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The Perfect Mosaic of Bees
DIY (Do-It-Yourself) Math Workshop

Dr. Ralph deLaubenfels

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Teacher-Scholar Institute

Columbus, Ohio

2021

DIY (Do-It YOURSELF)

THE PERFECT MOSAIC OF BEES WORKSHOP

As with all DIY Workshops,

Writing/drawings in red ______

are written on a chalkboard

& possibly spoken;

Writing in quotes in black

"______" is said out loud to

students & not written;

Writing not in quotes in black

______ is suggested & not

spoken or written.

PREREQUISITES:

Arithmetic, meaning addition, subtraction, multiplication & division, and

TERMINOLOGY of fractions, e.g.,

$\frac{30}{5}$ MEANS $(30 \div 5)$,

"thirty divided by 5".

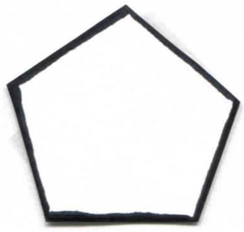
MATERIALS NEEDED:

Two Chalkboards, that we will call Board 1 & Board 2

Large number of colorful pattern blocks, as on duckduckgo.com or google.com; Lakeshore collection of 250 blocks in 6 shapes is approximately how many you need for 5 students.

You will also need templates of regular pentagons, that you can use to make cardstock pentagons; 10 per student should be enough.

Here is a possible template



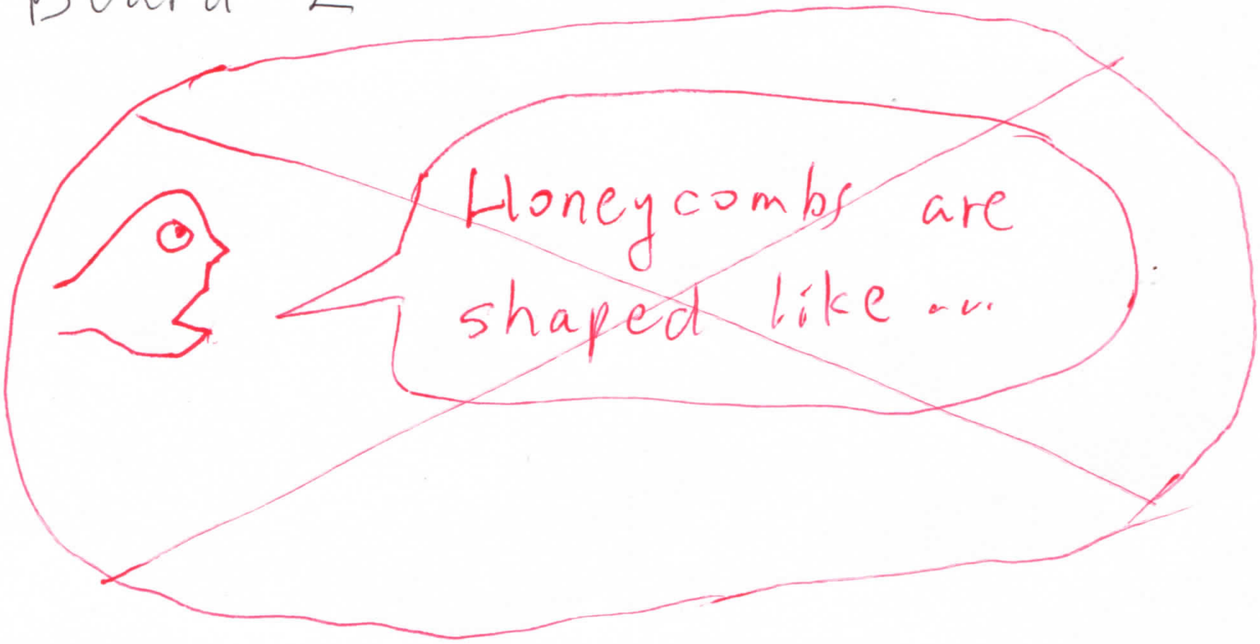
"Today we'd like to understand beehives; in particular"

Board 1

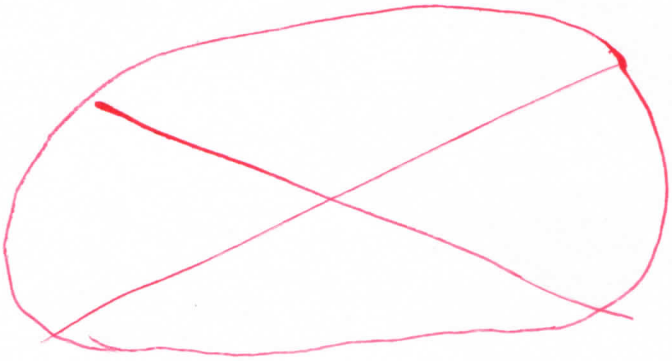
honeycombs: "(inside beehives)"
cells made of beeswax that store food & baby bees

"If a student already knows about honeycombs, please keep it secret; we will derive the (optimal) shape of beehive cells."

