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Calculus
MATHematics MAGnification™

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CALCULUS MAGNIFICATION

This is one of a series of very short books on math, statistics, and physics called "Math Magnifications." The "magnification" refers to focusing on a particular topic that is pivotal in or emblematic of mathematics.

OUTLINE

After some informal remarks, we will introduce, and hopefully demystify, the two biggest deals of "freshman [in college] calculus," the derivative and the integral.

The derivative will be seen to be an extension of the idea of slope of a straight line, while the integral extends the idea of area of a rectangle.

A deeper, root idea of calculus, that of approximations that become arbitrarily good, will be ubiquitous throughout our constructions.

Prerequisites for this magnification are first-year high school algebra, such as may be found in [7]. We will review slope of a straight line and area of a rectangle in Chapter I.

See [3, Section IV] for a more pedagogical discussion of the contents of Chapters III and IV of this magnification.

BRIEF HISTORICAL, PEDAGOGICAL, and PHILOSOPHICAL INTRODUCTION.

In assigning credit for the creation of calculus, the classical Greeks need to be given a large portion for identifying, and constantly frustrating themselves trying to resolve, problems and contradictions, such as Zeno's paradoxes (see [1, Chapter 11], [2, pp. 22-27],[4], and [5, Section 4.1]) that can be understood only with calculus. On a more positive note, Archimedes performed integration, using triangles instead of rectangles (see [8, Sections 4.4 and 4.5]).

The classical Greeks are also justifiably famous for exploring intensely almost every possible philosophical outlook. The purely philosophical duality of constant flux (e.g., Heraclitus said that you cannot step in the same river twice) versus an unchanging world (e.g., Parmenides said that "there is no such thing as change") (see [1, Chapter 11) is also resolved by calculus, dealing as it does with change and motion: calculus produces *patterns* that are unchanging in their description of change.

Derivatives and integrals were defined and calculated by many mathematicians in the early 1600s. When we say Newton and Leibniz (independently) *created* calculus, we mean the creation of the Fundamental Theorem of Calculus, that relates differentiation and integration (except for a homework problem that cryptically indicates something of that sort, we won't go into that relation in this magnification).

See the references for histories of both calculus and classical Greek mathematics.

Pedagogically, calculus begins with *motion.* In math classes that traditionally precede calculus, a great deal of impressive detective work should appear. With trigonometry, for example, one may figure out the width of a river purely by making measurements on only one side of the river (see, e.g., [5, Example 3.9]). But that's not in the style of calculus, because everything you care about is static, and sits around passively and obligingly waiting for you to make measurements and calculate. A typical calculus problem is to drop an orange off a building and worry about its height above the ground at different times. Not only is the height of the orange changing every moment, the *rate* at which it changes (the *velocity* of the orange) is itself changing. That constant motion of the orange and its rate of change is disconcerting.

The motion of calculus might refer to nimble mental activity, especially making approximations that you hope will get arbitrarily close to a mysterious and elusive quantity of concern. For example, the number π is essential for so many things, yet we can never write it down explicitly. In particular, its decimal expansion (whatever that means!) is infinite and nonrepeating. Thus we approximate π and worry about whether our approximation is good enough. One of many such sequences of approximations that follow is from the decimal expansion:

$$\pi \sim 3,\ \pi \sim 3.1,\ \pi \sim 3.14,\ \pi \sim 3.141, \ldots.$$

The idea of such a sequence is that, although it's possible that no member of the sequence will equal π, we can get approximations of π of any desired accuracy.

It should be mentioned that Zeno's paradoxes involve motion, and, probably not coincidentally, Zeno was a student of Parmenides, mentioned above in the second paragraph of this introduction. Let's conclude this introduction with a quick description of three of those paradoxes.

Dichotomy. To walk to a chair a yard away, I must first cover half a yard, then one fourth of a yard (half the remaining distance), then one eighth of a yard (half of one fourth of a yard), ... Actually reaching the chair thus requires traveling a distance equal to the sum

$$\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\right) \quad \text{yards}$$

of infinitely many terms, which our intuition tells us (falsely, but Zeno didn't know this) must be an infinite distance, requiring an infinite amount of time to travel. Thus I will never reach the chair.

Achilles. Achilles is following a tortoise walking (slowly) in a straight line, trying to catch up with said tortoise.

Let's say Achilles starts out a_1 yards from the tortoise. When Achilles has traveled those a_1 yards, the tortoise has also moved, say to a distance a_2 yards from Achilles. Achilles then patiently travels those a_2 yards, only to find the tortoise a new distance, call it a_3 yards, from Achilles.

Proceeding this way produces an infinite sequence of numbers

$$a_1, a_2, a_3, a_4, \ldots$$

such that, for Achilles to reach the tortoise, Achilles must travel

$$(a_1 + a_2 + a_3 + a_4 + \ldots) \quad \text{yards.}$$

As with *Dichotomy*, this sum appears, to the uncalculussed layman, to equal an infinite distance, thus Achilles will never reach the tortoise.

Arrow. If an arrow is fired through the air, and we take a picture of it at a particular "instant" of time, the arrow appears to be stationary. This means its intended target is safe; the arrow cannot go anywhere.

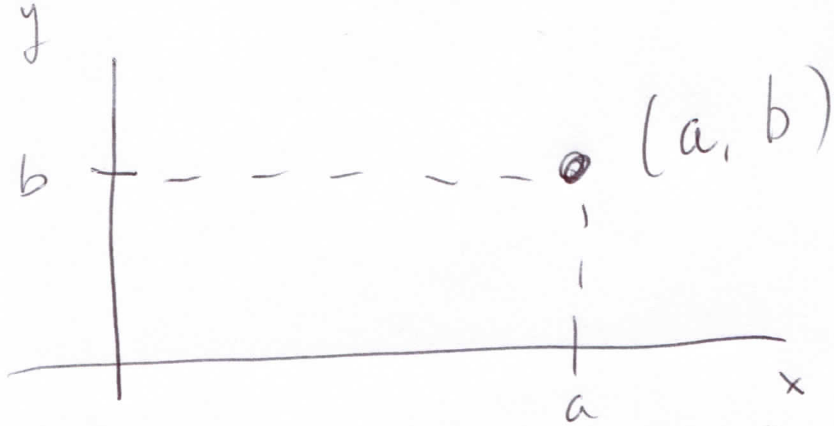
Dichotomy and *Achilles* involve arbitrarily large things (number of terms to add) while *Arrow* involves an arbitrarily small thing (an instant of time). To make sense of the seeming paradoxes requires the fundamental calculus idea of a *limiting process*, as in the comments after Tables 2.9 and 3.8.

CHAPTER I: REVIEW OF LINES.

We will briefly review the Cartesian plane and (straight) lines therein, partly to standardize terminology, but primarily with an eye to derivatives and integrals, that take on simple, familiar and believable forms in the setting of lines in the plane.

Definitions 1.1. The **Cartesian plane**, hereafter referred to as "**the plane**," is the set of all ordered pairs $\{(a, b) \mid a, b \text{ real}\}$, represented as points: for any real a, b, (a, b) is associated with the point a units to the right of the origin and b units above the origin. a is the **x coordinate** of (a, b) and b is the **y coordinate** of (a, b).

DRAWING 1.2



Definitions 1.3. A **line** (in the plane) is $\{(x, y) \mid y = mx + b\}$ or $\{(x, y) \mid x = c\}$, for some fixed real number m, b, or c.

$y = mx + b$ or $x = c$ is the **equation** of the line.

Examples 1.4. In each part, draw the line with the given equation, by finding two ordered pairs on the line.

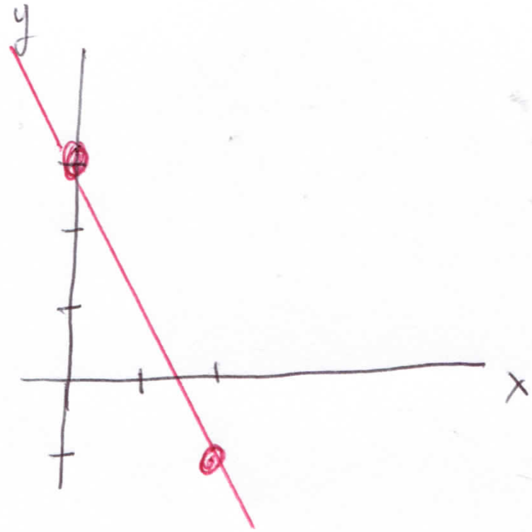
(a) $y = -2x + 3$

(b) $x = 1$

Solutions. For each line, make a small table of values, put the two ordered points from said table on the Cartesian plane, then draw the unique line through those points.

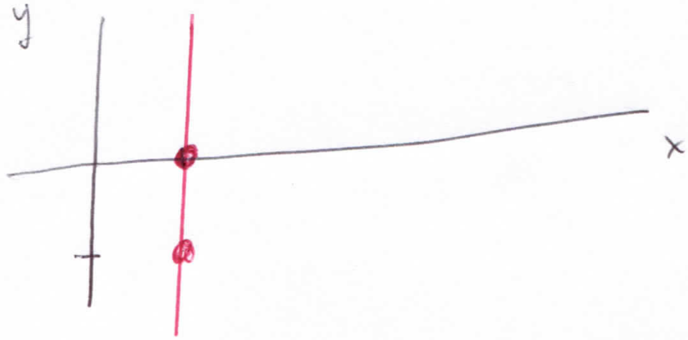
(a)

x	y
0	3
2	-1



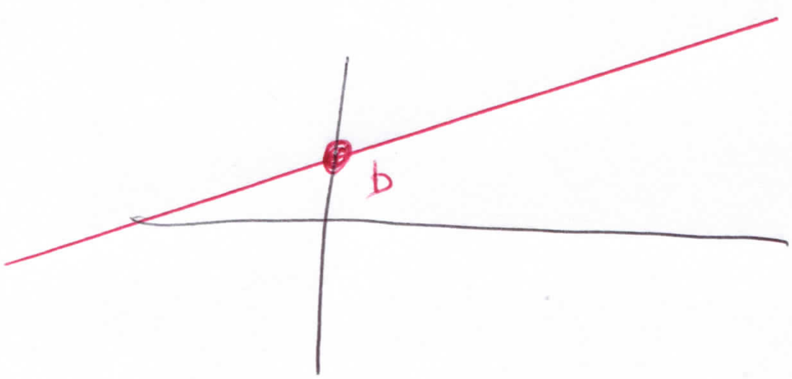
(b)

x	y
1	-1
1	0



Definitions 1.5. Suppose a line has equation $y = mx + b$. The number b is the **y intercept** of the line. For example, the line in Examples 1.4(a) has a y intercept of 3; in general, the y intercept is where the line crosses the y axis.

