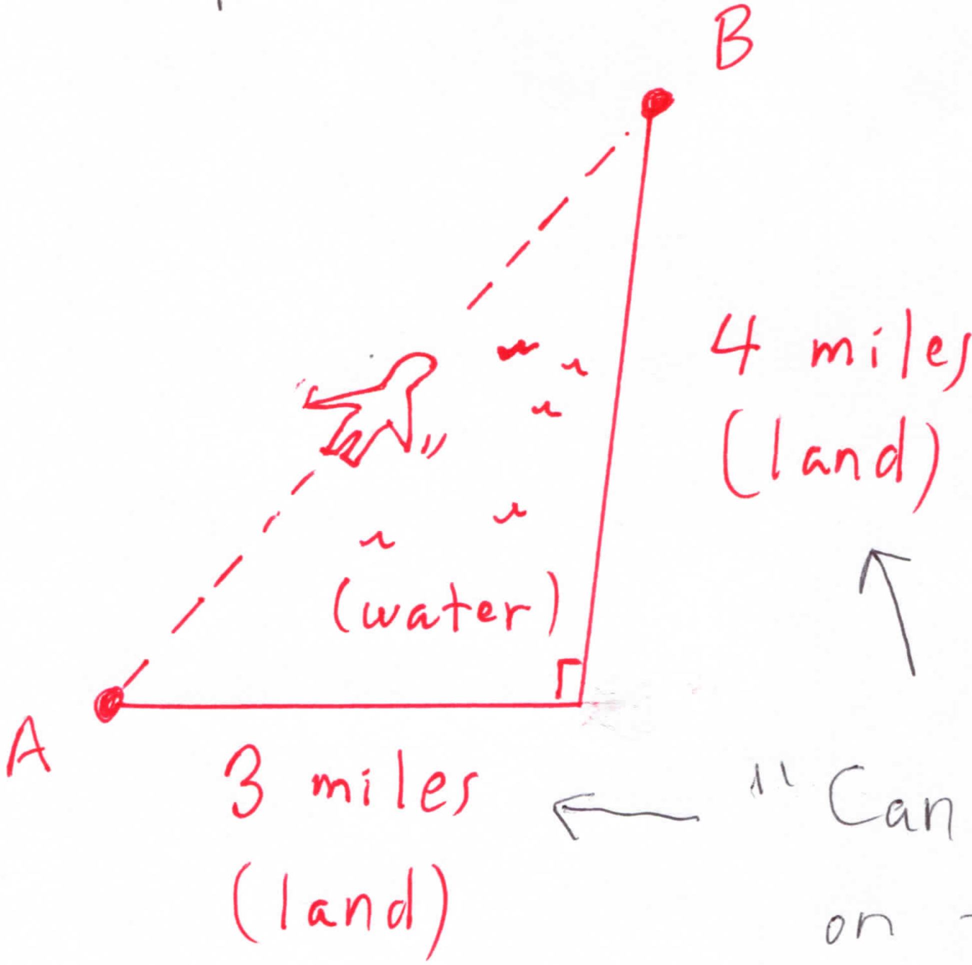
new Board 2:

A right triangle is a triangle with a right angle:



"For example, consider the following picture of a penguin who wants to travel through water from point A to point B"

new Board 1




"Can measure on foot"

"Penguins are much faster and smoother in water."

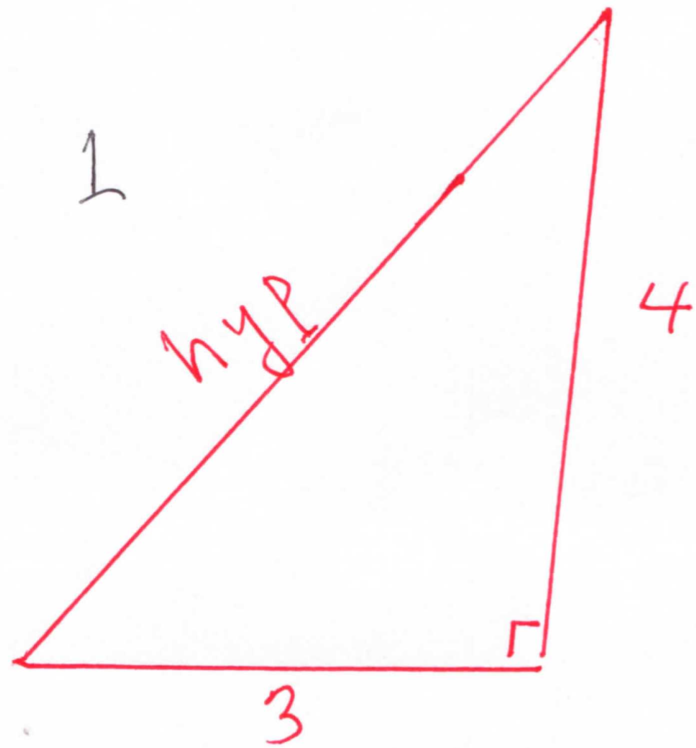
new Board 2:

WANT TO KNOW length of line from A to B

"preferably before tossing penguin in water; we don't want the penguin to get exhausted."

"Let's simplify our picture"

new Board 1

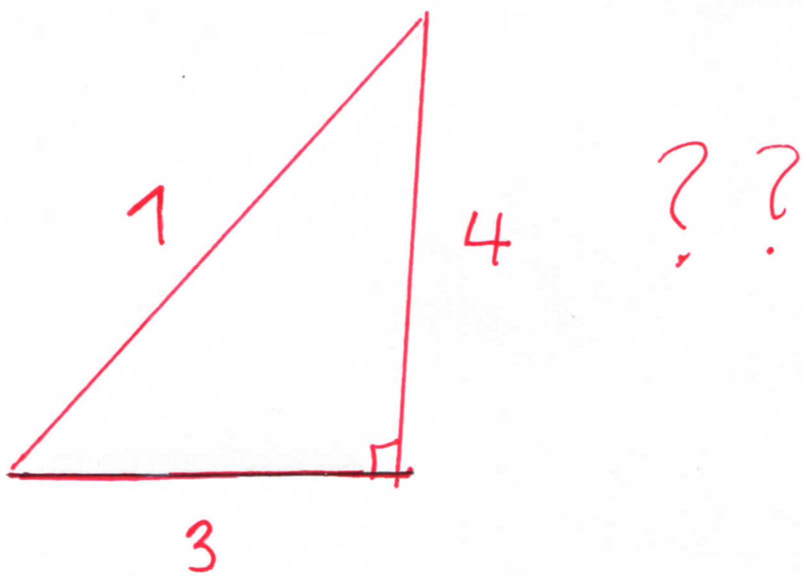



GOAL: Get "hyp"

(short for <u>hypotenuse</u>)

"WHY NOT

hyp = 3 + 4 = 7?"