Board 2

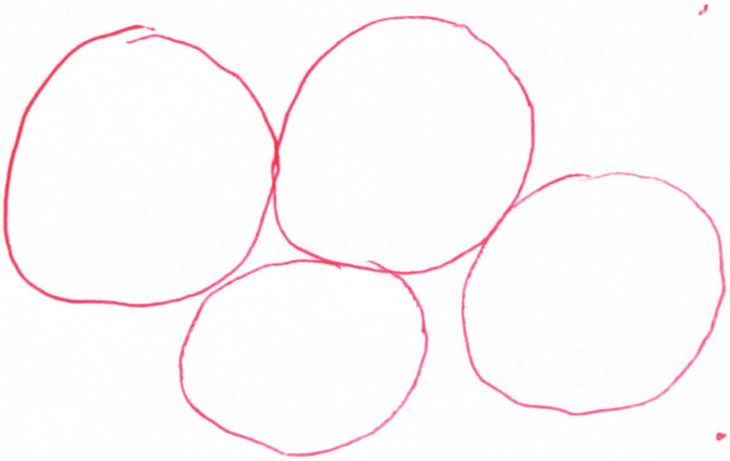


"Note the Math Busters symbol"



"How about discs for cells?"

new Board 1



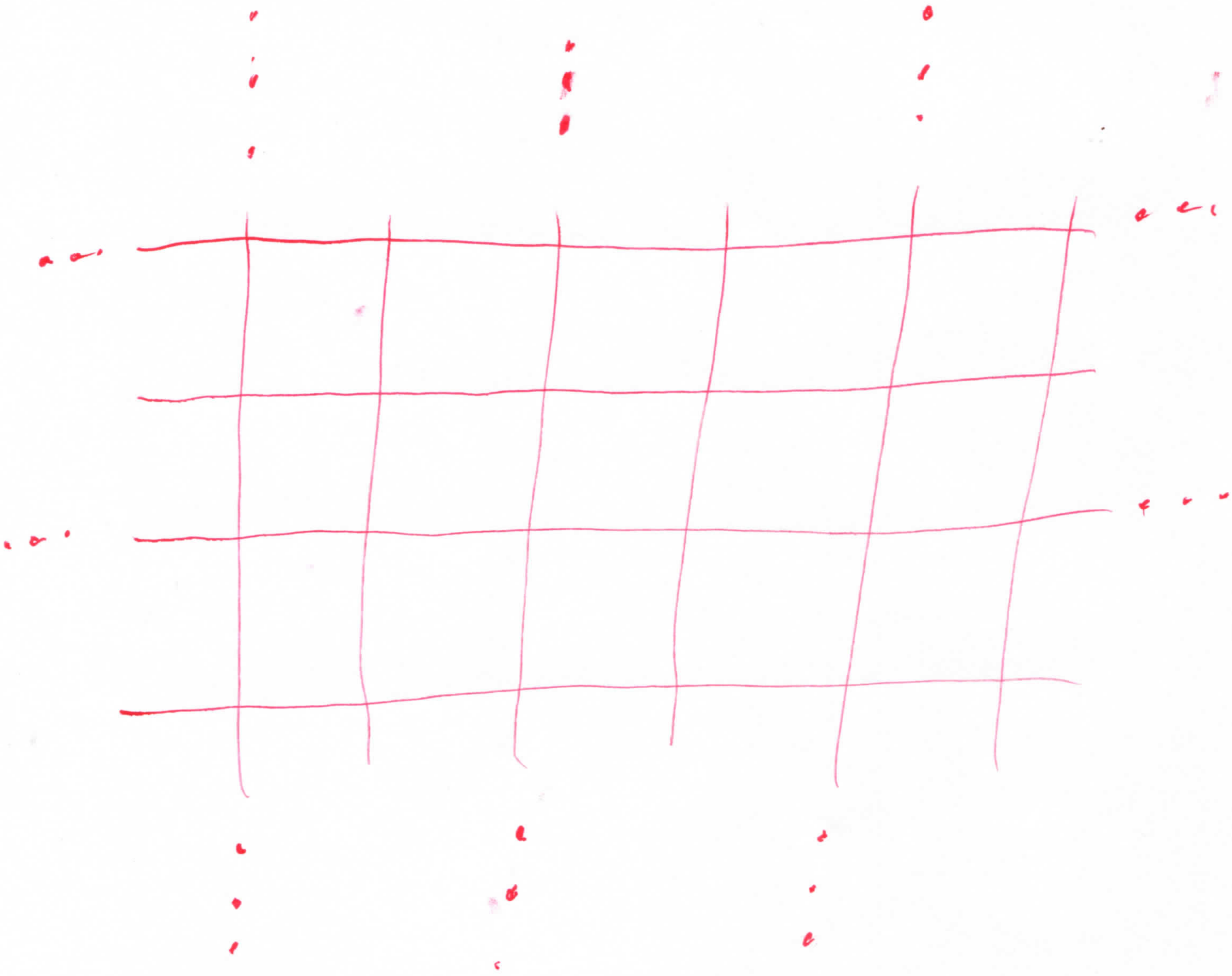
"What problems do you see?"