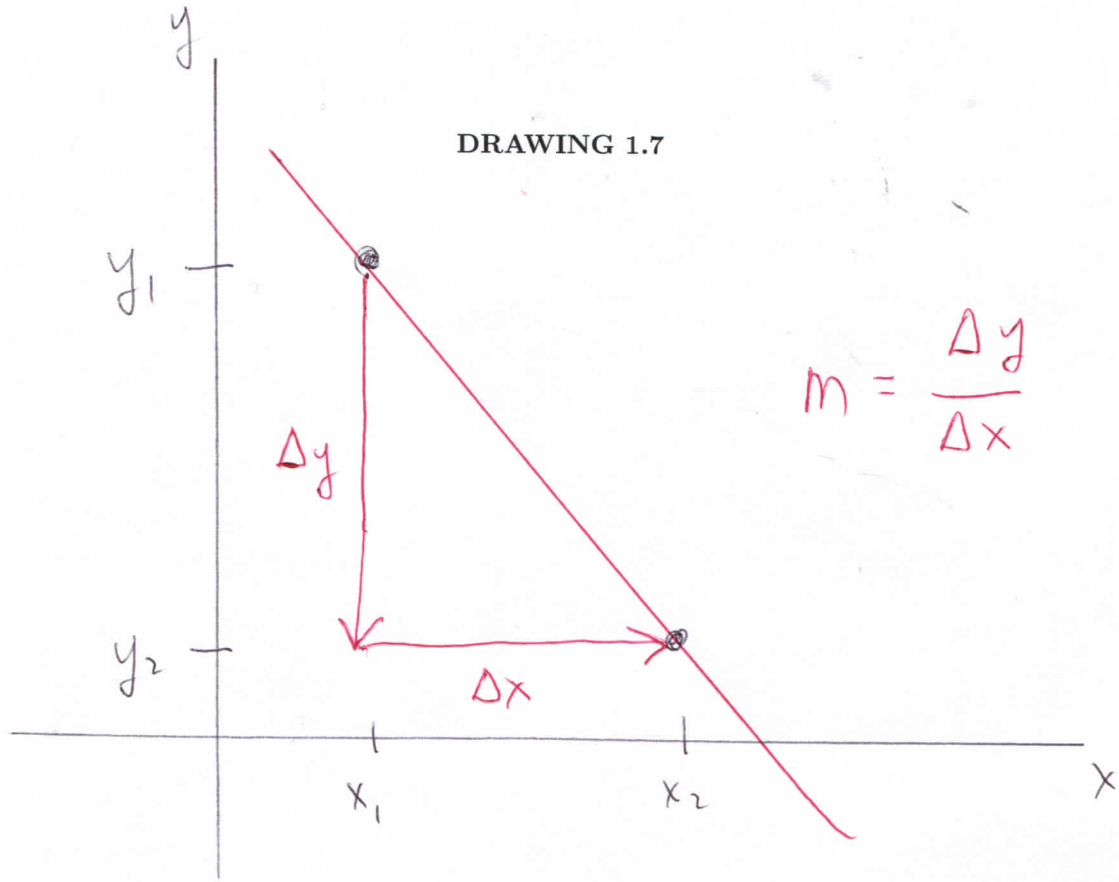
DRAWING 1.6



The number m is the **slope** or **rate of change** of $y = mx + b$ or the line, and has a more subtle picture (DRAWING 1.7), which we will now describe.

The capital Greek letter "delta" is denoted Δ, and translates roughly as *difference* or *change.* In our setting, take two points (x_1, y_1) and (x_2, y_2) on a line; then

$$\Delta x \equiv (\text{change in } x) \equiv (x_2 - x_1) \quad \text{and} \quad \Delta y \equiv (\text{change in } y) \equiv (y_2 - y_1).$$



Then it is not hard to show that $m = \frac{\Delta y}{\Delta x}$. Δy is sometimes called **rise** (think of a hot-air balloon), Δx **run** (think of running on the ground keeping up with the balloon), so that slope is dynamically called

$$m = \frac{\text{rise}}{\text{run}}, \quad \text{“rise over run.”}$$

Example 1.8. Let $y = 2x - 1$ be the equation of a line. For the pair of points $(0, -1), (3, 5)$ on the line,

$$\frac{\Delta y}{\Delta x} = \frac{5 - (-1)}{3 - 0} = \frac{6}{3} = 2;$$

for the pair of points $(-1, -3), (1, 1)$ on the line,

$$\frac{\Delta y}{\Delta x} = \frac{1 - (-3)}{1 - (-1)} = \frac{4}{2} = 2.$$

Notice that 2 is the slope of the line with equation $y = 2x - 1$.

Lines are characterized by the fact that $\frac{\Delta y}{\Delta x} \equiv \frac{(y_2 - y_1)}{(x_2 - x_1)}$ is the same for *any* pair of points $(x_1, y_1), (x_2, y_2)$ on the line; no other curve in the plane has this property.

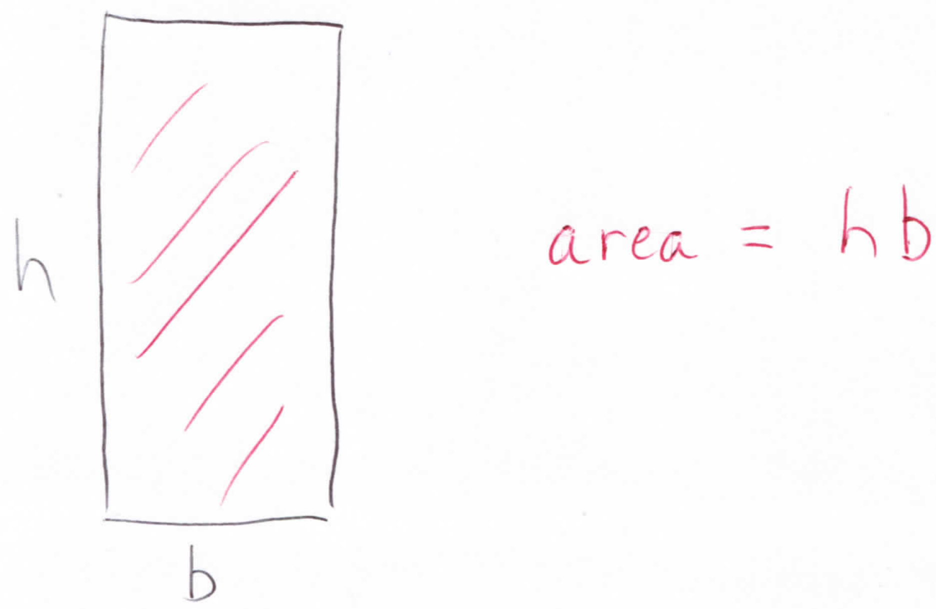
Example 1.9. Suppose a work drone charges fifty dollars to show up, then charges twelve dollars per hour for work done. Let y be the money made by the drone and let x be the number of hours the drone works. Then

$$y = 50 + 12x;$$

note that 12 is the slope of the line with the equation y = 50 + 12x and 50 is the y intercept.

Discussion 1.10. Up until now, all our work about lines in this chapter has been a precursor for the derivative, introduced in Chapter II. For the integral, we only need one result involving horizontal and vertical lines; the analogous preparation for the integral, to appear in Chapter III, consists solely of the following formula for the area of a rectangle:

DRAWING 1.11

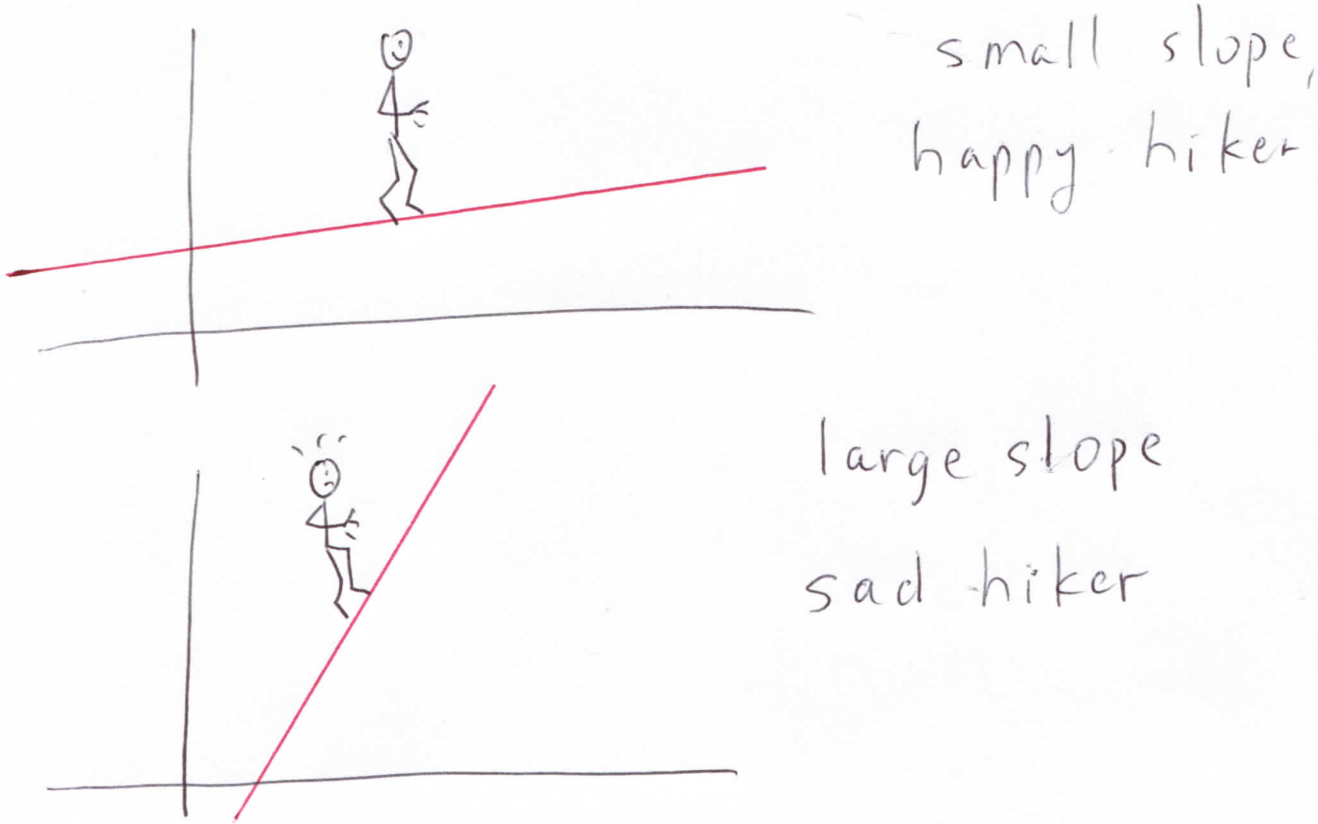


The derivative will be a natural extension of slope of a straight line, while the integral will be a natural extension of area of a rectangle.