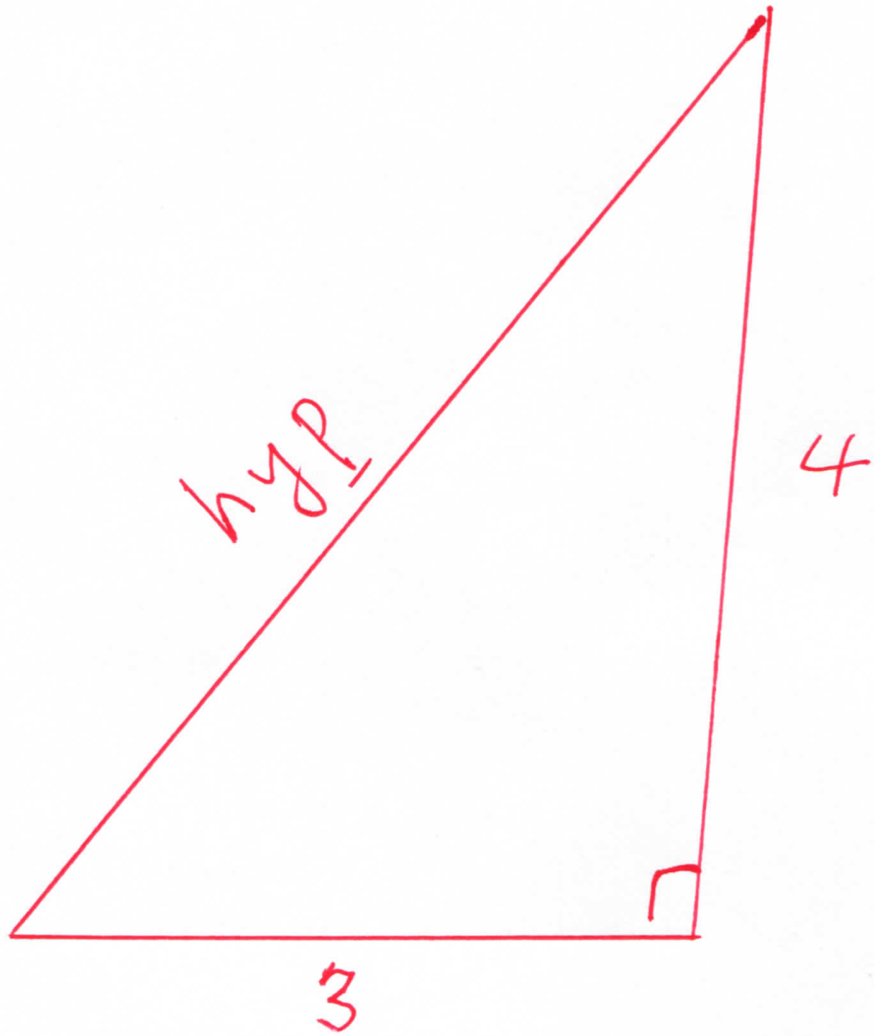



new

Board 2

"Problem with picture: shortest path between two points is a straight line, in this case the hypotenuse."

new Board 1

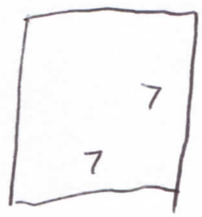


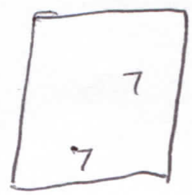

hyp = ??

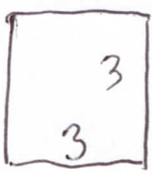
hyp < (3 + 4) = 7

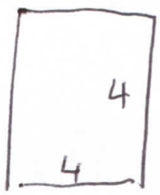
# V. JIGSAW PUZZLES LEAD TO PYTHAGOREAN THEOREM

GIVE, to each participant,

five squares: 

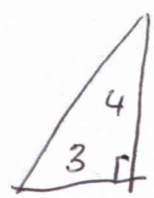
, 

, 

, 

& 

;

and eight right triangles 


(SEE Materials Needed for Part 1)

Have each participant
make two jigsaw puzzles:

new Board 2

# JIGSAW PUZZLES

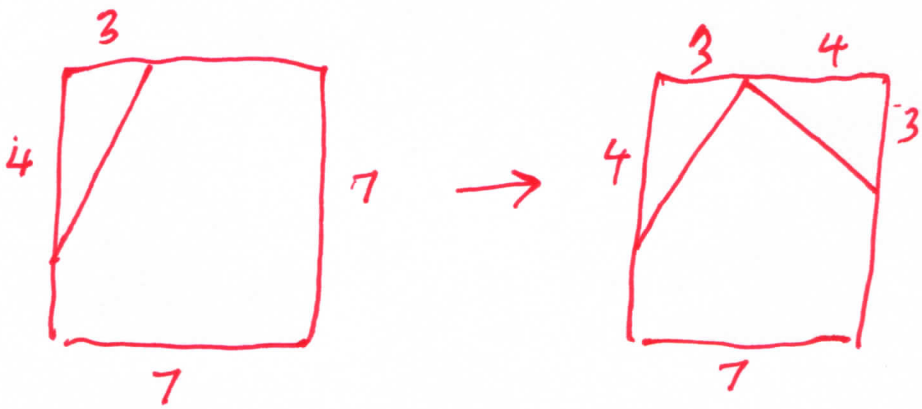
(1) Make 4 $\triangle$s & 1 $\boxed{\text{hyp}}$ cover
one of $\boxed{7\ 7}$ with no overlap AND

(2) Make 4 $\triangle$s & smaller squares
$\boxed{3\ 3}$ & $\boxed{4\ 4}$ cover one of $\boxed{7\ 7}$
with no overlap.

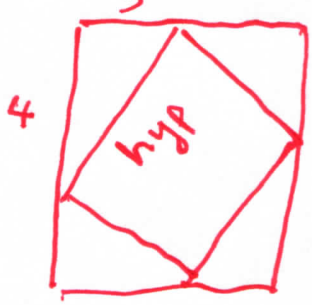
Walk around, see how students are doing, ad lib assistance if needed...

After some time of students working:  new Board 1

HINT for jigsaw (1):

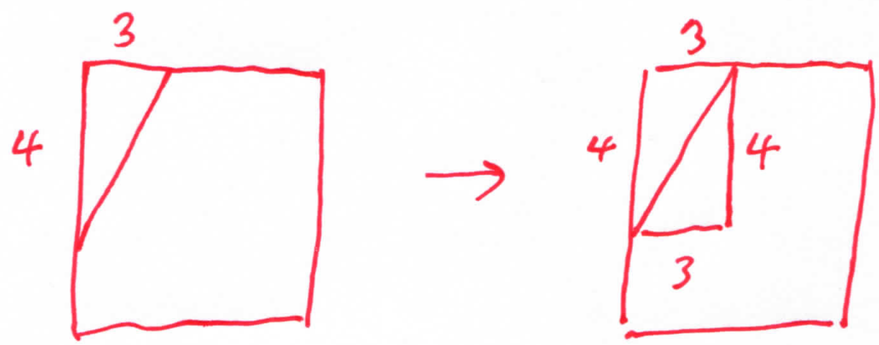



→ . . .
let some time elapse

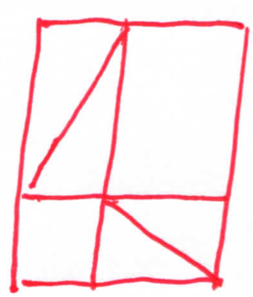


Similarly ad lib
assistance on jigsaw(2)
as students work on it,
& eventually: new Board 2

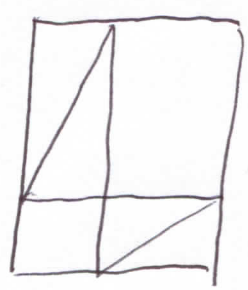
HINT for jigsaw(2):



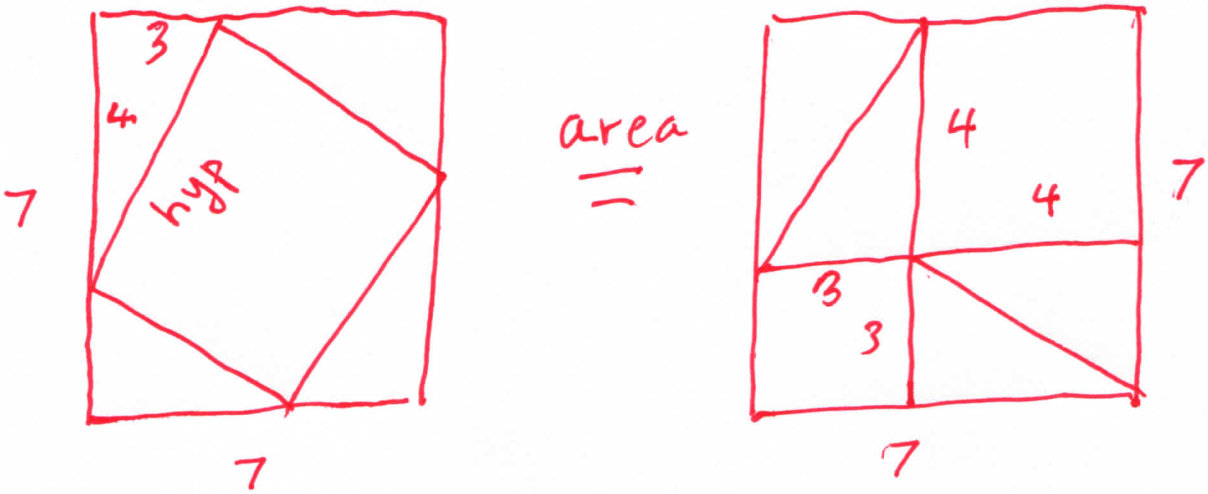

→ . . .
let some time elapse



OR



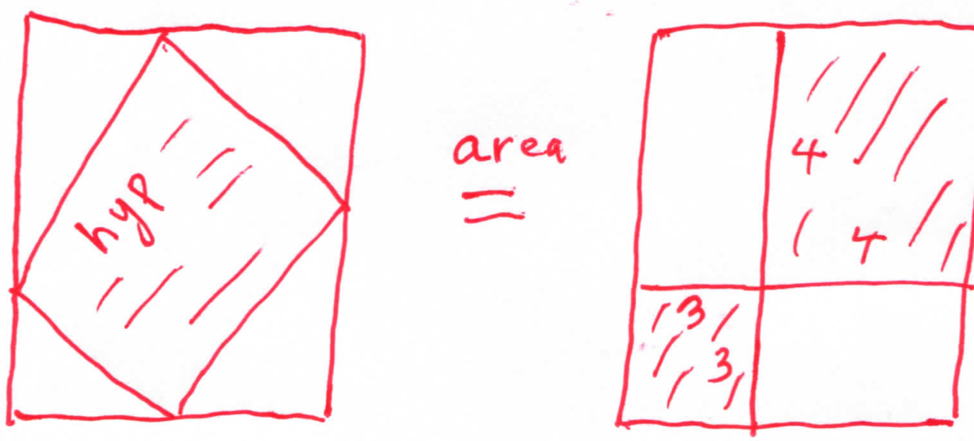
new Board 1




"What could be removed from both sides of equality while keeping the areas equal?"