Hopefully students will be concerned about gaps or wasted space.

"How about squares for cells?"

Hand out square pattern blocks; "try to put them together with no gaps"; eventually should draw on the board, as on the next page

new Board 2



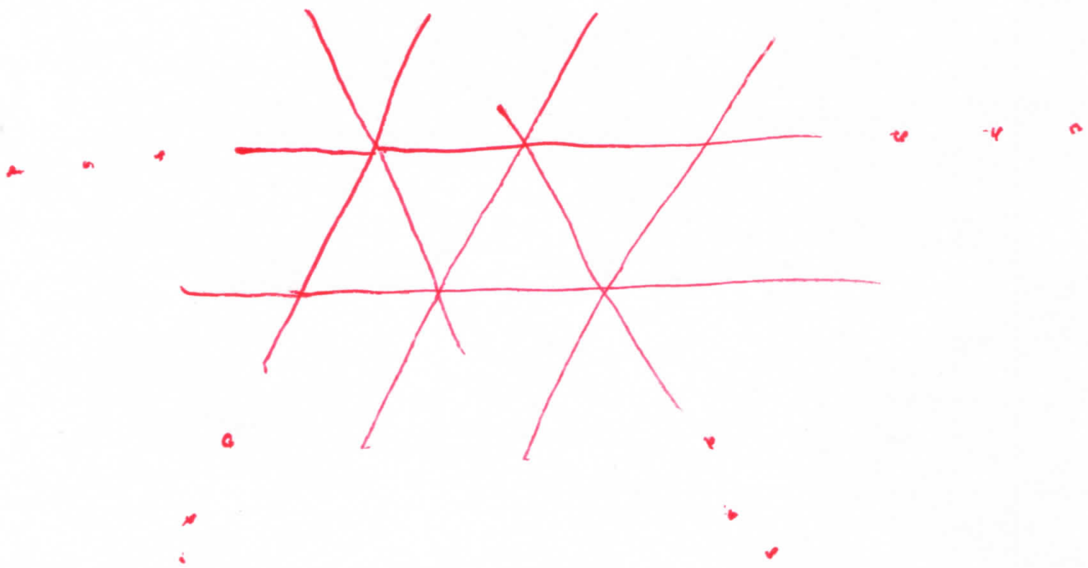
"This is called a"

new Board 1

tesselation or (mathematical) mosaic: arrangement of flat things that fit together, without gaps or overlap, to cover all or some of a flat surface

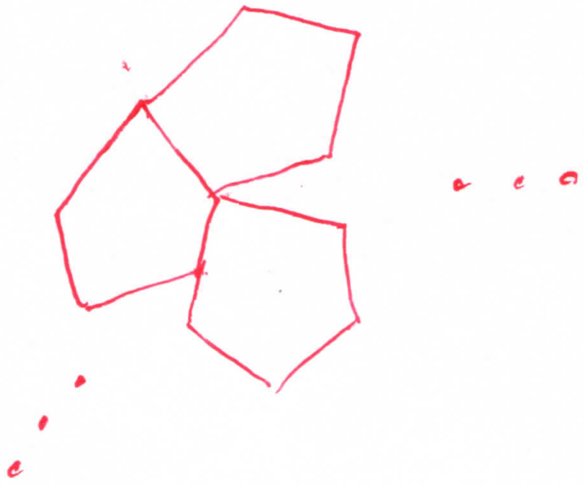
Take squares away, hand out triangle pattern blocks & ask students to "tesselate"; eventually should get

new Board 2



Take triangles away,
hand out pentagon pattern
blocks, have students
TRY to tesselate

new Board 1



SHOULD FAIL eventually

"Triangles, squares & pentagons are examples of"

new Board 2

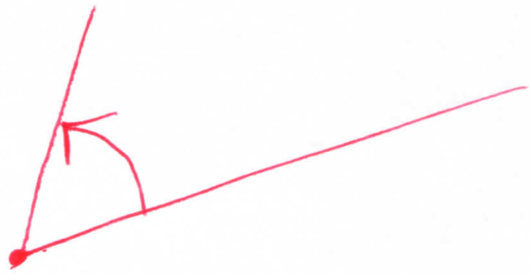
polygon: anything with straight sides & no holes that can be cut from a piece of paper;

e.g.,

"The key idea to tesselating with polygons is"

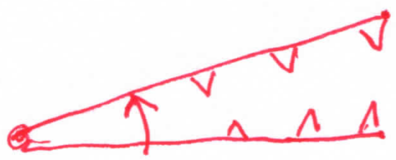
new Board 1

angle : refers to how far one line is rotated from another



"A crocodile's jaws are an important example"

new Board 2



small angle

"(just starting to be interested)"

large angle

"(very hungry)"

zero angle

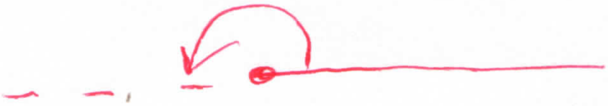
"(mouth closed)"