CHAPTER II: SLOPE OF LINES LEADS TO DERIVATIVE

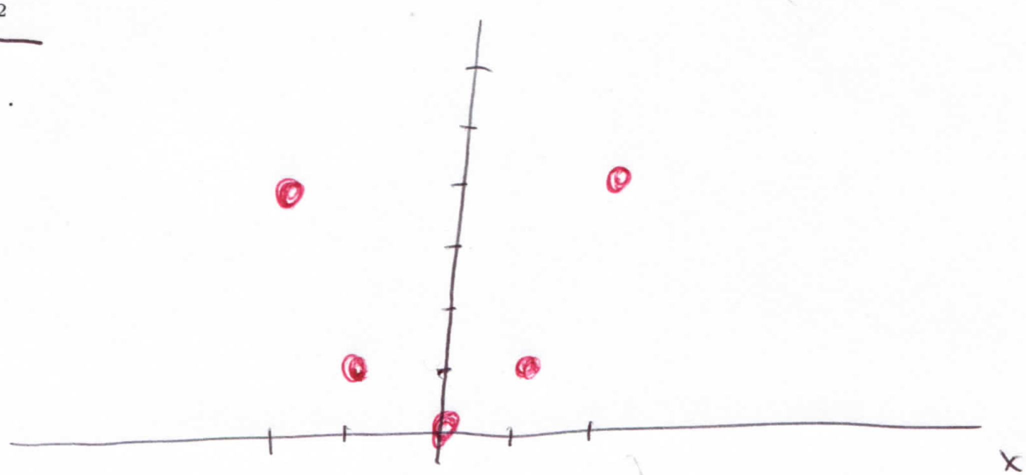
The word "slope," from Chapter I, has visceral physical meaning, as in the slope of a hill you wish to ascend. See DRAWINGS 2.1 below: you *feel* a steep slope (meaning the slope of Chapter I is a large positive number) in your heart rate and lung inhalation.

DRAWINGS 2.1



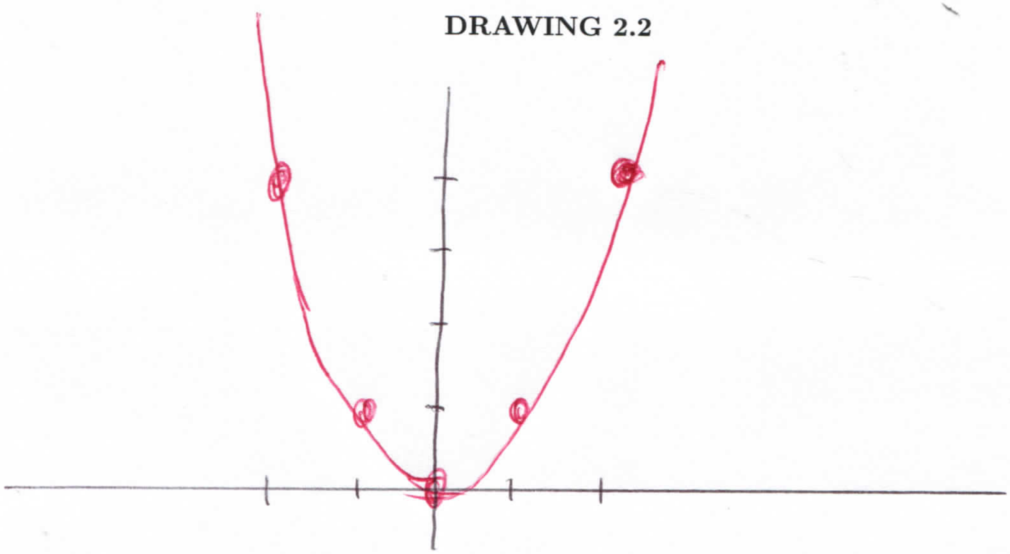
Consider now the parabola with equation $y = x^2$. As we did with lines, I'll draw the parabola by starting with a small table of values.

x	$y = x^2$
-2	4
-1	1
0	0
1	1
2	4



Draw the parabola by connecting the five dots smoothly.

DRAWING 2.2



If our hillside has this parabolic shape, hiking will feel different at different points.

DRAWING 2.3

From the point of view of hiker happiness, there appear to be different slopes at different points on the parabola.

Discussion 2.4. But don't take the stick figures' word. Recall our rigorous characterization of slope of a straight line: $\frac{\Delta y}{\Delta x} \equiv \frac{(y_2 - y_2)}{(x_2 - x_1)}$ (see Definitions 1.5).

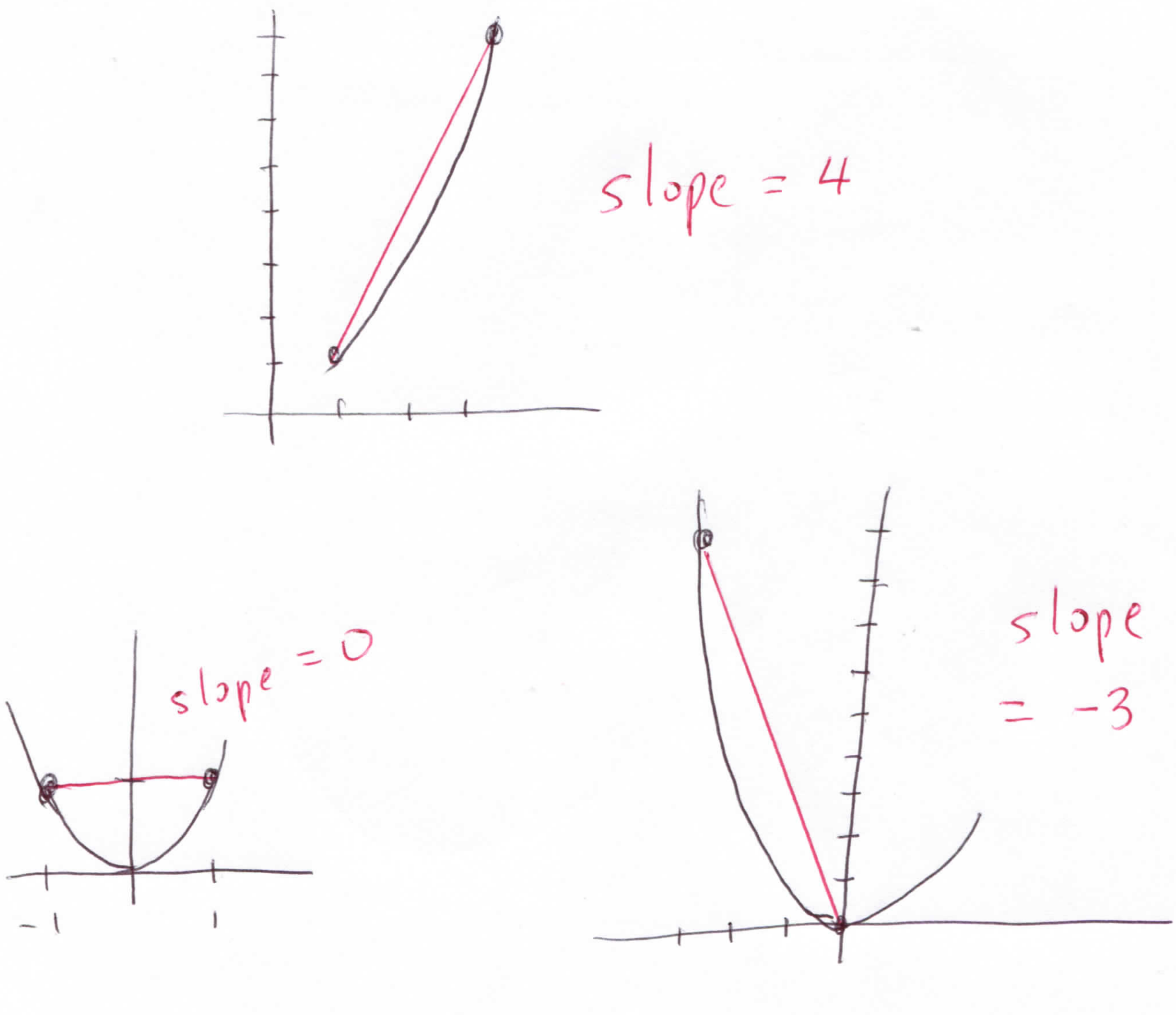
Let's calculate $\frac{\Delta y}{\Delta x}$, for different pairs of points $(x_1, y_1), (x_2, y_2)$ on the parabola.

$$(x_1, y_1) = (1, 1), (x_2, y_2) = (3, 9) \rightarrow \frac{\Delta y}{\Delta x} = \frac{9-1}{3-1} = \frac{8}{2} = 4.$$

$$(x_1, y_1) = (-1, 1), (x_2, y_2) = (1, 1) \rightarrow \frac{\Delta y}{\Delta x} = \frac{1-1}{1-(-1)} = \frac{0}{2} = 0.$$

$$(x_1, y_1) = (-3, 9), (x_2, y_2) = (0, 0) \rightarrow \frac{\Delta y}{\Delta x} = \frac{0-9}{0-(-3)} = \frac{-9}{3} = -3.$$

DRAWINGS 2.4(a)



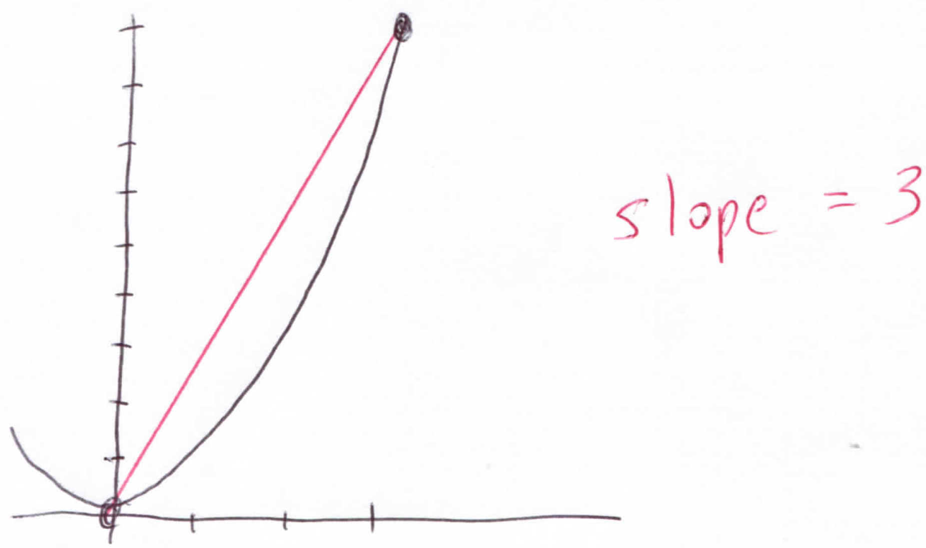
Let's keep x_1 constant, and try $\frac{\Delta y}{\Delta x}$:

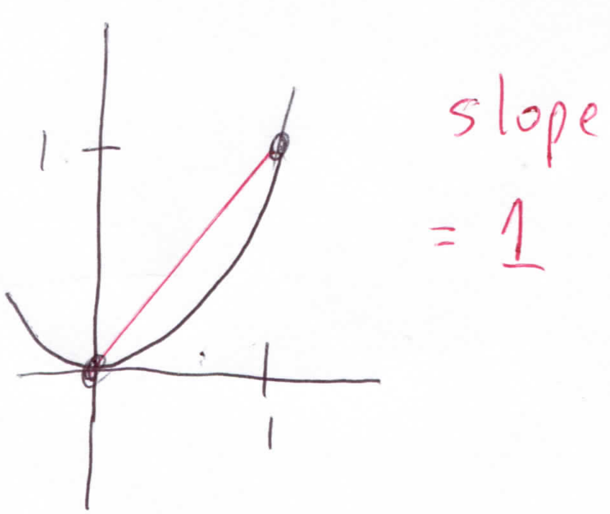
$$(x_1, y_1) = (0,0), (x_2, y_2) = (3,9) \rightarrow \frac{\Delta y}{\Delta x} = \frac{9-0}{3-0} = \frac{9}{3} = 3.$$