Students should physically remove right triangles

new Board 2




"Look at areas of squares"

new Board 1

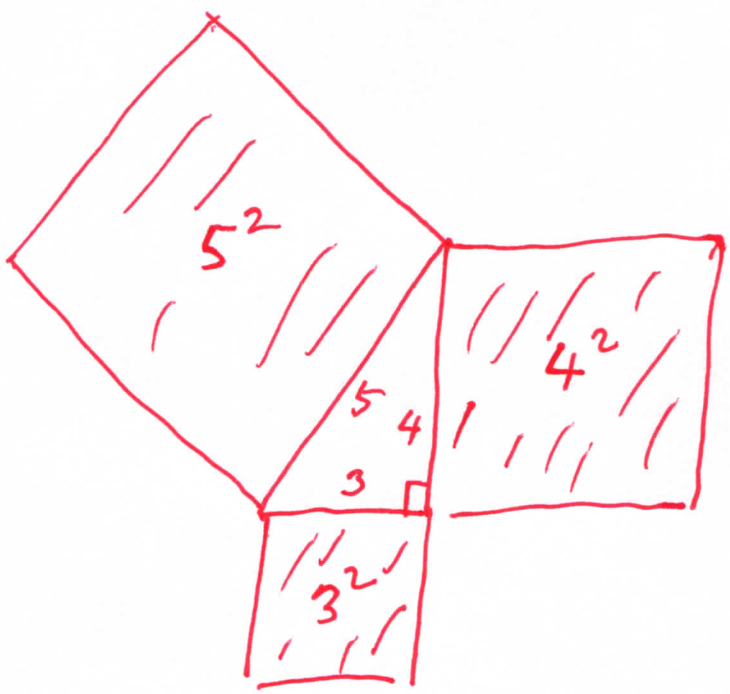
$$(hyp)^2 = 3^2 + 4^2 = 9 + 16 = 25$$

$$\rightarrow (hyp) = \sqrt{25} = 5.$$

(since $5^2 = 25$)

"Here is a popular picture to emphasize squaring & areas of squares"

new Board 2




$5^2 = 3^2 + 4^2$

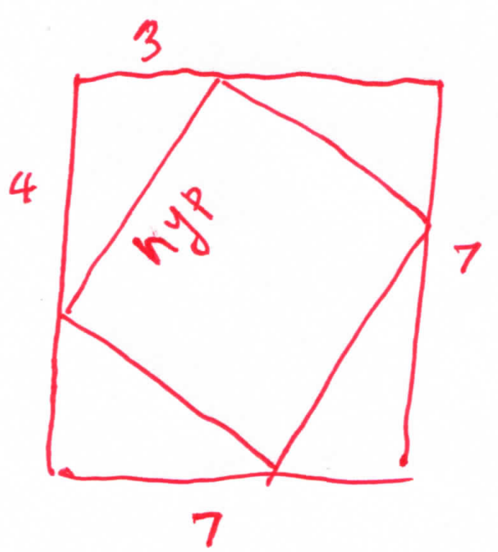
"Here is..."

(new Board 1)

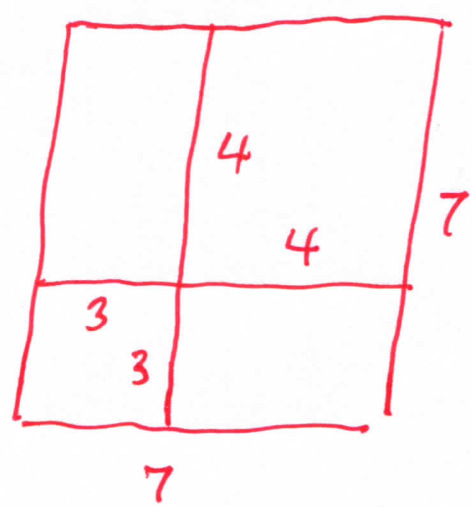
Another right triangle


(hyp) = ??

"Recall" (new Board 2)

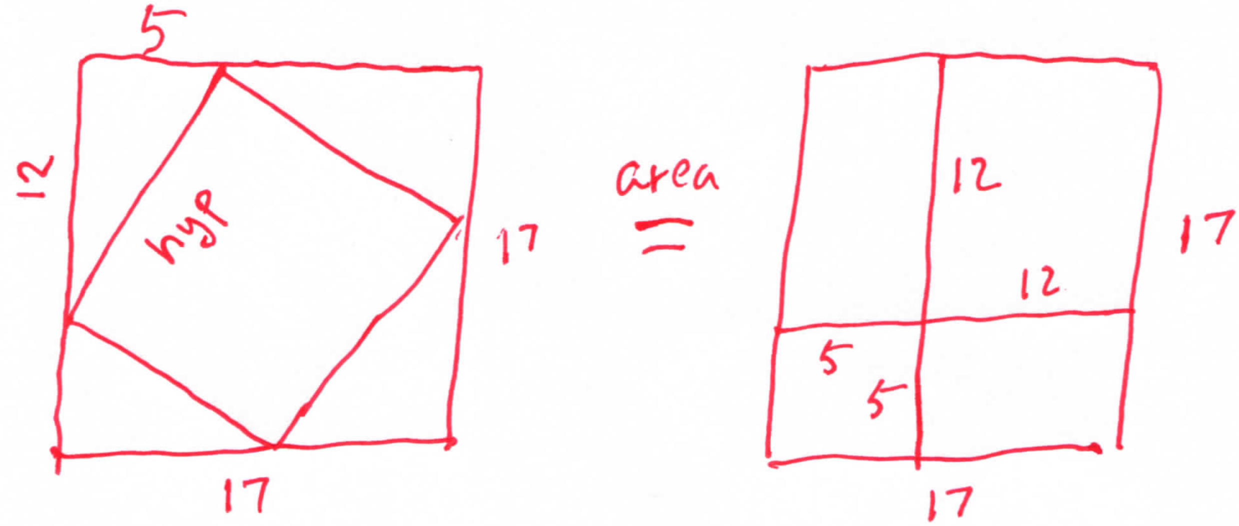



$\stackrel{\text{area}}{=}$



"Similarly"

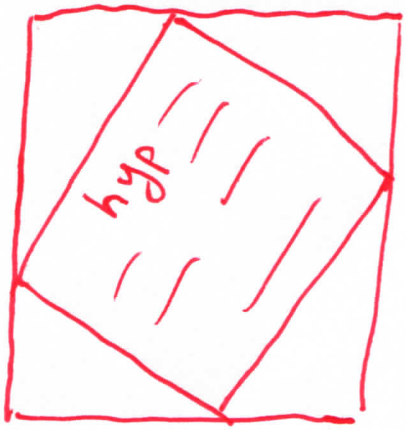
(new Board 1)



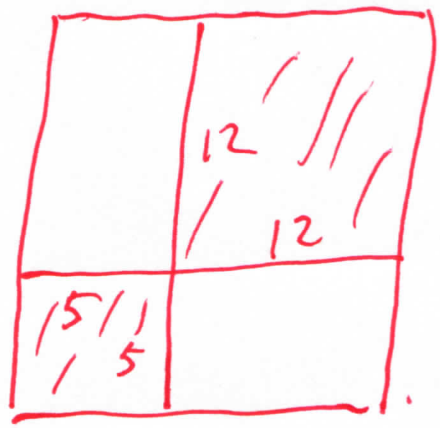

Have students do analogous right triangle removals, to get

new Board 2





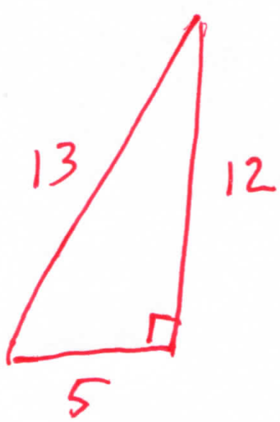

$\overset{\text{area}}{=}$




$\rightarrow (hyp)^2 = 5^2 + 12^2 = 25 + 144$

$= 169$

$\rightarrow (hyp) = \sqrt{169} = 13 \quad \left(\begin{array}{c}\text{since} \\ 13^2 = 169\end{array}\right)$