Could also illustrate angle with a door or a book

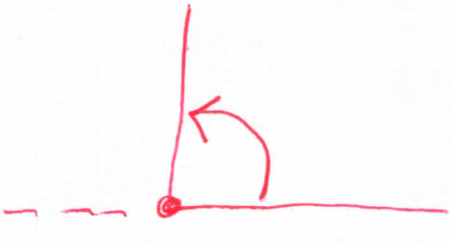
new Board 1

Full revolution is 360° (360 degrees)

Half revolution is $\frac{360^\circ}{2} = 180^\circ$



Quarter revolution (right angle) is



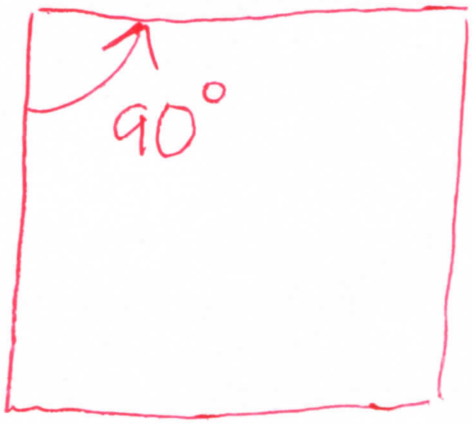
$\frac{360^\circ}{4} = 90^\circ$

"Degrees are due to the Babylonians; they also gave us 60 seconds to a minute & 60 minutes to an hour."

new Board 2

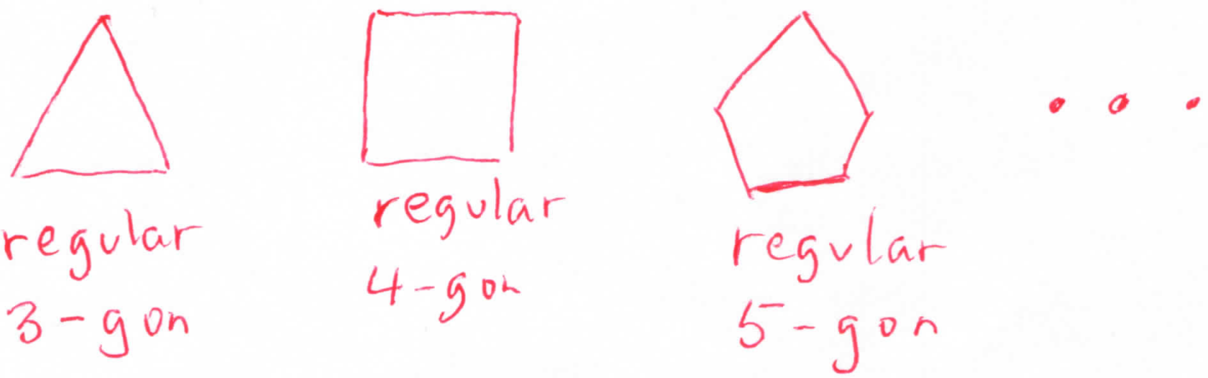
<u>Interior</u> <u>angle</u> of a polygon is between consecutive sides on the inside

"E.g., a square's interior angles are each 90°"