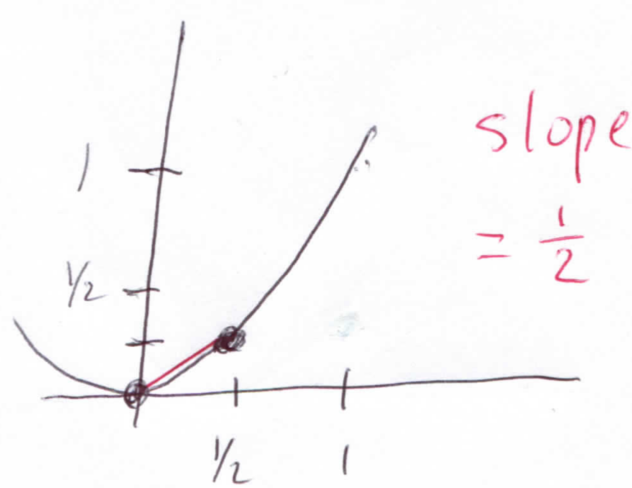
$$(x_1, y_1) = (0,0), (x_2, y_2) = (1,1) \rightarrow \frac{\Delta y}{\Delta x} = \frac{1-0}{1-0} = \frac{1}{1} = 1.$$

$$(x_1, y_1) = (0,0), (x_2, y_2) = (\frac{1}{2}, \frac{1}{4}) \rightarrow \frac{\Delta y}{\Delta x} = \frac{\frac{1}{4}-0}{\frac{1}{2}-0} = \frac{1}{2}.$$

DRAWINGS 2.4(b)





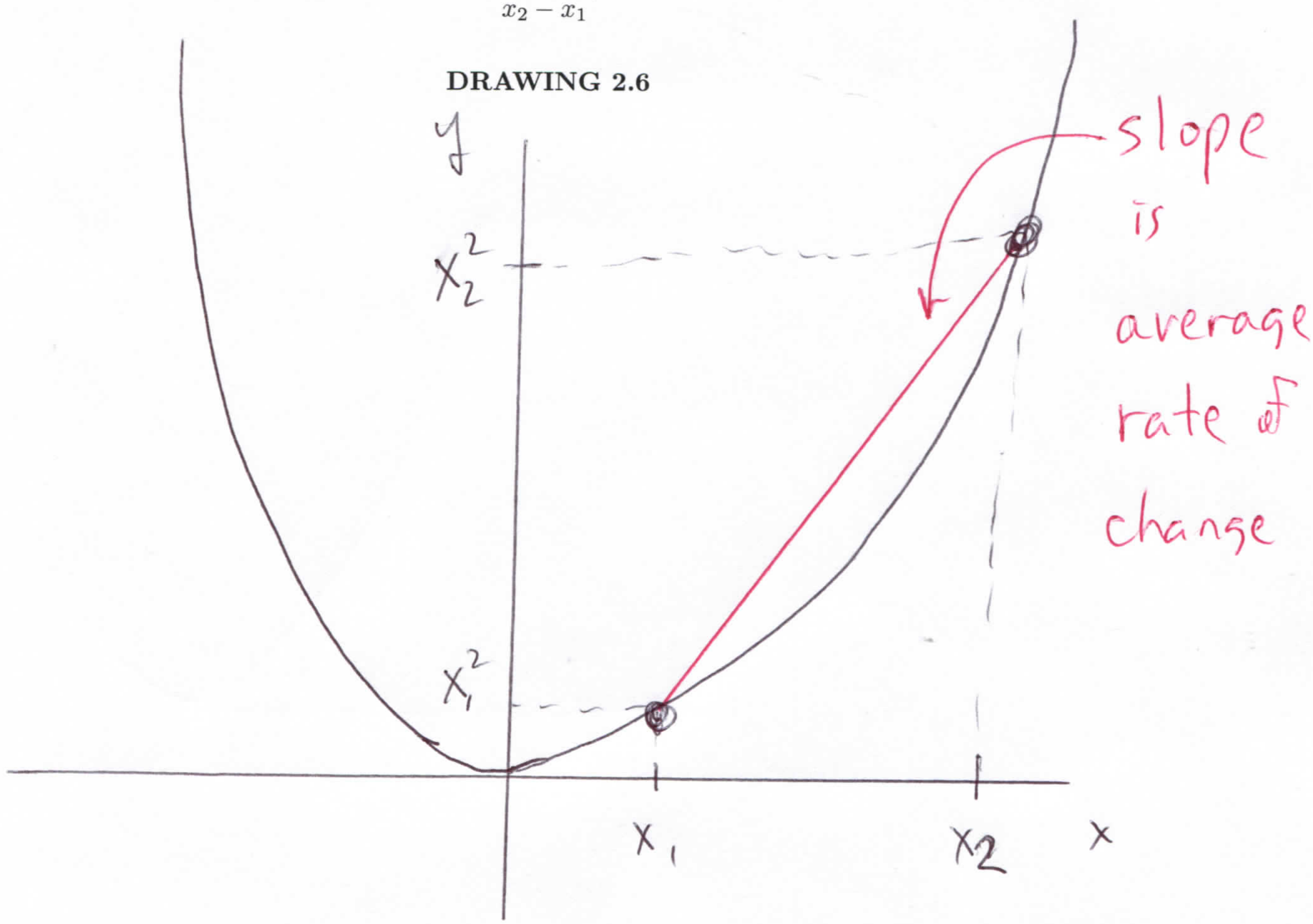


Definition 2.5. For the parabola, or any curve other than a straight line, $\frac{\Delta y}{\Delta x}$ will not be the same, for all pairs of points $(x_1, y_1), (x_2, y_2)$; thus *slope*, or *rate of change*, as in Definitions 1.5, is not a uniquely defined number for the entire curve. The best we can do, before wading into the waters of calculus, is talk about *average rate of change*, over an interval of x values $x_1 \leq x \leq x_2$.

For the parabola with equation $y = x^2, x_1 < x_2$ real numbers, the **average rate of change** over $x_1 \leq x \leq x_2$ is

$$\frac{x_2^2 - x_1^2}{x_2 - x_1}.$$

DRAWING 2.6



For example, we've calculated in Discussion 2.4 that the average rate of change over $1 \leq x \leq 3$ is 4, while the average rate of change over $-1 \leq x \leq 1$ is 0.

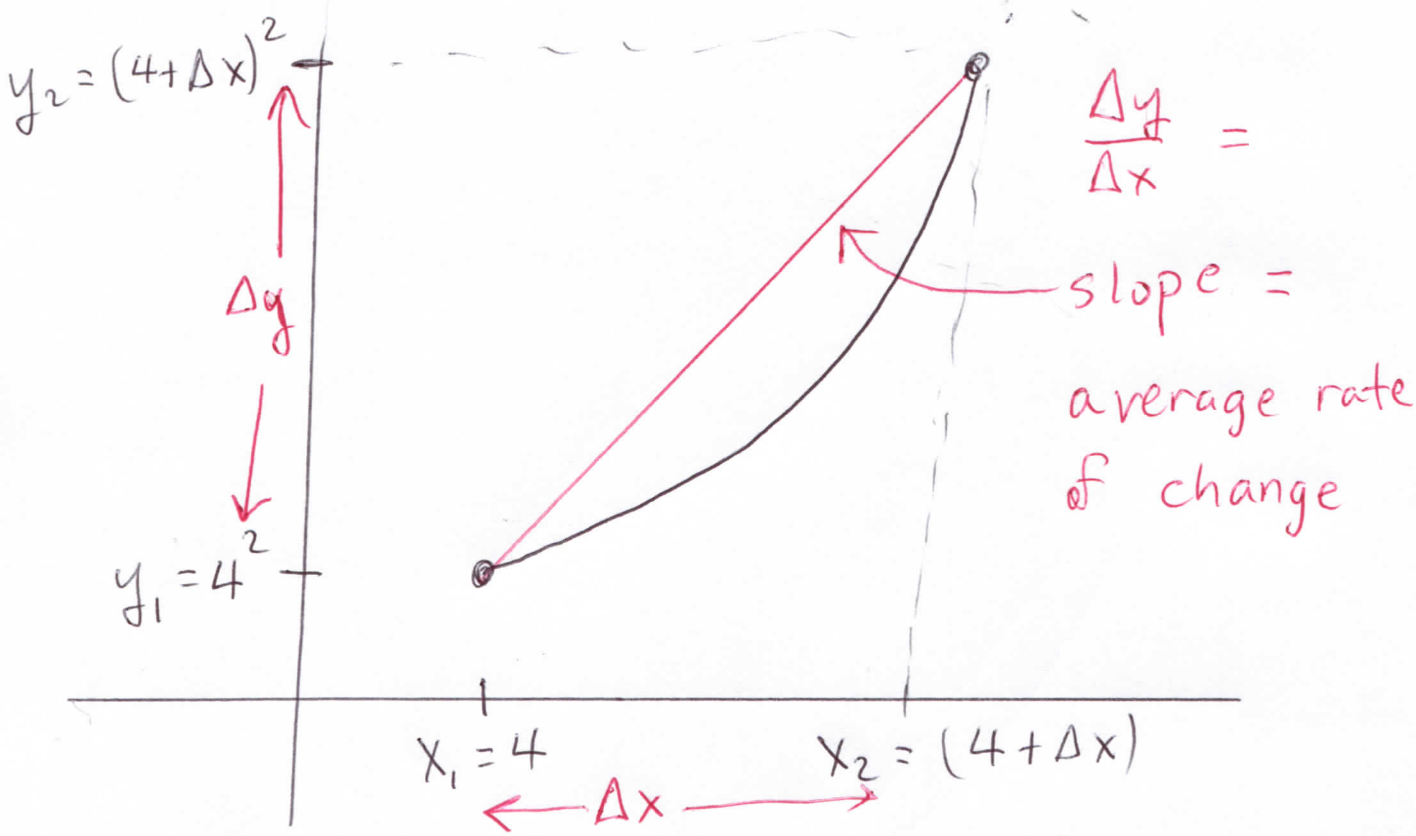
Discussion 2.7. Average rate of change is the rate of change over an interval. We want to address the rate of change *at a point*. For example, we want to know at what rate a hiker is ascending at a particular moment, as in DRAWING 2.3.

Let's say we want the rate of change of the parabola $y = x^2$ at $x = 4$. Here is our strategy. For arbitrary Δx, we will calculate the average rate of change over

$$4 \leq x \leq (4 + \Delta x);$$

see DRAWING 2.8.

DRAWING 2.8



Our intuition is, that as Δx gets very small, we should be getting very good approximations of a number that we could reasonably call *slope* or *(instantaneous) rate of change* of $y = x^2$ at $x = 4$.