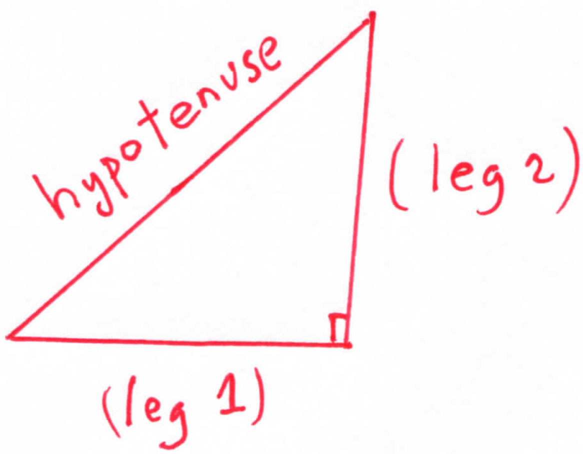



$13^2 = 5^2 + 12^2$

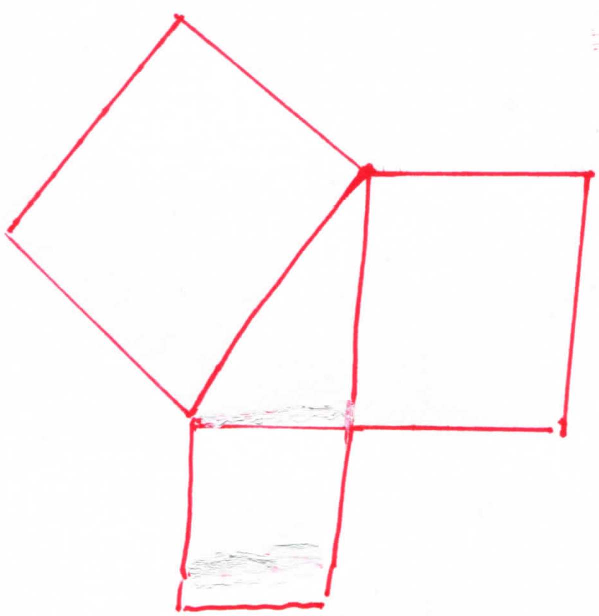
new Board 1



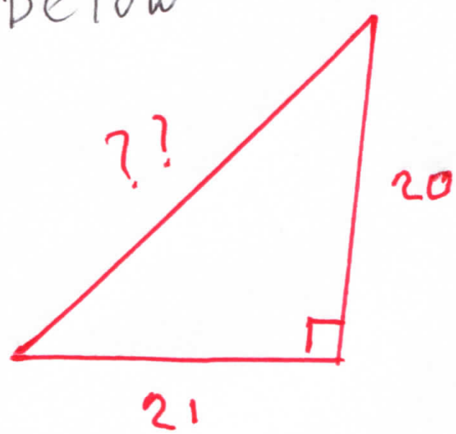
# Pythagorean Theorem




$$(\text{hypotenuse})^2 = (\text{leg } 1)^2 + (\text{leg } 2)^2$$



"For example, to get the missing hypotenuse below"




new Board 2

"add together the squares of the legs; this will be the square of the hypotenuse"

Board 2 continued:

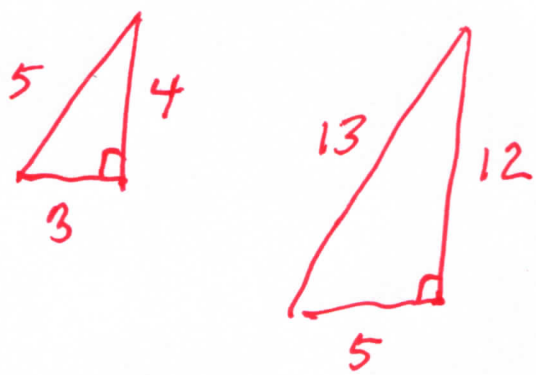
$$(??)^2 = 21^2 + 20^2 = 441 + 400 = 841$$

$$\rightarrow (??) = \sqrt{841} = 29$$

# A <u>Pythagorean</u> <u>Triple</u>

is three integers that form the sides of a right triangle.

<u>Examples</u>: (3, 4, 5) & (5, 12, 13) we have seen are Pythagorean triples.



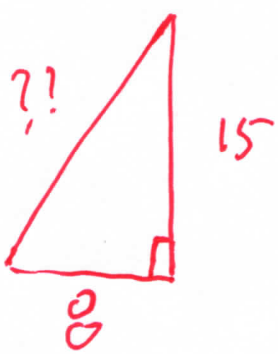
p. 42

new Board 2

(8, 15, ??) ; (7, 24, ??)

"Fill in the missing numbers to make Pythagorean triples"

new Board 1



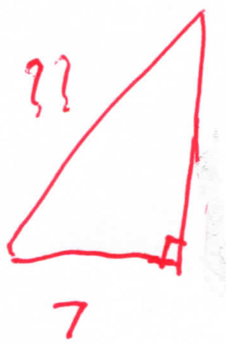

$$(??)^2 = 8^2 + 15^2 =$$

$$64 + 225 = 289$$

$$\to ?? = \sqrt{289} = 17$$

→ (8, 15, 17) is a Pythagorean triple.

new Board 2




$(??)^2 = 7^2 + 24^2 =$

$49 + 576 = 625$

$\rightarrow (??) = 25,$

(7, 24, 25) is a Pythagorean triple.

"From Babylonia, ~ between 1,000 & 2,000 BC, comes the following formula for Pythagorean triples"

new Board 1



Integers $m > n \rightarrow$

$(m^2 - n^2) = (\text{leg } 1)$

$(2mn) = (\text{leg } 2)$

$(m^2 + n^2) = (\text{hyp})$

new Board 2

Examples $m = 2,\ n = 1 \rightarrow$

$(\text{leg } 1) = 3,\ (\text{leg } 2) = 4,\ (\text{hyp}) = 5$

$m = 3,\ n = 2 \longrightarrow$

$(\text{leg } 1) = 5,\ (\text{leg } 2) = 12,\ (\text{hyp}) = 13$

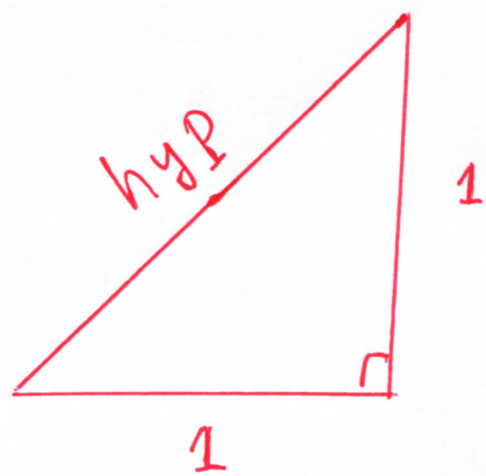
You could hand out copies of the Pythagorean triples below

| m | n | (leg 1) | (leg 2) | (hyp) |
|---|---|---|---|---|
| 2 | 1 | 3 | 4 | 5 |
| 3 | 2 | 5 | 12 | 13 |
| 4 | 1 | 8 | 15 | 17 |
| 4 | 3 | 7 | 24 | 25 |
| 5 | 2 | 20 | 21 | 29 |
| 6 | 1 | 12 | 35 | 37 |
| 5 | 4 | 9 | 40 | 41 |
| 7 | 2 | 28 | 45 | 53 |
| 6 | 5 | 11 | 60 | 61 |
| 8 | 1 | 16 | 63 | 65 |
| 7 | 4 | 33 | 56 | 65 |
| 8 | 3 | 48 | 55 | 73 |
| 7 | 6 | 13 | 84 | 85 |
| 9 | 2 | 36 | 77 | 85 |
| 8 | 5 | 39 | 80 | 89 |
| 9 | 4 | 65 | 72 | 97 |
| 10 | 1 | 20 | 99 | 101 |
| 10 | 3 | 60 | 91 | 109 |
| 8 | 7 | 15 | 112 | 113 |
| 11 | 2 | 44 | 117 | 125 |
| 11 | 4 | 88 | 105 | 137 |
| 9 | 8 | 17 | 144 | 145 |
| 12 | 1 | 24 | 143 | 145 |

p. 46

"Consider the right triangle whose legs each measure 1"

new Board 1




By the Pythagorean theorem,

$(hyp)^2 = 1^2 + 1^2 = 1 + 1 = 2$

$\rightarrow (hyp) = \sqrt{2}$;

"that is, the hypotenuse is $\sqrt{2}$, the square root of 2."

new Board 2

CAN SHOW: $\sqrt{2}$ is not rational; that is, is not a ratio of integers.