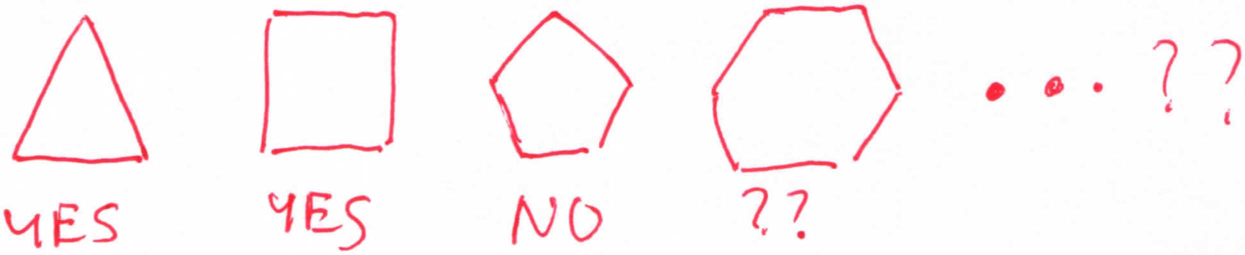


new Board 1

A regular polygon has equal sides & equal interior angles



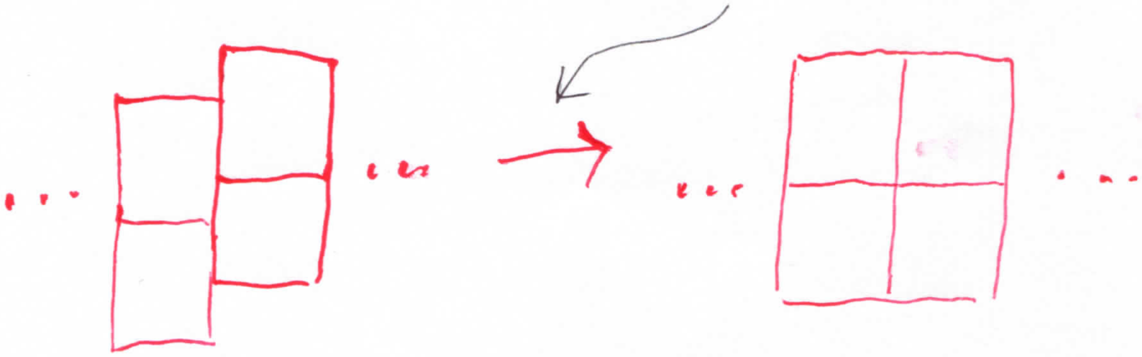
"We want to tesselate with copies of a fixed regular polygon"



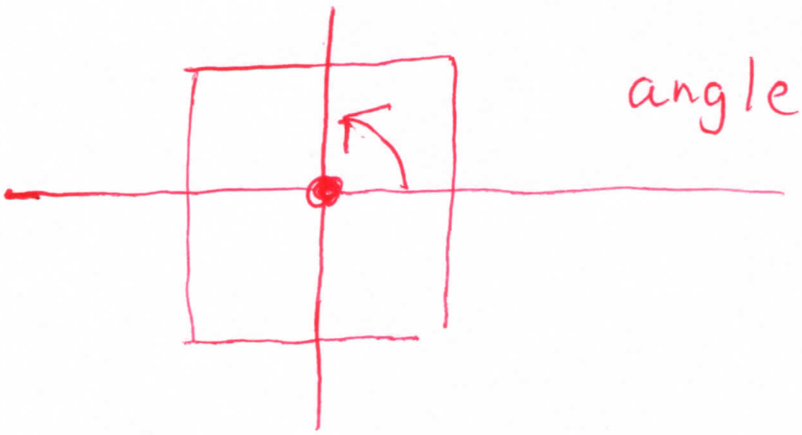
Hand out square pattern blocks

new Board 2

Have students rearrange (see next page)



"By sliding squares in tesselation around, can make 4 squares meet at a shared vertex"



angle $= \frac{360^\circ}{4} = 90^\circ$

"Thus interior angles must be $\frac{360^\circ}{4} = 90^\circ$"

Take back squares & hand out triangles to tesselate. Eventually can get

new Board 1

angle $= \frac{360^\circ}{6}$

$= 60^\circ$

"Now 6 triangles meet at a shared vertex, so interior angles must be $\frac{360^\circ}{6} = 60^\circ$"

p. 21

"Let's start making a table of regular polygons meeting at a shared vertex."

new Board 2

REQUIRED ANGLES

# of polygons meeting at vertex	Interior Angle
2	 $\left(\frac{360}{2}\right)^\circ = 180^\circ$
3	students fill in
4	 $\left(\frac{360}{4}\right)^\circ = 90^\circ$
5	students fill in
6	 $\left(\frac{360}{6}\right)^\circ = 60^\circ$
7	$\left(\frac{360}{7}\right)^\circ \simeq 51^\circ$
8	students fill in
9	↓ decreasing
⋮	

Board 2 completed

REQUIRED ANGLES

# of polygons meeting at vertex	Interior Angle
2	$\left(\frac{360}{2}\right)^{\circ} = 180^{\circ}$
3	$\left(\frac{360}{3}\right)^{\circ} = 120^{\circ}$
4	$\left(\frac{360}{4}\right)^{\circ} = 90^{\circ}$
5	$\left(\frac{360}{5}\right)^{\circ} = 72^{\circ}$
6	$\left(\frac{360}{6}\right)^{\circ} = 60^{\circ}$
7	$\left(\frac{360}{7}\right)^{\circ} \sim 51^{\circ}$
8	$\left(\frac{360}{8}\right)^{\circ} = 45^{\circ}$
9	$\downarrow$ decreasing
⋮	⋮

"We want to compare the angles we just calculated to the interior angles of regular polygons."

new Board 1

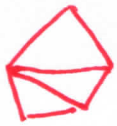
FACTOID: In a triangle, the sum of the interior angles is 180°.

Thus each angle in a regular triangle is $\frac{180°}{3} = 60°$.

Square? 

Two triangles → sum of angles is $(2 \times 180)^\circ = 360^\circ$ → each angle is $\left(\frac{360}{4}\right)^\circ = 90^\circ$.