TABLE 2.9

x_2	change in x	change in y	average rate of change
5	$(5-4) = 1$	$(4+1)^2 - 4^2 = 5^2 - 4^2 = 9$	$\frac{9}{1} = 9$
4.1	$(4.1-4) = 0.1$	$(4+0.1)^2 - 4^2 = 4.1^2 - 4^2 = 0.81$	$\frac{0.81}{0.1} = 8.1$
4.01	$(4.01-4) = 0.01$	$(4+0.01)^2 - 4^2 = 4.01^2 - 4^2 = 0.0801$	$\frac{0.0801}{0.01} = 8.01$
4.001	$(4.001-4) = 0.001$	$(4+0.001)^2 - 4^2 = 4.001^2 - 4^2 = 0.008001$	$\frac{0.008001}{0.001} = 8.001$
$(4+\Delta x)$	$(4+\Delta x) - 4 = \Delta x$	$(4+\Delta x)^2 - 4^2 = (8\Delta x + (\Delta x)^2)$	$\frac{(8\Delta x+(\Delta x)^2)}{\Delta x} = (8+\Delta x)$

Focusing on the right-hand column of the table above, we see average rate of change getting arbitrarily close to 8, as Δx gets small. In fact, the bottom entry

$$(8 + \Delta x)$$

in the right-hand column would *equal* 8 if we let $\Delta x = 0$.

The number 8 is the **limit, as Δx goes to zero** of the average rate of change over

$$4 \leq x \leq (4 + \Delta x),$$

denoted

$$\lim_{\Delta x \to 0} [\text{average rate of change over } 4 \leq x \leq (4 + \Delta x)] = \lim_{\Delta x \to 0} \left[\frac{(4 + \Delta x)^2 - 4^2}{\Delta x} \right] = \lim_{\Delta x \to 0} [8 + \Delta x] = 8.$$

That limiting number 8 is called the **(instantaneous) rate of change** or **slope** or **derivative** of $y = x^2$, at $x = 4$.

We also say that the average rate of change **converges** to the derivative.

Remark 2.10. When x is time and y is distance traveled, the average rate of change of y is average velocity and the instantaneous rate of change is instantaneous velocity, the number you'd see if a speedometer were attached to the moving body.

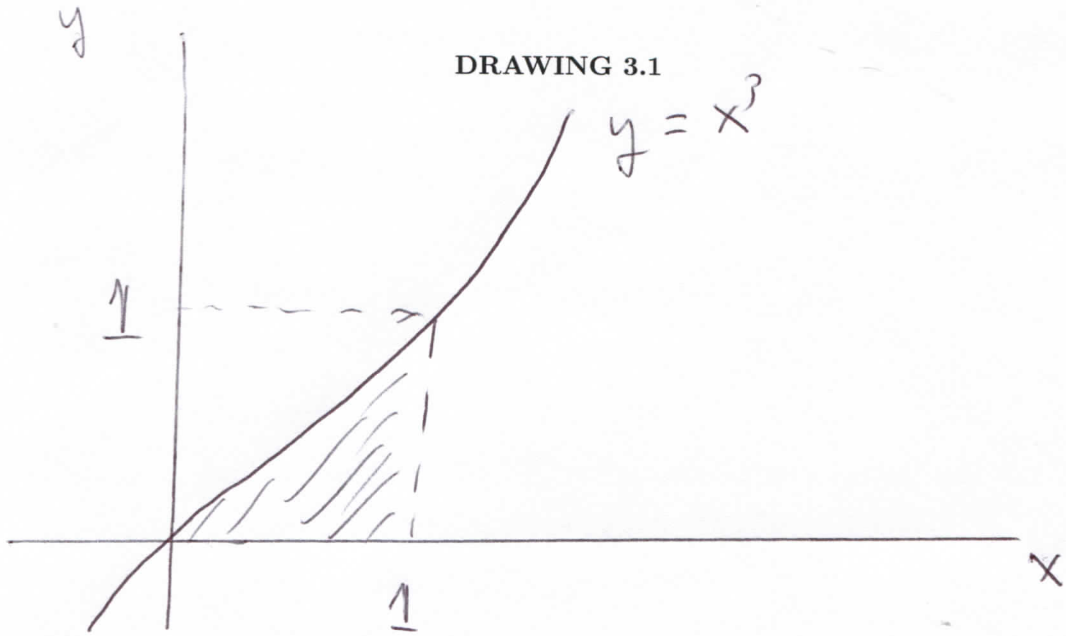
CHAPTER III: AREA OF RECTANGLES LEADS TO INTEGRAL

In this chapter we will get the area between $y = x^3, y = 0$ and $x = 1$, that is, the area of

$$\{(x, y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq x^3\},$$

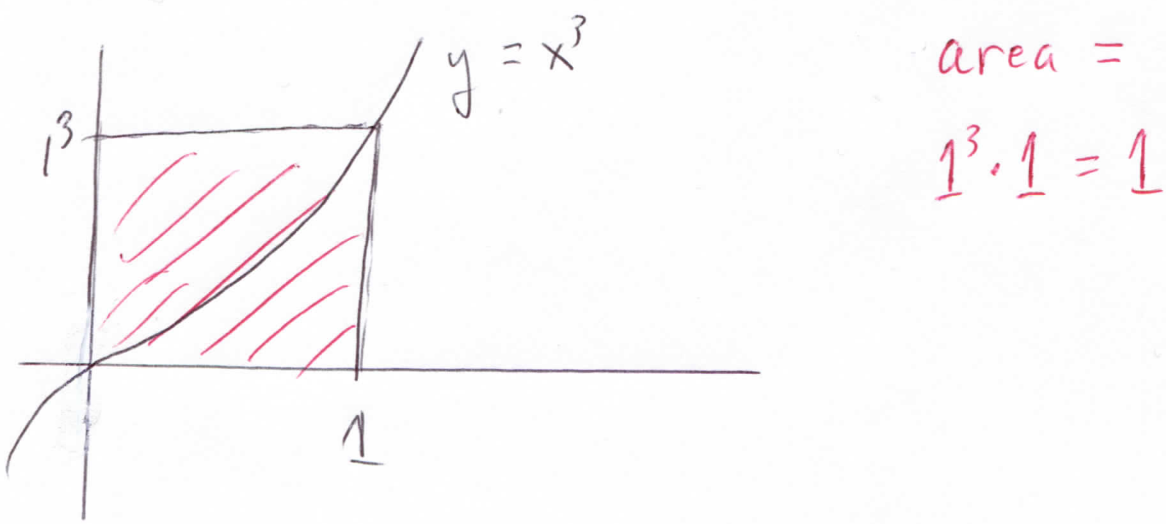
by approximating with rectangles.

DRAWING 3.1



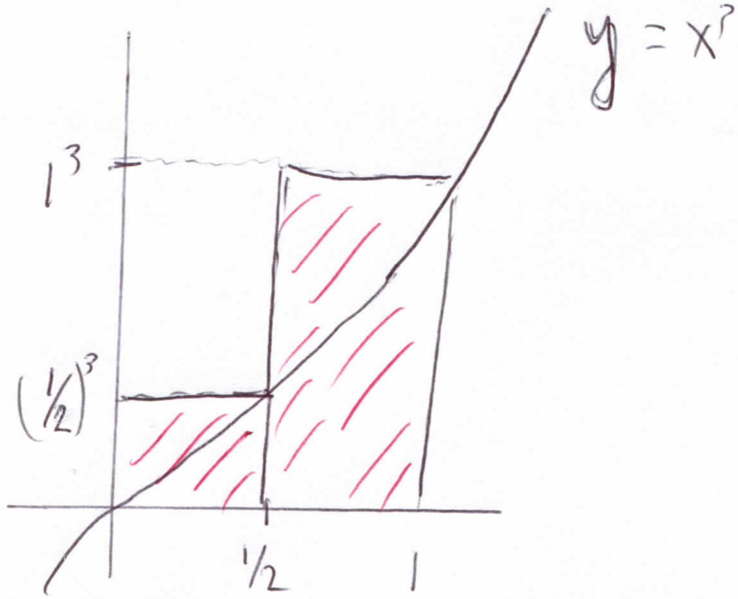
All we need to begin is the area of a rectangle, as in DRAWING 1.12. We can approximate the area in DRAWING 3.1 by putting it inside rectangles. In DRAWINGs 3.2–3.5, the red-shaded area is approximating the black-shaded area of DRAWING 3.1 that we seek. Let n be the number of rectangles. Here is an approximation with $n = 1$, only one rectangle.

DRAWING 3.2



Here is an approximation with $n = 2$, two rectangles.

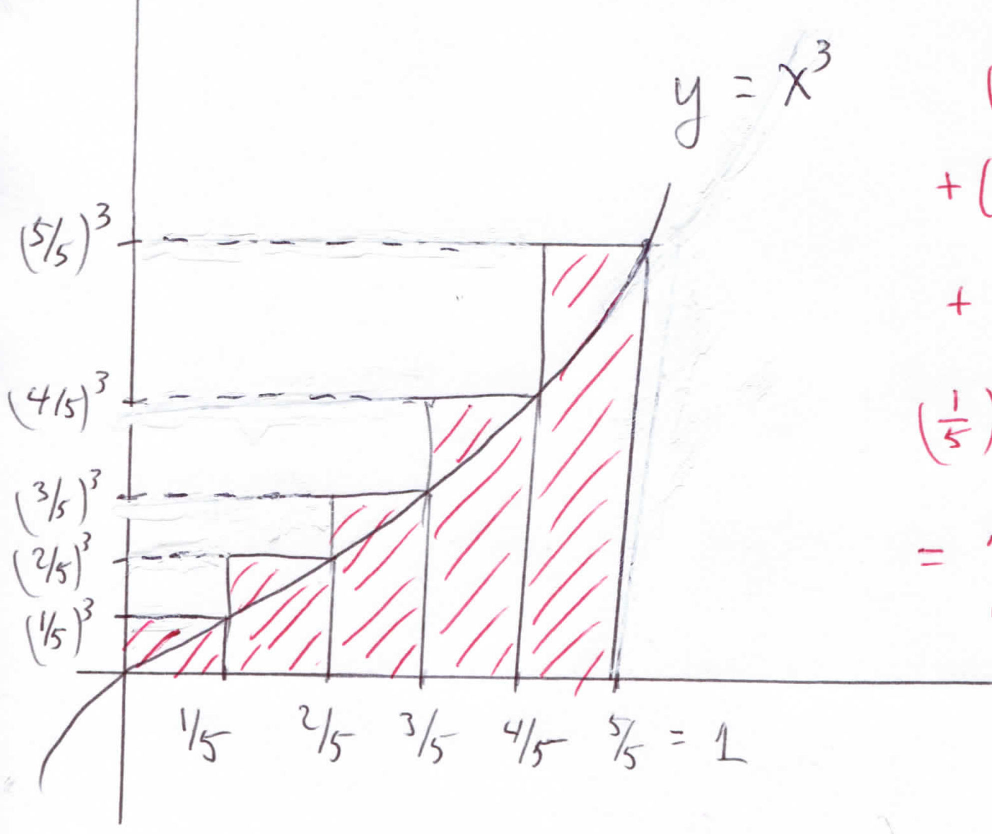
DRAWING 3.3



$$\text{area} = \left(\tfrac{1}{2}\right)^3 \cdot \tfrac{1}{2} + 1^3 \cdot \tfrac{1}{2} = \tfrac{1}{2}\left(\tfrac{1}{8} + 1\right) = \tfrac{9}{16}$$

This is an improvement over one rectangle, but still not so good. The more rectangles, the closer we can hug the curve and the better the approximation. Here's an approximation with $n = 5$, five rectangles.