"This was very disturbing to the classical Greeks; $\sqrt{2}$ exists geometrically but not algebraically."

# VI MATERIALS NEEDED FOR PART 2

For each participant:

(1) A piece of paper, to be taped on a surface; and

(2) EITHER (a) a compass and ruler; OR (b) Two cardstock rectangles of width one inch, one rectangle five inches long, the other four inches long.

# VII CONVERSE of PYTHAGOREAN THEOREM STATED

"We've talked earlier about the importance of having a right angle. Yet the Pythagorean Theorem <u>assumes</u> a right angle. It would be nice to know when we have a right angle, or, better yet, <u>construct</u> a right angle.

First, let's restate the Pythagorean Theorem more succinctly."

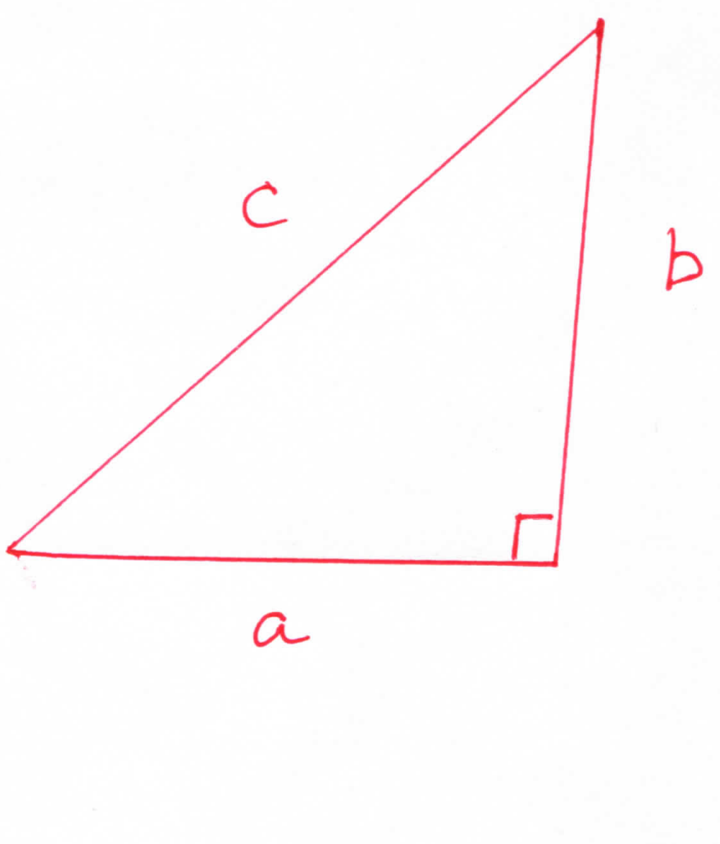
new Board 1

p. 50

## Pythagorean Theorem

If a right triangle has a hypotenuse of length c & legs of length a & b, then

$$a^2 + b^2 = c^2$$



"What we want now is the converse of the Pythagorean theorem."

new Board 2

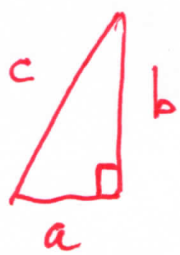
Converse of Pythagorean Theorem:

If a triangle has sides of length a, b, & c, with

$$a^2 + b^2 = c^2$$

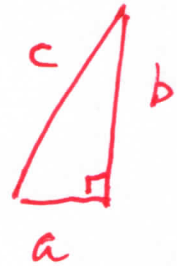
then said triangle is a right triangle with hypotenuse of length c.

Here is an informal picture, of Pythag. versus converse of Pythag., that could go on either board:




$\longrightarrow$ $a^2+b^2 = c^2$ (Pythag.)

[ 


& $a^2+b^2 = c^2$ ] $\longrightarrow$ 


(Pythag. converse)

# VIII CONSTRUCTION of RIGHT ANGLES

## (a) With Compass & Ruler

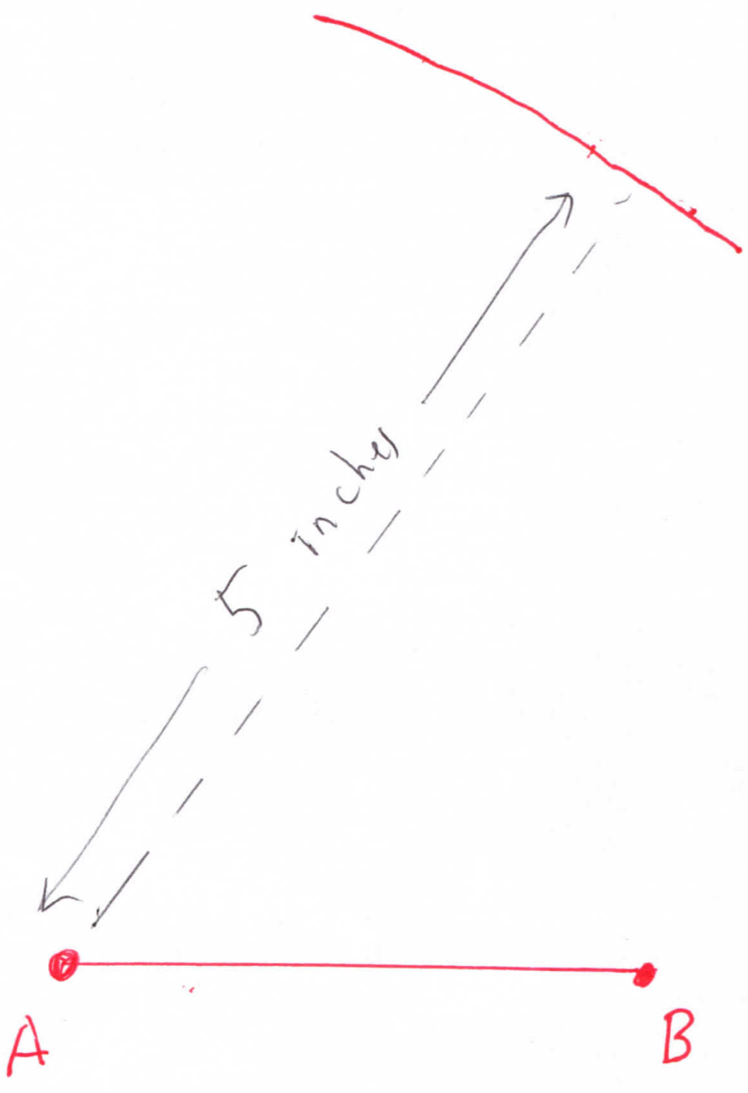
"Draw a horizontal line segment 3 inches long near the bottom of the taped piece of paper. Label the left end of the line segment A, the right endpoint B."