Pentagon? Three triangles → sum of angles is $(3 \times 180)^\circ = 540^\circ$ → each angle is $\left(\frac{540}{5}\right)^\circ = 108^\circ$



"Let's put all these angles in a table"

new Board 1

POSSIBLE ANGLES

# of sides	Interior Angle
3	60°
4	90°
5	108°
6	120°
7	$\left(\frac{900}{7}\right)^\circ \sim 129^\circ$
8	135°
⋮	⋮ $\downarrow$ increasing

Have students stare at both POSSIBLE ANGLES & REQUIRED ANGLES boards; either you or students should circle matching angles, ending up eventually with completed tables like the following

Board 1 completed

POSSIBLE ANGLES

# of sides	Interior Angle	
3	60°	1
4	90°	2
5	108°	
6	120°	3
7	$\left(\frac{900}{7}\right)^\circ \sim 129^\circ$	
8	135°	
⋮	⋮ increasing ↓	

new Board 2 ~~completed~~

REQUIRED ANGLES

# of polygon meeting	Interior Angle
2	180°
3	120°
4	90°
5	72°
6	60°
7	$\left(\frac{360}{7}\right)^\circ \sim 51^\circ$
8	45°
⋮	⋮ decreasing ↓

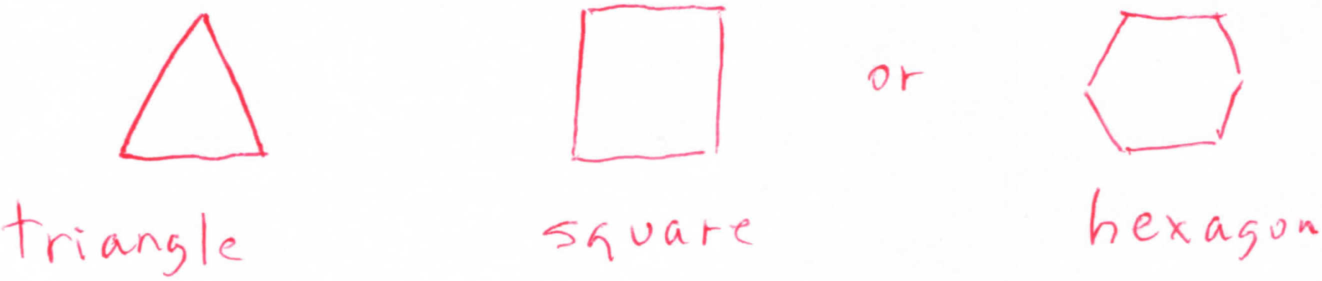
3

2

1

ASK STUDENTS what the possible polygons for tesselating are, by comparing REQUIRED ANGLES & POSSIBLE ANGLES; should get

new Board 2



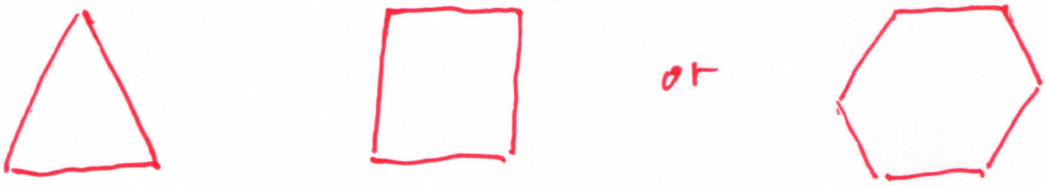
"This is surprising: there are infinitely many numbers of sides possible in a regular polygon, but only 3, 4, or 6 sides ~~are~~ allow tesselation."

"This is the nature of math research: stare at data, look for patterns."

"Let's return to our favorite tesselation:"

new Board 1

Cells of a honeycomb
(cross section)



Hand out those pattern blocks & have each student tesselate, only with triangles, only with squares & only with hexagons.

"Which of those tesselations is best?"

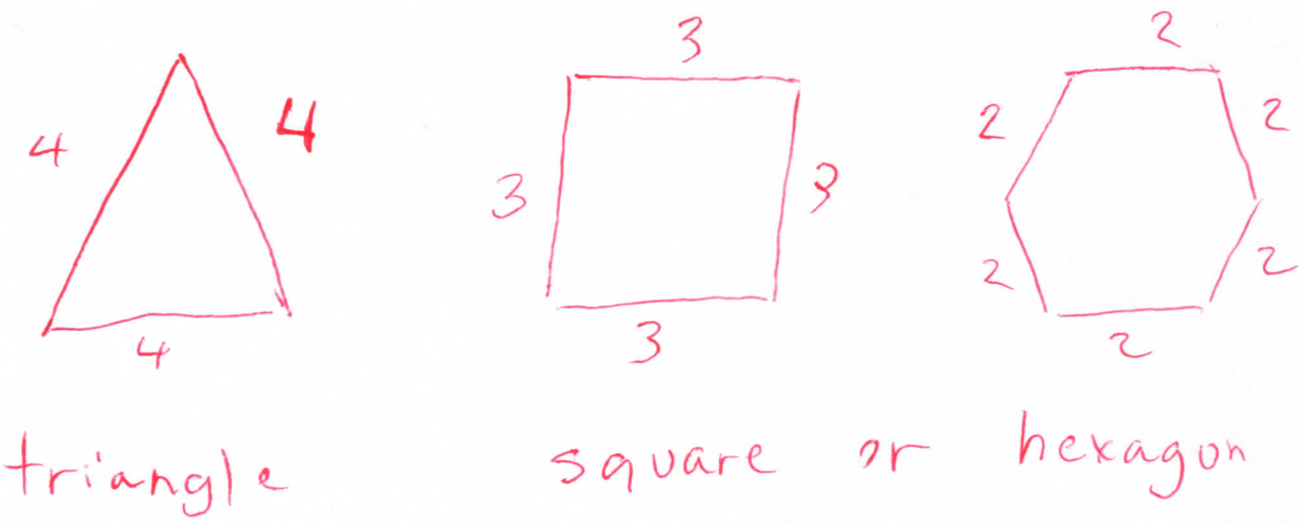
"The sides of each cell are made from beeswax, HARD to produce. The area inside each cell holds good stuff, food & baby bees."

new Board 2

WANT: biggest area of cell for a fixed amount of beeswax

"Say we have 12 inches of beeswax for sides of a cell."