DRAWING 3.4

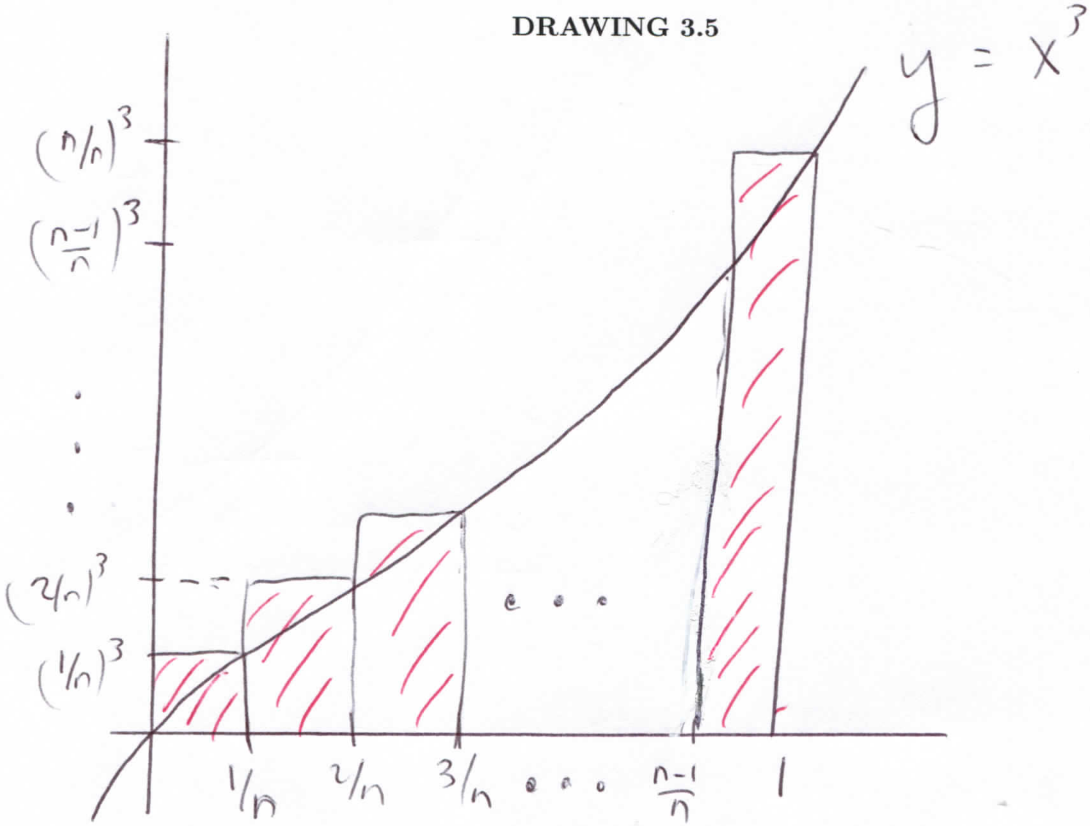


$$\text{area} = \left(\tfrac{1}{5}\right)^3 \cdot \tfrac{1}{5} + \left(\tfrac{2}{5}\right)^3 \cdot \tfrac{1}{5} + \left(\tfrac{3}{5}\right)^3 \cdot \tfrac{1}{5} + \left(\tfrac{4}{5}\right)^3 \cdot \tfrac{1}{5} + \left(\tfrac{5}{5}\right)^3 \cdot \tfrac{1}{5} =$$

$$\left(\tfrac{1}{5}\right)^4 \left[1^3 + 2^3 + 3^3 + 4^3 + 5^3\right] = \tfrac{225}{625} = \tfrac{9}{25}$$

In general, let's put n red-shaded rectangles of equal bases with upper right vertex on the curve of $y = x^3$, covering the black-shaded area in DRAWING 3.1, and calculate their total area.

DRAWING 3.5



The total sum of areas of red-shaded rectangles in DRAWING 3.5 is, adding up height times base for each rectangle, from left to right (compare with DRAWINGs 3.3 and 3.4),

$$\left(\frac{1}{n}\right)^3 \frac{1}{n} + \left(\frac{2}{n}\right)^3 \frac{1}{n} + \left(\frac{3}{n}\right)^3 \frac{1}{n} + \ldots \left(\frac{n}{n}\right)^3 \frac{1}{n} = \frac{1}{n^4}\left(1^3 + 2^3 + 3^3 + \ldots n^3\right)$$

Now we need the following formula.

Sum of cubes 3.6. For $n = 1, 2, 3, \ldots,$

$$\left(1^3 + 2^3 + 3^3 + \cdots + n^3\right) = \frac{n^2(n+1)^2}{4}.$$

For example (compare to DRAWINGS 3.2–4),

$$n = 1 \to 1^3 = \frac{1^2 \times 2^2}{4} = 1, \;\; n = 2 \to (1^3+2^3) = \frac{2^2 \times 3^2}{4} = 9, \;\; n = 5 \to (1^3+2^3+3^3+4^3+5^3) = \frac{5^2 \times 6^2}{4} = 225.$$

Combine this with our calculation of sums of areas of rectangles after DRAWING 3.5, to get the following.

3.7. SUM of AREAS of RECTANGLES in DRAWING 3.5 is

$$\frac{(n+1)^2}{4n^2}.$$

Let's put this in a table. The first three entries will be from DRAWINGs 3.2—4, and what remains will use 3.7.

TABLE 3.8.

number of rectangles	total area of rectangles
1	1
2	$\frac{9}{16} = 0.5625$
5	$\frac{9}{25} = 0.36$
10	$\frac{11^2}{4\times 10^2} = \frac{121}{400} = 0.3025$
100	$\frac{101^2}{4\times 10^4} = \frac{10,201}{40,000} = 0.255025$
$1,000$	$\frac{1,001^2}{4\times 10^6} = \frac{1,002,001}{4,000,000} = 0.25050025$
n	$\frac{(n+1)^2}{4n^2} = (\frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2})$

As the number of rectangles, denoted n, gets arbitrarily large, the fractions $\frac{1}{2n}$ and $\frac{1}{n^2}$ in the bottom of the right-hand column get arbitrarily small (the reader should experiment with a calculator to believe this). In words, the **limit, as** $n \to \infty$, **of the total area of** n **rectangles equals** $\frac{1}{4}$. The shorthand is

$$\lim_{n\to\infty} [\text{ total area of } n \text{ rectangles }] = \lim_{n\to\infty} \left(\frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2}\right) = \frac{1}{4}.$$

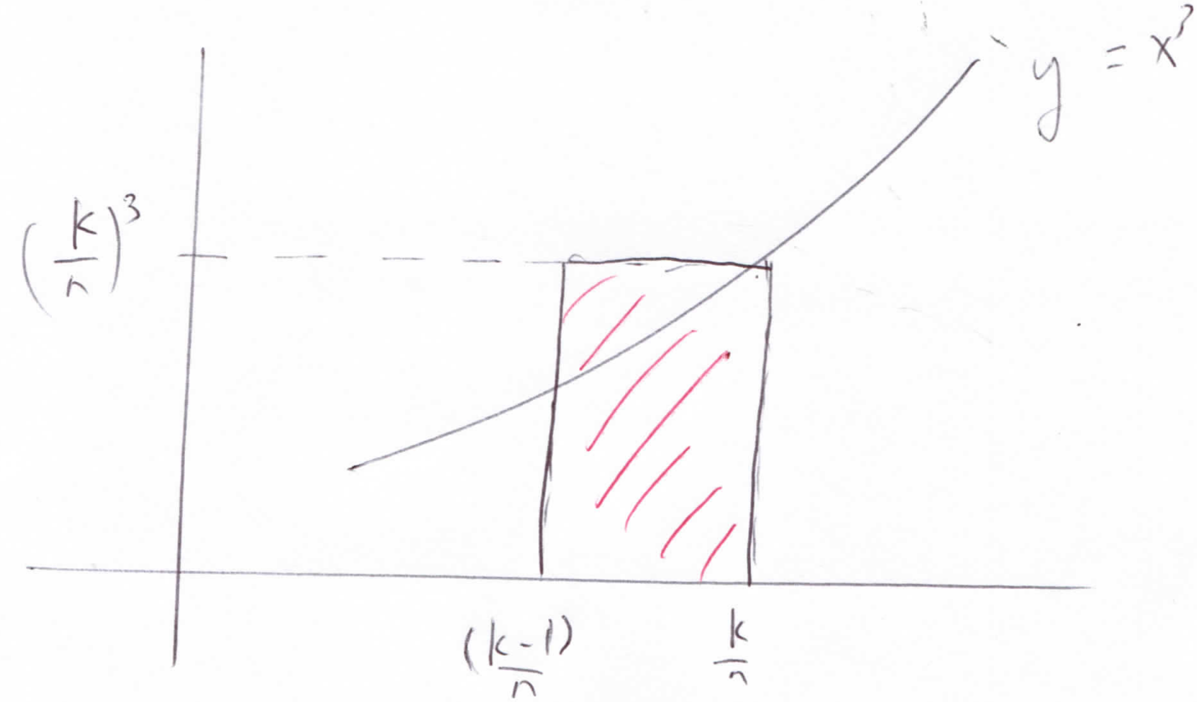
That $\frac{1}{4}$ is the **area** of the figure in DRAWING 3.1. In the language of calculus, the **integral of** $y = x^3$, **from** 0 **to** 1, is $\frac{1}{4}$, denoted

$$\int_0^1 x^3\,dx = \frac{1}{4}.$$

The integral sign $\int$ is meant to be an "S" for "Sum" (of areas of rectangles).