new Board 1 p. 54

p. 55

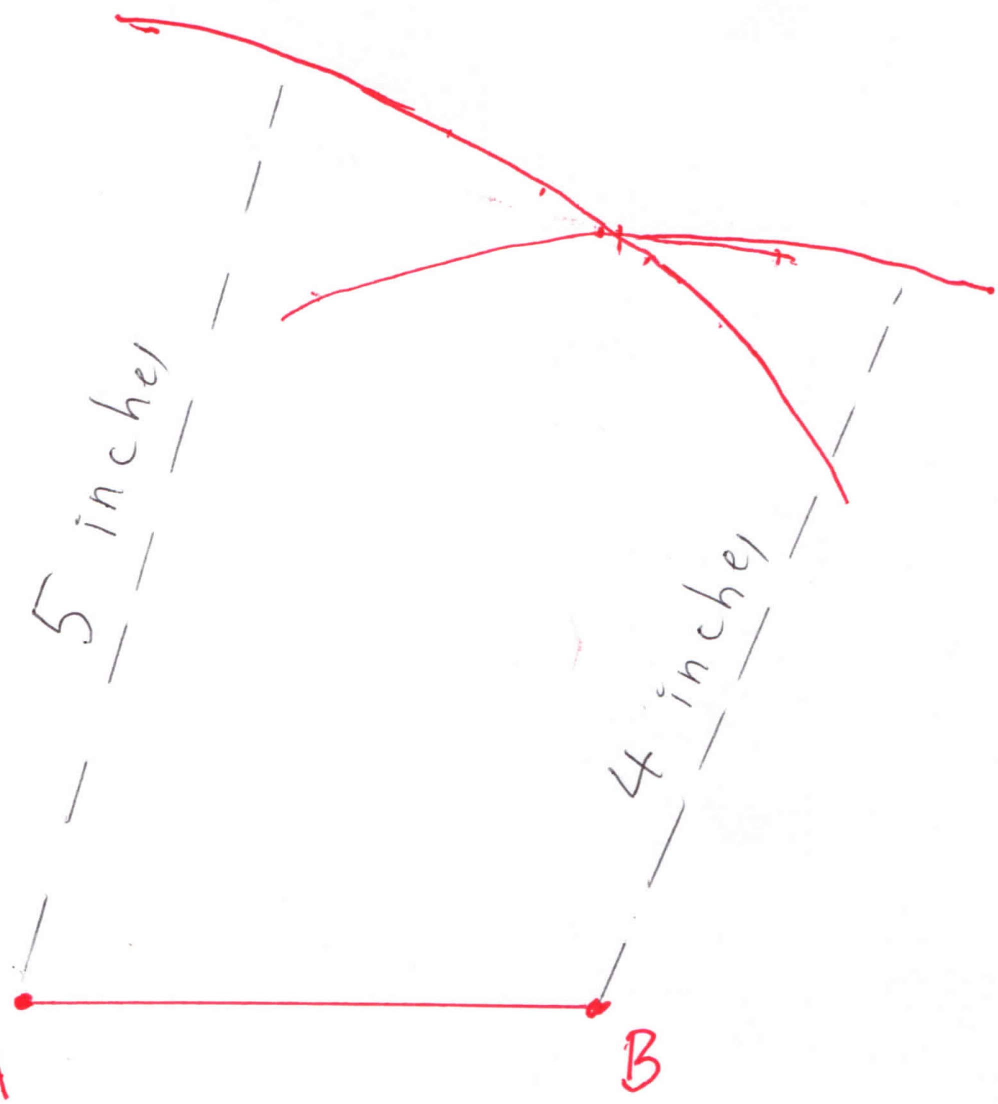
"Set the compass at 5 inches & draw an arc centered at A"

new Board 1



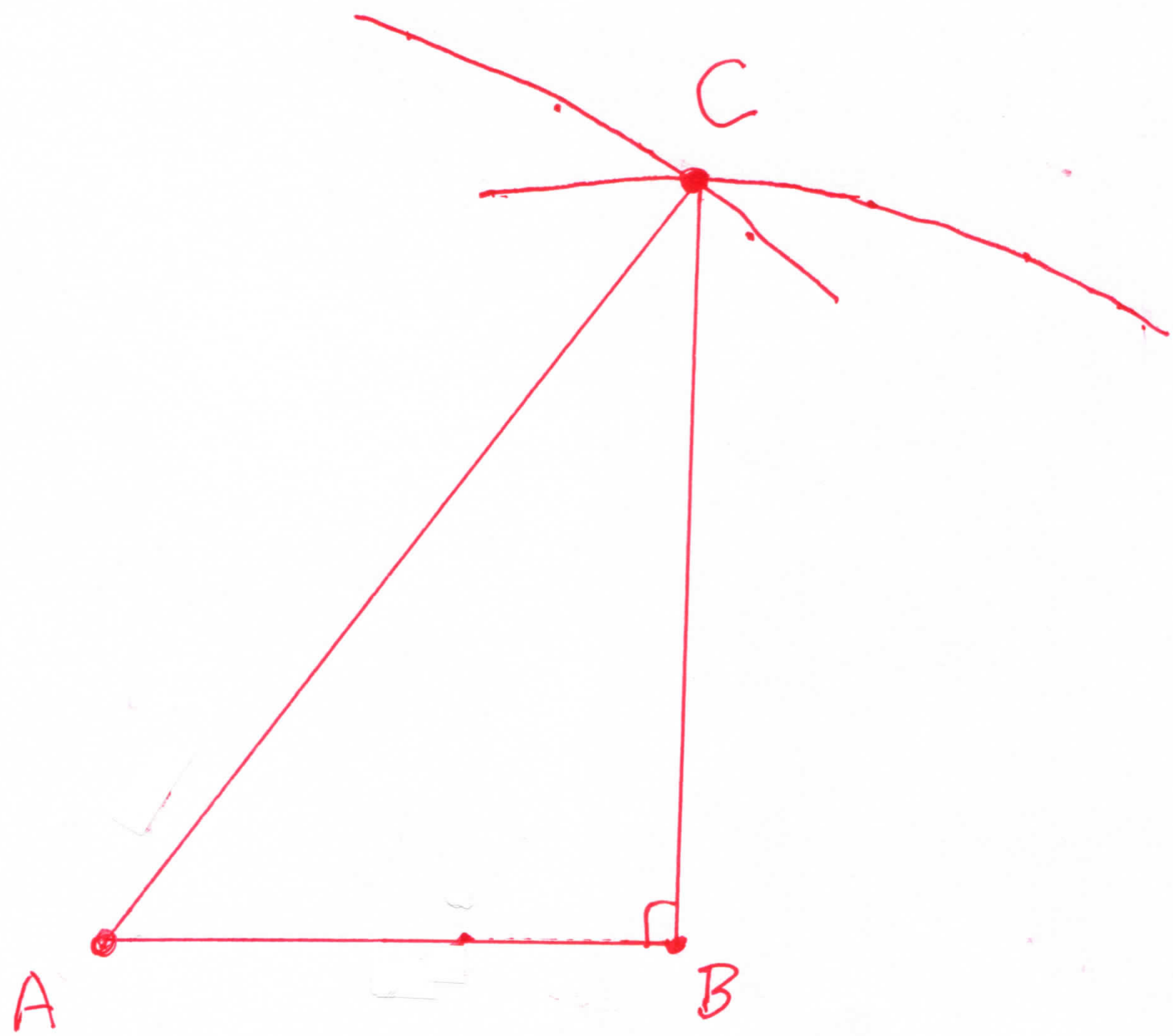
"Set the compass at 4 inches + draw an arc centered at B"

new Board 1



"Label the intersection of the two arcs C. Draw the triangle ABC. The angle at B should look like a right angle."

new Board 1



# CONSTRUCTION of RIGHT ANGLES

(b) With two cardstock rectangles of width one inch, one rectangle five inches long, the other four inches long.

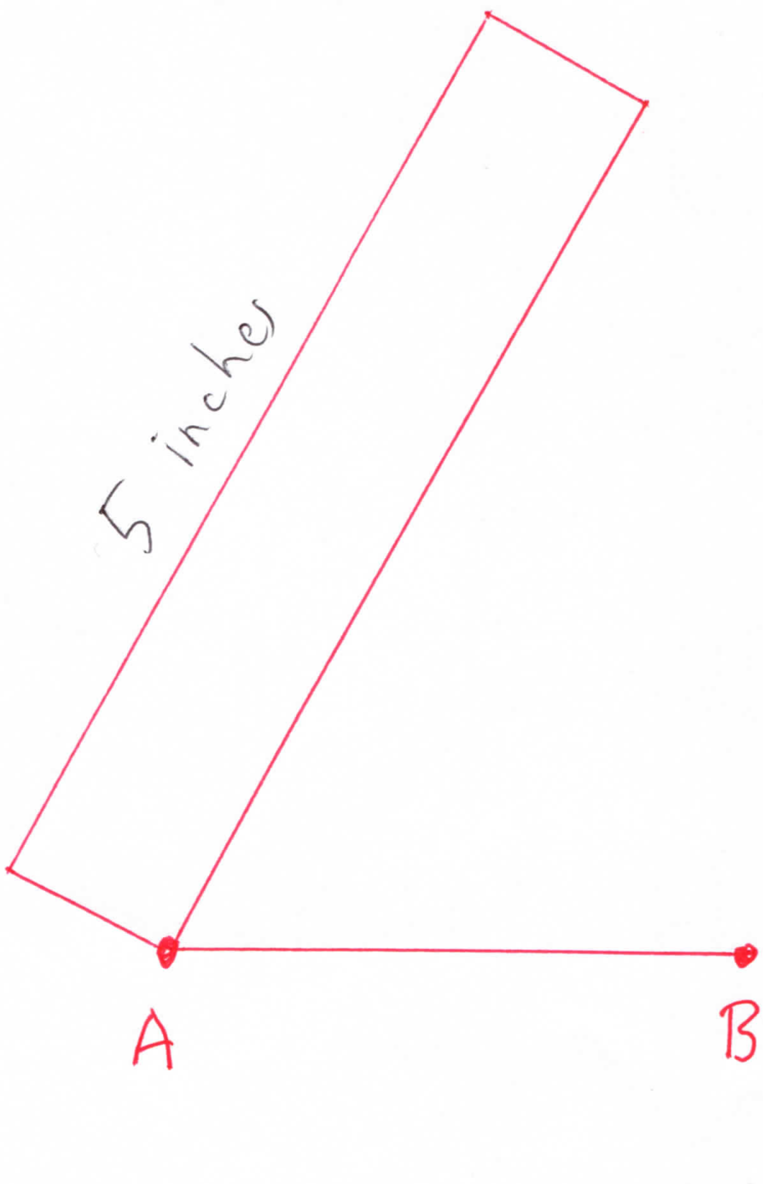
"Draw a hoizontal line segment 3 inches long near the bottom of the taped piece of paper. Label the left end of the line segment A, the right endpoint B."