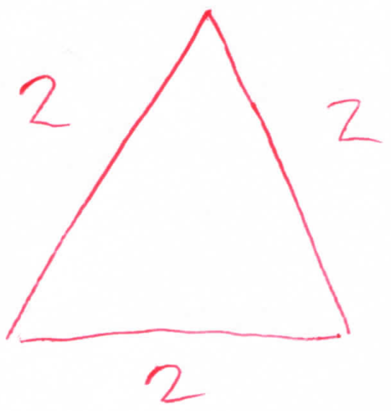
Board 2 continued



Ask students which polygon has greatest area.

"We will find it convenient p. 35
to introduce the"

new Board 1

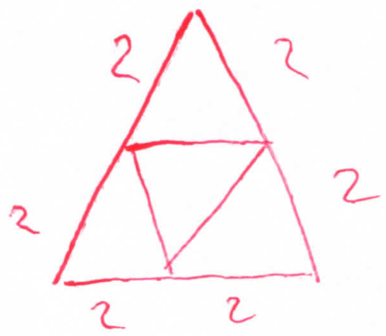


triangle
cubit

"an equilateral triangle
with sides of length 2."

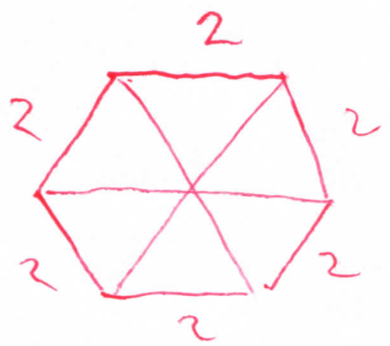
" Note that our triangle cell is 4 triangle cubits "

new Board 2



" while our hexagon cell is 6 triangle cubits "

Board 2 continued



"Thus our hexagon cell area is greater than our triangle cell area"

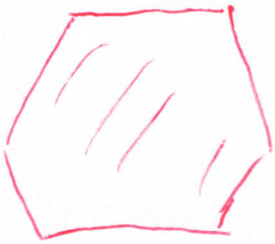
new Board 1

"It can also be shown that our square cell area is less than our hexagon area & more than our triangle area"

new Board 2

less than

less than



4 triangle cubits

~5 triangle cubits

6 triangle cubits

new Board 1

<u>Hexagons</u> are <u>best</u>

(most area per beeswax)

NATURE FACT: cells of a honeycomb are regular hexagons.

p. 39

"Bees are smart, in fact, perfect: their beehive construction is the best it can possibly be."

MENTION Math Magnifications <u>Bees and Hexagons</u>, on

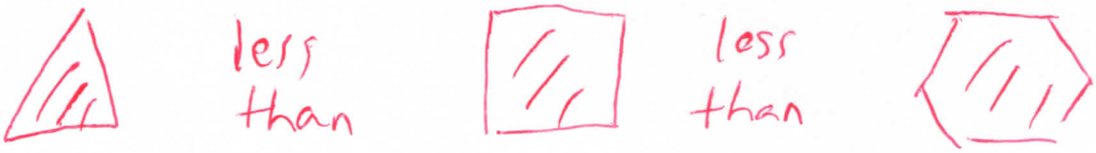
teacherscholarinstitute.com

Hand out all pattern blocks, let students do any tesselation they want (for example, see p. 3 of <u>Bees and Hexagons</u>, already mentioned).

OPTIONAL (if you have students who know square roots & the Pythagorean theorem):

new Board 2

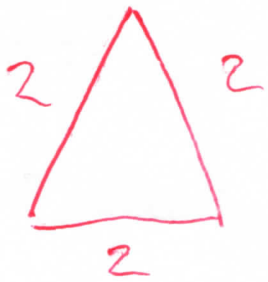
Demonstrate



You could do this during the final tesselations at the bottom of the previous page.

"Let's get the area of our triangle cubit"

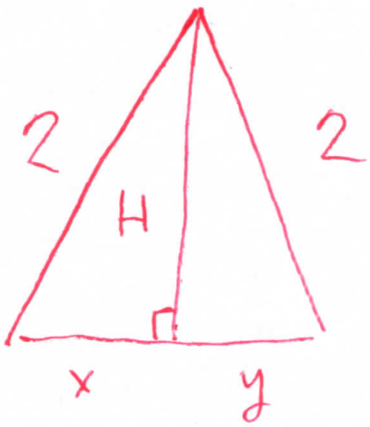
new Board 1



area = ??