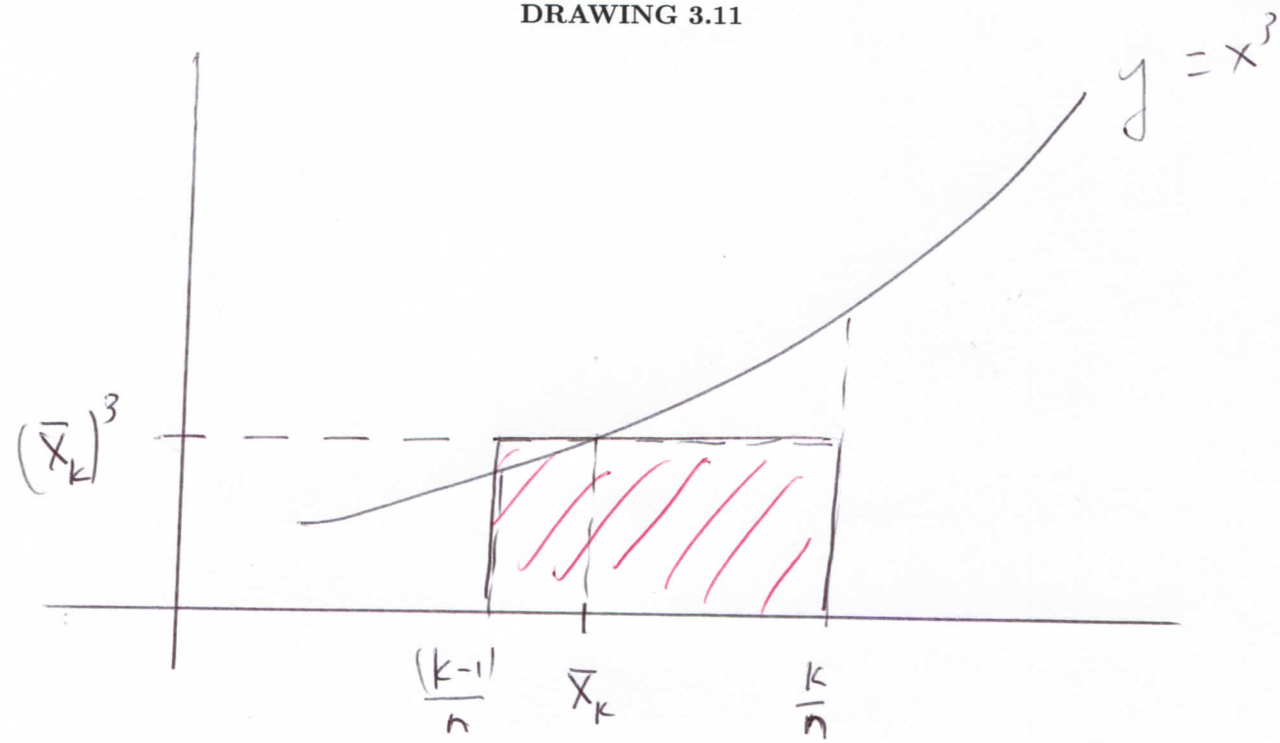
Remarks 3.9. In our construction of rectangles, as in DRAWING 3.5, each subinterval of the x axis used its right endpoint to determine the height of the rectangle; see DRAWING 3.10 below, where, for $1 \leq k \leq n$, we have drawn the k^{th} rectangle from DRAWING 3.5, with base $\frac{k-1}{n} \leq x \leq \frac{k}{n}$ and height $(\frac{k}{n})^3$.

DRAWING 3.10



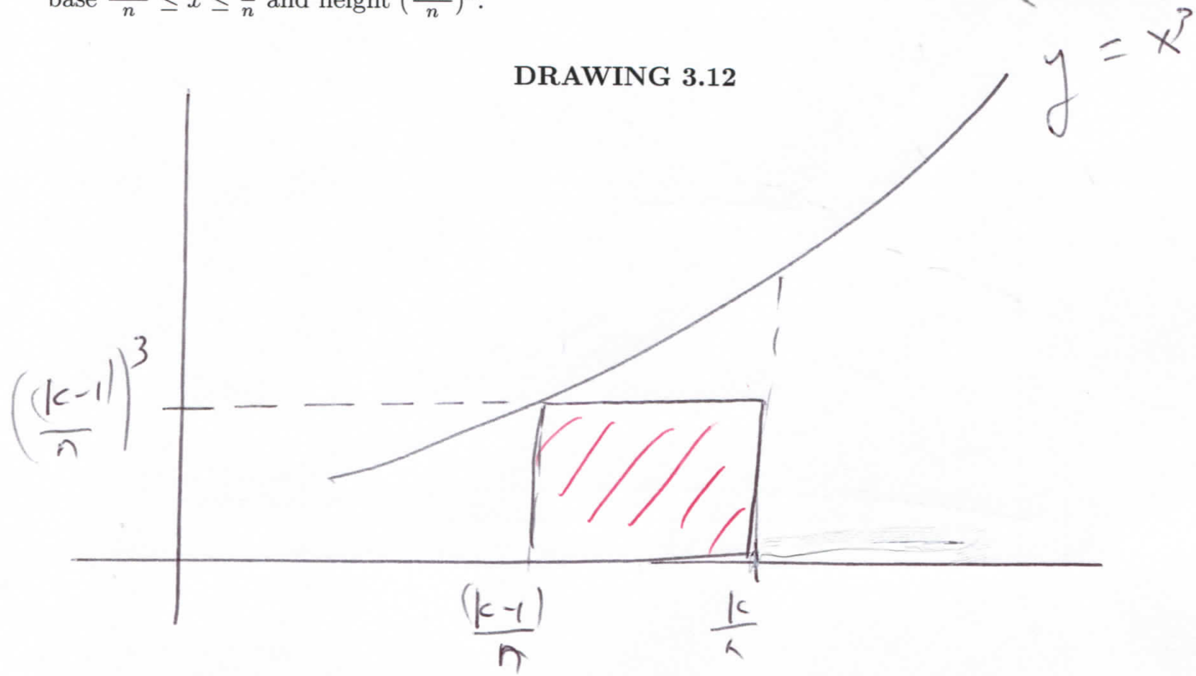
For each k, instead of the right endpoint, we could choose *any* point in that subinterval $\frac{k-1}{n} \leq x \leq \frac{k}{n}$, call said point $\overline{x_k}$, to determine the height of the k^{th} rectangle; that is, the k^{th} rectangle has base $\frac{k-1}{n} \leq x \leq \frac{k}{n}$ and height $(\overline{x_k})^3$, as in DRAWING 3.11 below.

DRAWING 3.11



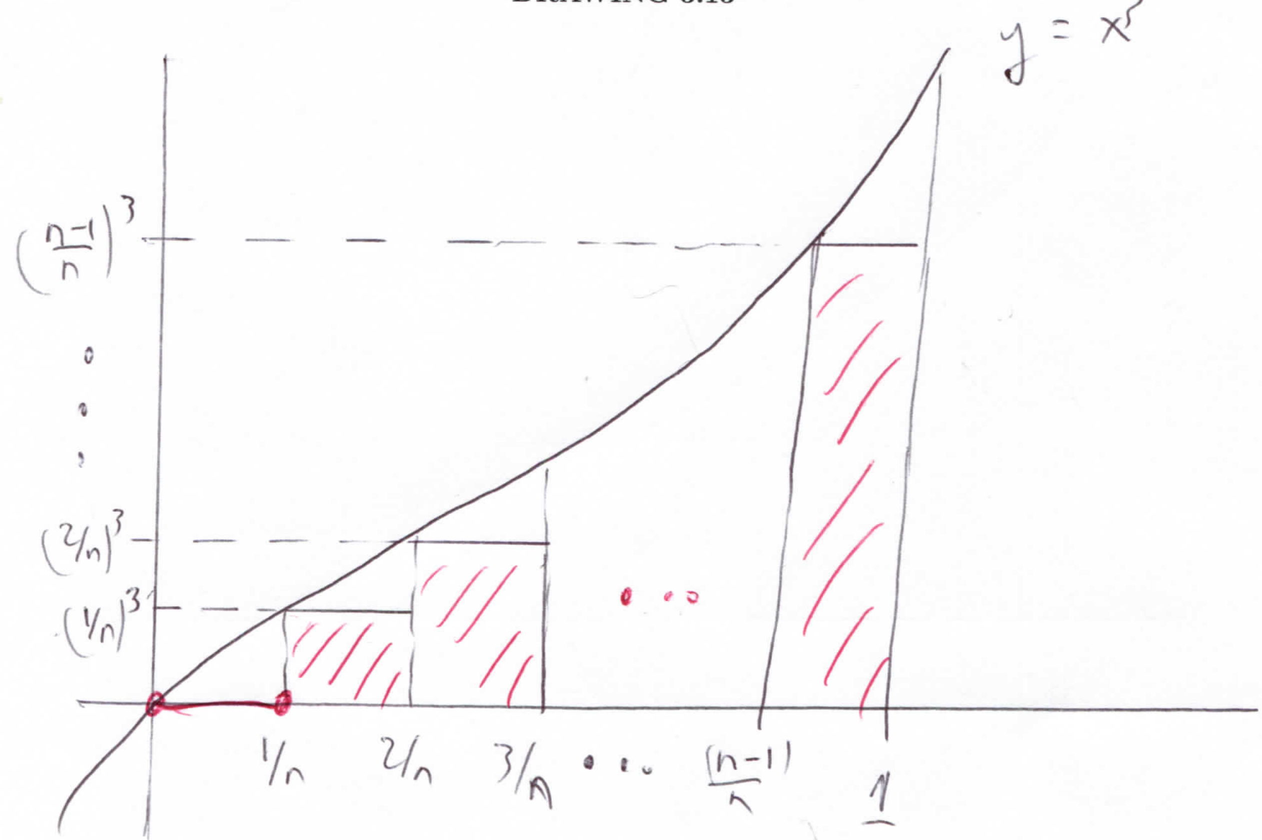
In particular, let's try using the left endpoint of each subinterval for the height of the approximating rectangle; in DRAWING 3.12 below we have, for $1 \leq k \leq n$, drawn the k^{th} rectangle, with base $\frac{k-1}{n} \leq x \leq \frac{k}{n}$ and height $(\frac{k-1}{n})^3$.

DRAWING 3.12



Putting together all the rectangles, $1 \leq k \leq n$, from DRAWING 3.12, produces the following analogue of DRAWING 3.5. Notice that the rectangles in DRAWING 3.5 produce an overestimate of the desired black-shaded area from DRAWING 3.1, while DRAWING 3.13 below produces an underestimate.

DRAWING 3.13



As with 3.7 and the bottom line of TABLE 3.8, let's calculate, using 3.6, the sum of areas of rectangles in DRAWING 3.13:

3.14.

$$\left(\frac{0}{n}\right)^3\frac{1}{n}+\left(\frac{1}{n}\right)^3\frac{1}{n}+\left(\frac{2}{n}\right)^3\frac{1}{n}+\dots\left(\frac{(n-1)}{n}\right)^3\frac{1}{n}=\frac{1}{n^4}\left(0^3+1^3+2^3+\dots(n-1)^3\right)$$

$$=\frac{1}{n^4}\left(\frac{(n-1)^2n^2}{4}\right)=\frac{(n-1)^2}{4n^2}=\left(\frac{1}{4}-\frac{1}{2n}+\frac{1}{4n^2}\right),$$

which, as with the calculations after TABLE 3.8, has limit, as $n \to \infty$, of $\frac{1}{4}$.

Remark 3.15. Combining the overestimate of 3.7 and the underestimate of 3.14 tells us that, for $n = 1, 2, 3, \dots$, the desired black-shaded area in DRAWING 3.1, denoted $\int_0^1 x^3\,dx$, is betweeen

$$(1_n)\left(\frac{1}{4}-\frac{1}{2n}+\frac{1}{4n^2}\right) \quad \text{and} \quad (2_n)\left(\frac{1}{4}+\frac{1}{2n}+\frac{1}{4n^2}\right).$$

We can make both (1_n) and (2_n) as close to $\frac{1}{4}$ as we like. This implies that $\int_0^1 x^3\,dx = \frac{1}{4}$. This argument is an example of the *Method of Exhaustion*, practiced by the classical Greeks. Although they did not have the idea of limit, the Method of Exhaustion has many similarities to limit; see [8, Section 4.3].