p. 59

new Board 1

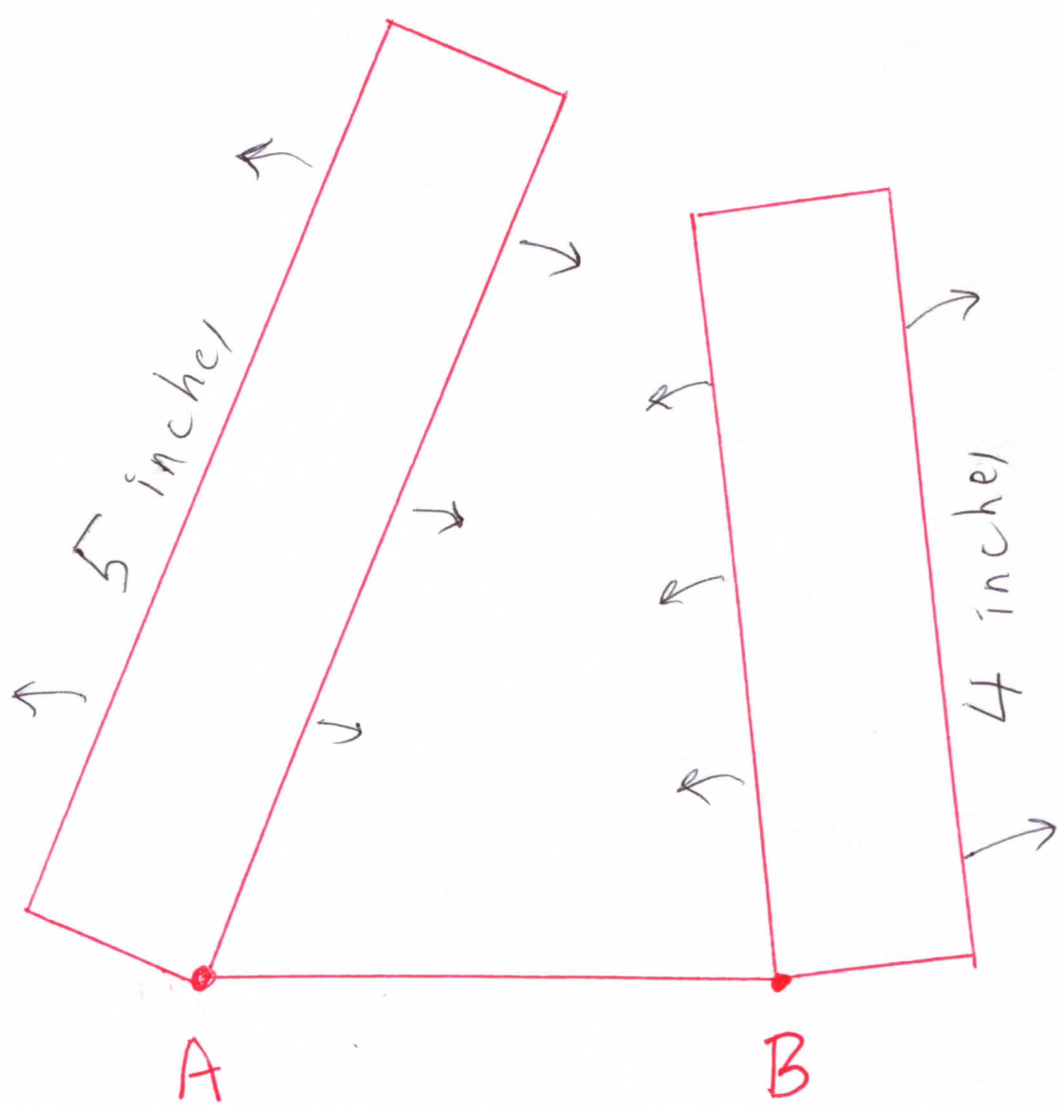
"Place the lower right corner of the 5 inch rectangle at A."

new Board 1



" Place the lower left corner of the 4 inch rectangle at B. "

new Board 1

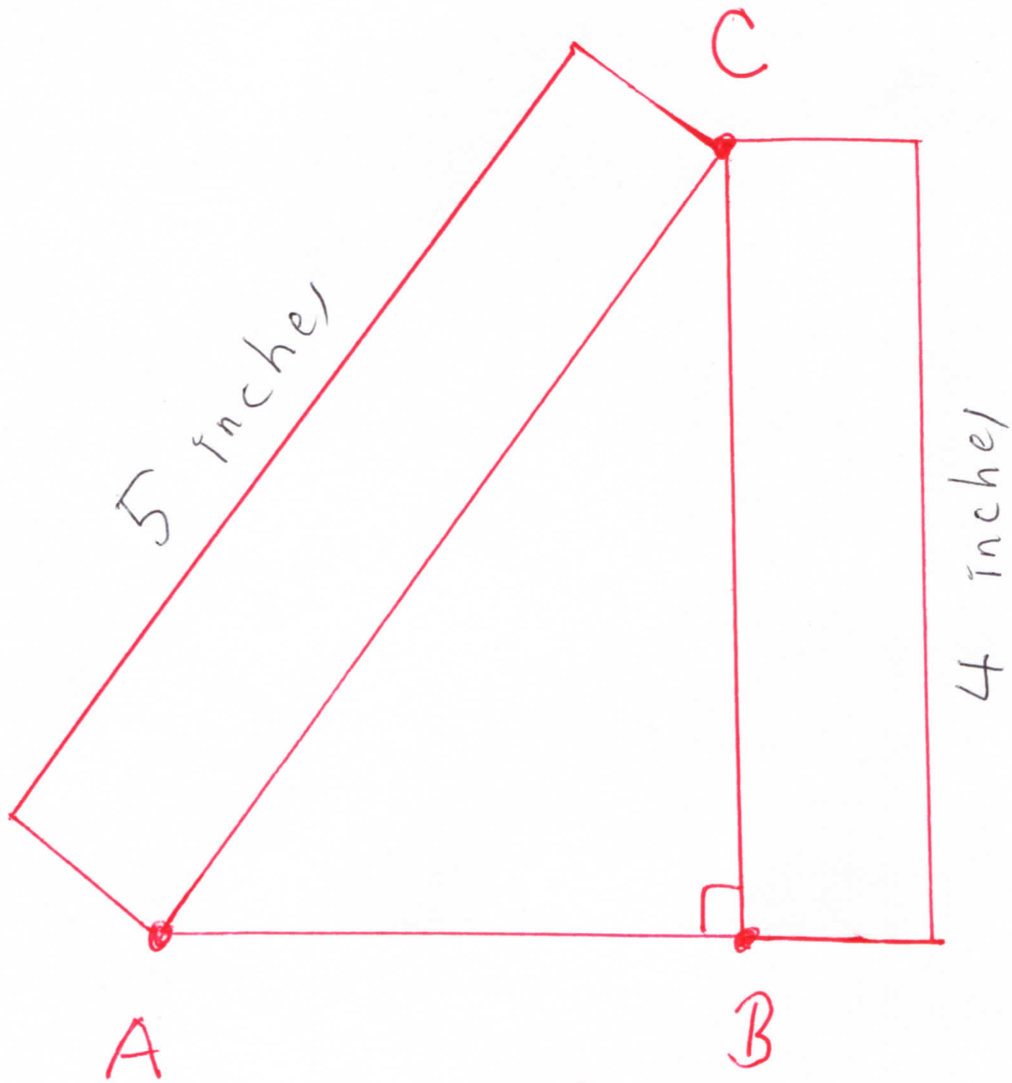


p. 62

"Rotate both rectangles, maintaining their pivots on the line segment AB, until the upper right corner of the 5 inch rectangle touches the upper left corner of the 4 inch rectangle; label that point where they touch C. Draw the triangle ABC. The angle at B should look like a right angle."