"Drop a perpendicular of unknown height H."

new Board 2



Pythagoras →

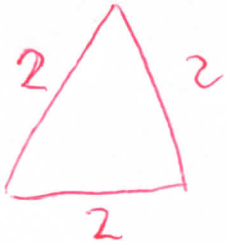
$x^2 + H^2 = 2^2 = y^2 + H^2$

$\rightarrow x = 1 = y$

Board 2 continued

$1^2 + H^2 = 2^2 \rightarrow$

$H = \sqrt{3} \rightarrow$

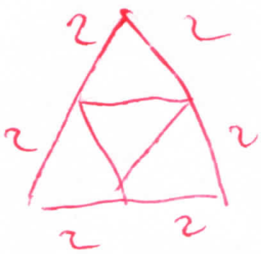
(area of 

) $= \frac{1}{2} \times \sqrt{3} \times 2$

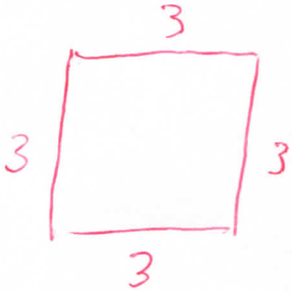
$= \sqrt{3}$

new Board 1

Possible cell areas



$4\sqrt{3}$



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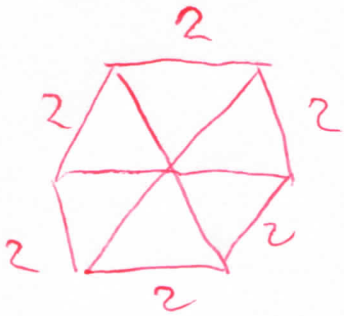


$6\sqrt{3}$

new Board 2

Is $4\sqrt{3} < 9 < 6\sqrt{3}$?

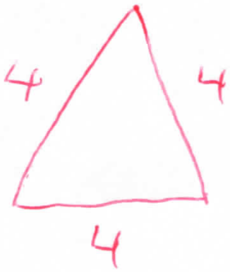
This is equivalent to

$(4\sqrt{3})^2 < 9^2 < (6\sqrt{3})^2$ (true?)

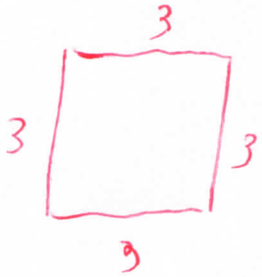
or

$48 < 81 < 108$ (true?),

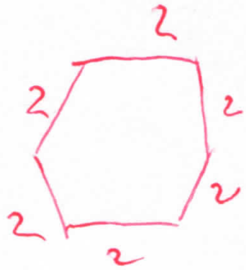
which is true, thus



less than



less than



new Board 1

"Can show"

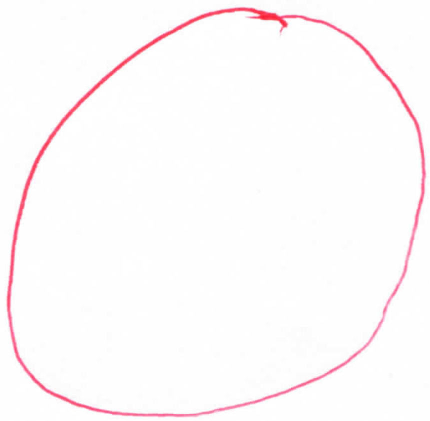
$4\sqrt{3} \sim 6.9, \quad 6\sqrt{3} \sim 10.4$

OPTIONAL (for students who know algebra, perimeter, & the relationship between radius & circumference of a circle & area of a disc)

"What do you think happens to regular polygons when the number of sides gets large?"

new Board 2

"(MIGHT BELIEVE)"

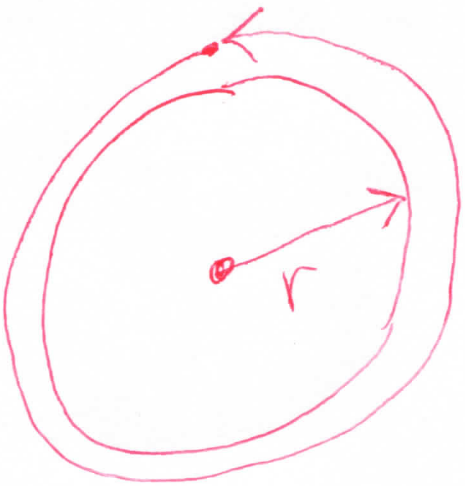


"Assuming perimeter remains 12, what will area of this disc be?"

new Board 1

perimeter of polygon

→ circumference of circle



$2\pi r$ = circumference,

radius r

$2\pi r = 12 \to r = \frac{6}{\pi} \to$

area $= \pi r^2 = \pi \left(\frac{6}{\pi}\right)^2 = \frac{36}{\pi}$

~ 11.5