HOMEWORK

1. Find the derivative (slope) of $y = x^2$ at $x = 3$ by approximating with the slope of the line between

$$(3,9) \text{ and } (3+\Delta x, (3+\Delta x)^2),$$

for Δx small.

You will need the formula

$$(a+b)^2 = a^2 + 2ab + b^2.$$

2. Find the derivative of $y = x^3$ at $x = 2$ by approximating with the slope of the line between

$$(2,8) \text{ and } (2+\Delta x, (2+\Delta x)^3),$$

for Δx small.

You will need the formula

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

3. Find the area $\int_0^1 x^2\,dx$ between the x axis, $x = 0, x = 1$, and $y = x^2$, that is, the area of

$$\{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le x^2\},$$

by approximating with rectangles.
You will need the formula

$$(1^2 + 2^2 + 3^2 + \cdots + n^2) = \frac{n(n+1)(2n+1)}{6},$$

for $n = 1, 2, 3, \ldots$.

4. Let $b > 0$ be a fixed number. By using the approximations of slope, rate of change, or derivative from Chapter II and the approximations of area, or integral, from Chapter III (3.9–3.14, with x^3 replaced by b^x, recommended), express

$$(\text{derivative of } b^x \text{ at } x = 0)\left(\int_0^1 b^x\,dx\right)$$

in terms of b.

You may find the following formula useful:

$$\left(1 + r + r^2 + \ldots r^{n-1}\right) = \frac{(1-r^n)}{(1-r)},$$

for $r \neq 1, n = 0, 1, 2, \ldots$.

HOMEWORK HINTS

1. See Discussion 2.7, and replace $x = 4$ with $x = 3$, to produce the following analogue of TABLE 2.9.

TABLE I

x_2	change in x	change in y	average rate of change
4	$(4-3) = 1$	$(3+1)^2 - 3^2 = 4^2 - 3^2 = 7$	$\frac{7}{1} = 7$
3.1	$(3.1-3) = 0.1$	$(3+0.1)^2 - 3^2 = 3.1^2 - 3^2 = 0.61$	$\frac{0.61}{0.1} = 6.1$
3.01	$(3.01-3) = 0.01$	$(3+0.01)^2 - 3^2 = 3.01^2 - 3^2 = 0.0601$	$\frac{0.0601}{0.01} = 6.01$
3.001	$(3.001-3) = 0.001$	$(3+0.001)^2 - 3^2 = 3.001^2 - 3^2 = 0.006001$	$\frac{0.006001}{0.001} = 6.001$
$(3+\Delta x)$	$(3+\Delta x) - 3 = \Delta x$	$(3+\Delta x)^2 - 3^2 = (6\Delta x + (\Delta x)^2)$	$\frac{(6\Delta x+(\Delta x)^2)}{\Delta x} = (6+\Delta x)$

2. See Discussion 2.7 and replace $x = 4$ with $x = 2$ and $y = x^2$ with $y = x^3$, to produce the following table, analogous to TABLE I above and TABLE 2.9.

TABLE II

x_2	change in x	change in y	average rate of change
3	$(3-2) = 1$	$(2+1)^3 - 2^3 = 3^3 - 2^3 = 19$	$\frac{19}{1} = 19$
2.1	$(2.1-2) = 0.1$	$(2+0.1)^3 - 2^3 = 2.1^3 - 2^3 = 1.261$	$\frac{1.261}{0.1} = 12.61$
2.01	$(2.01-2) = 0.01$	$(2+0.01)^3 - 2^3 = 2.01^3 - 2^3 = 0.120601$	$\frac{0.120601}{0.01} = 12.0601$
2.001	$(2.001-2) = 0.001$	$(2+0.001)^3 - 2^3 = 2.001^3 - 2^3 = 0.012006001$	$\frac{0.012006001}{0.001} = 12.006001$
$(2+\Delta x)$	$(2+\Delta x) - 2 = \Delta x$	$(2+\Delta x)^3 - 2^3 = (12\Delta x + 6(\Delta x)^2 + (\Delta x)^3)$	$\frac{(12\Delta x+6(\Delta x)^2+(\Delta x)^3)}{\Delta x}$ $= (12 + 6\Delta x + (\Delta x)^2)$

3. Throughout Chapter III, replace cubes with squares, including replacing the sum of cubes 3.6 with the sum of squares

$$(1^2 + 2^2 + 3^2 + \ldots n^2) = \frac{n(n+1)(2n+1)}{6},$$

for $n = 1, 2, 3, \ldots$, to produce the following table, analogous to TABLE 3.8.

TABLE III.

number of rectangles	total area of rectangles
1	$\frac{2\times 3}{6} = 1$
2	$\frac{3\times 5}{24} = \frac{5}{8} = 0.625$
5	$\frac{6\times 11}{6\times 25} = \frac{11}{25} = 0.44$
10	$\frac{11\times 21}{600} = \frac{231}{600} = 0.385$
100	$\frac{101\times 201}{60{,}000} = \frac{20{,}301}{60{,}000} = 0.33835$
1,000	$\frac{1{,}001\times 2{,}001}{6{,}000{,}000} = \frac{2{,}003{,}001}{6{,}000{,}000} = 0.3338335$
n	$\frac{(n+1)(2n+1)}{6n^2} = (\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2})$

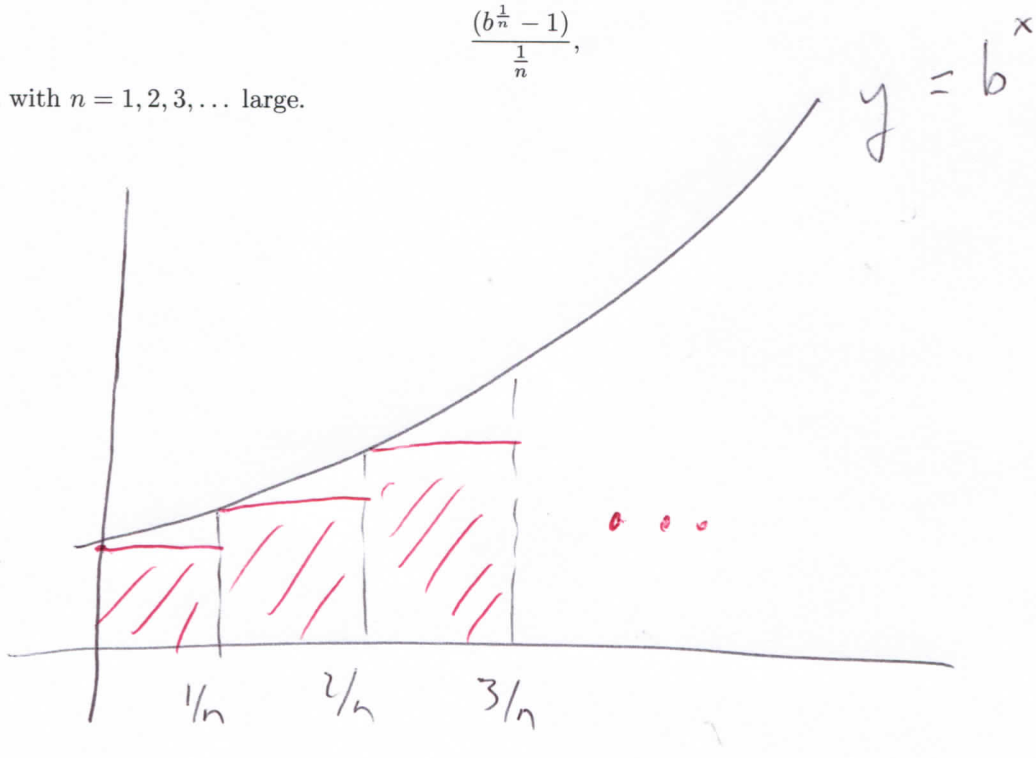
4. Approximate $\int_0^1 b^x\,dx$ with the sum of areas of rectangles

$$\frac{1}{n}\left(1 + b^{\frac{1}{n}} + b^{\frac{2}{n}} + b^{\frac{3}{n}} + \cdots + b^{\frac{n-1}{n}}\right)$$

and the derivative of b^x at $x = 0$ with the slope of the line from $(0, 1)$ to $(\frac{1}{n}, b^{\frac{1}{n}})$

$$\frac{(b^{\frac{1}{n}} - 1)}{\frac{1}{n}},$$

both with $n = 1, 2, 3, \ldots$ large.



HOMEWORK ANSWERS

1. In Table I in Homework Hint no. 1 let $\Delta x = 0$ in the lower right-hand corner:

$$\text{derivative } = 6.$$

2. In Table II in Homework Hint no. 2 let $\Delta x = 0$ in the lower right-hand corner:

$$\text{derivative } = 12.$$

3. See TABLE III in Homework Hint no. 3.

As with Table 3.8, as the number of rectangles, denoted n, gets arbitrarily large, the fractions $\frac{1}{2n}$ and $\frac{1}{6n^2}$ in the bottom of the right-hand column get arbitrarily small, so that

$$\int_0^1 x^2\,dx = \text{area of } \{(x,y)\,|\,0 \le x \le 1, 0 \le y \le x^2\} = \lim_{n\to\infty} [\text{ total area of } n \text{ rectangles }]$$

$$= \lim_{n\to\infty}\left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right) = \frac{1}{3}.$$

4. For $b \neq 1$, argue as follows.
Denote

$$I_n \equiv \frac{1}{n}\left(1 + b^{\frac{1}{n}} + b^{\frac{2}{n}} + b^{\frac{3}{n}} + \cdots + b^{\frac{n-1}{n}}\right)$$

for the approximation of $\int_0^1 b^x\,dx$ with n rectangles and the derivative of b^x at $x = 0$ with the slope of the line from $(0,1)$ to $(\frac{1}{n}, b^{\frac{1}{n}})$

$$D_n \equiv \frac{(b^{\frac{1}{n}} - 1)}{\frac{1}{n}}.$$

Then, for any $n = 1, 2, 3, \ldots$,

$$I_n D_n = \frac{1}{n}\left(1 + b^{\frac{1}{n}} + \left(b^{\frac{1}{n}}\right)^2 + \left(b^{\frac{1}{n}}\right)^3 + \cdots + \left(b^{\frac{1}{n}}\right)^{n-1}\right) D_n = \frac{1}{n}\left(\frac{(1-b)}{(1-b^{\frac{1}{n}})}\right)\left(\frac{(b^{\frac{1}{n}}-1)}{\frac{1}{n}}\right) = (b-1),$$

thus

$$(\text{derivative of } b^x \text{ at } x = 0)\left(\int_0^1 b^x\,dx\right) = (b-1).$$

A different argument is needed for $b = 1$. Then $D_n = 0$ for $n = 1, 2, 3, \ldots$.

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