new Board 1





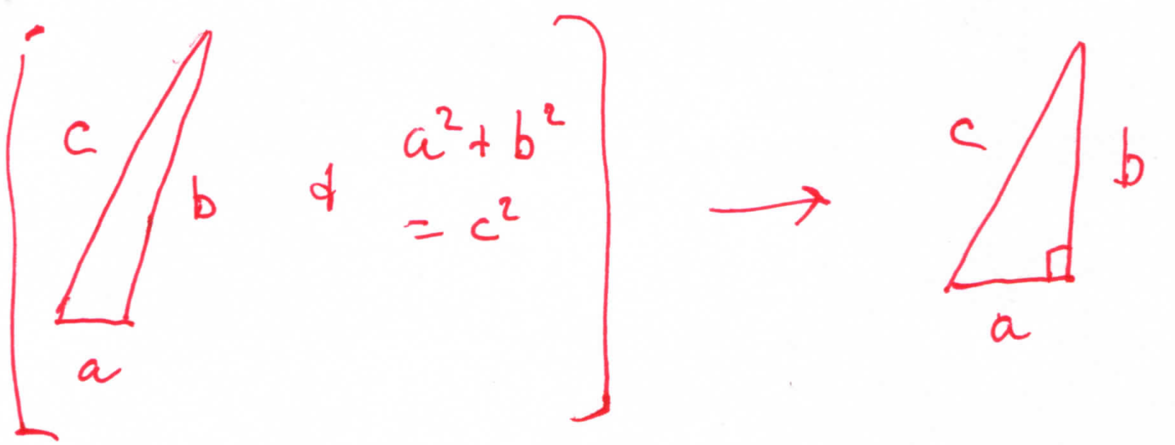
p. 64

"We could have identically drawn a right angle for a triangle with sides 5, 12, & 13 inches."

<u>IX</u> (OPTIONAL) PROOF of CONVERSE of PYTHA-GOREAN THEOREM

"Here is our informal picture of said CONVERSE"

new Board 1



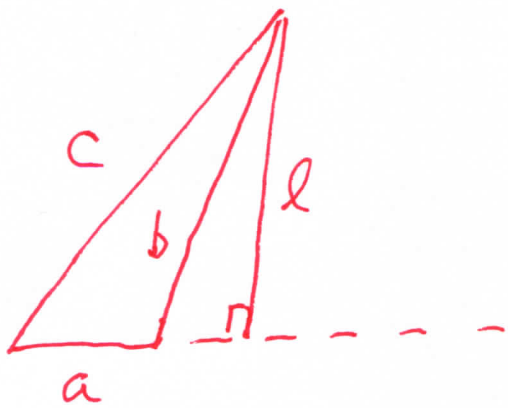
"Here is a verbal description of the beginning of our proof of said CONVERSE; we will then clarify with two pictures (one labelled CASE 1, the other CASE 2) of the beginning of our proof."

new Board 2

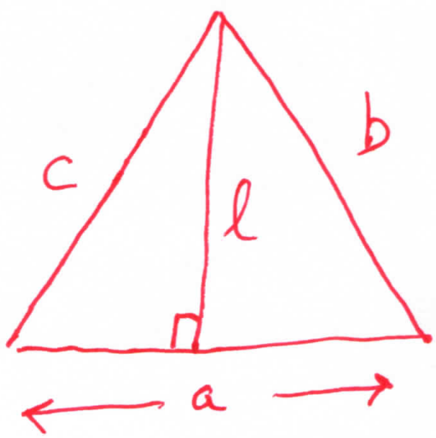
PROOF: Draw a line segment from the vertex opposite the side of length a, perpendicular to the line containing the side of length a. Let $\ell$ be the length of the perpendicular line segment just drawn.

new Board 1

CASE 1




CASE 2



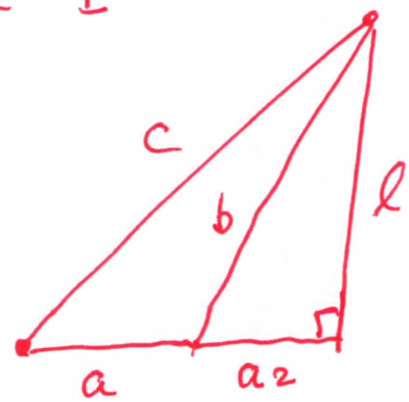

WHAT WE WANT TO SHOW:

the side of length b equals the side of length l.

new Board 2

p. 68

CASE 1




By the Pythagorean Theorem

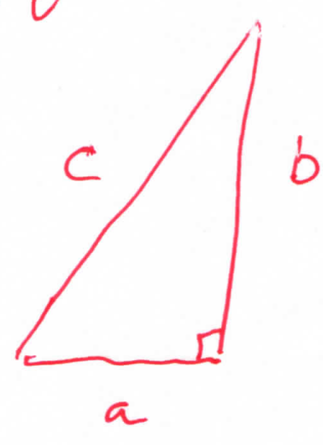
$(a+a_2)^2 + l^2 = c^2$; and

$a_2^2 + l^2 = b^2$

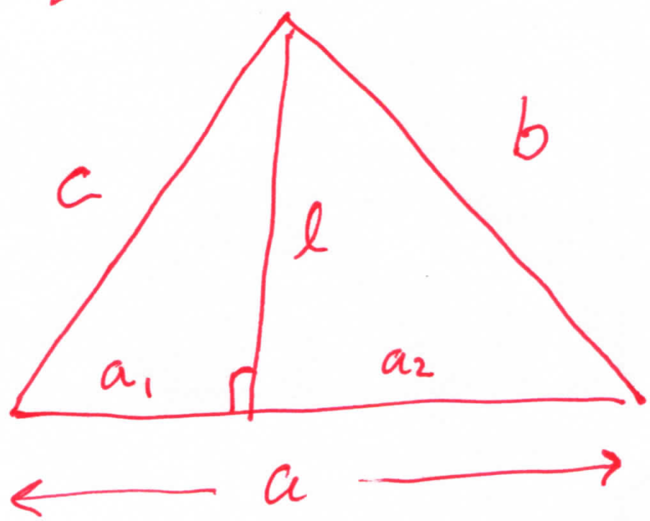
Thus $a^2+b^2 = c^2 = (a^2 + 2aa_2 + a_2^2) + l^2$

$= a^2 + 2aa_2 + b^2$

$\rightarrow 0 = 2aa_2 \rightarrow a_2 = 0$



CASE 2




By the Pythagorean Theorem,

$a_1^2 + l^2 = c^2$; and

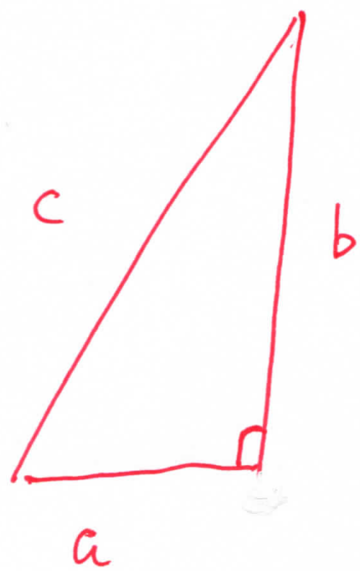
$a_2^2 + l^2 = b^2$

Thus $a^2 + b^2 = c^2 = a_1^2 + l^2$

$= (a - a_2)^2 + l^2 = (a^2 - 2aa_2 + a_2^2) + l^2$

$= a^2 - 2aa_2 + b^2$

$\rightarrow \quad 0 = -2aa_2 \rightarrow a_2 = 0$



(Last thing on Boards)

See, at

https://teacherscholarinstitute.com

"Pythgorean Theorem and More"

under MATH MAGNIFICATIONS

and

"Vectors Point to Geometry and Trigonometry"

under FREE MATH BOOKS, HIGH-SCHOOL LEVEL