TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI

Fibonacci Numbers and the Golden Ratio
MATHematics MAGnification™

Dr. Ralph deLaubenfels

TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI

Teacher-Scholar Institute

Columbus, Ohio

2018

FIBONACCI NUMBERS and THE GOLDEN RATIO MAGNIFICATION

This is one of a series of very short books on math, statistics, and physics called "Math Magnifications." The "magnification" refers to focusing on a particular topic that is pivotal in or emblematic of mathematics.

OUTLINE

This magnification will introduce golden rectangles and the golden ratio, a geometry subject of great interest to the classical Greeks, and Fibonacci numbers, arising from a population model, from the 13^{th} century. Linear algebra, from the 19^{th} century, reveals a surprising relationship between the golden ratio and Fibonacci numbers. This relationship is in the form of the fundamental calculus concept of arbitrarily good approximations (known as a *limit*); specifically, the ratios of consecutive Fibonacci numbers get arbitrarily close to the golden ratio.

The definition of the Fibonacci numbers is recursive; specifically, each term is the sum of the previous two terms. We will outline how linear algebra produces a much more practical explicit expression for each Fibonacci number.

The only prerequisites for this magnification (at least prior to the appendix) are first-year high school algebra. We also assume the reader knows the definitions of polygons, pentagons, rectangles, and squares. Reference [4] is more than sufficient.

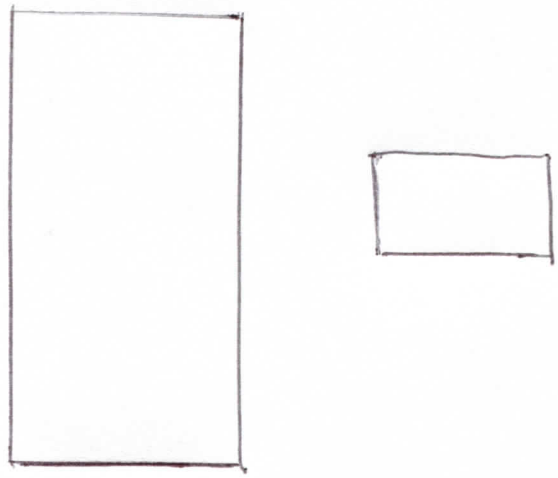
For this magnification students will need a calculator that can calculate square roots and arbitrary integral powers.

CHAPTER I: Golden rectangles, golden sections, and the golden ratio.

Definitions 1.1. The **length** of a rectangle is the measure of the longer side, while the **width** is the measure of the shorter side. If the rectangle is a square, this means the length and width are the same, mainly the measure of any side.

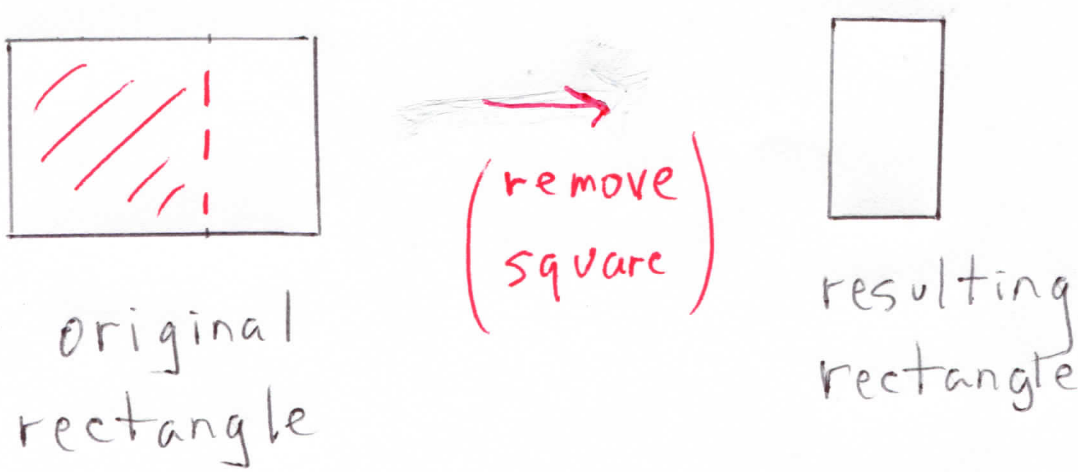
Two rectangles are **similar** if they have the same ratio of length to width.

Examples 1.2. A 3×6 rectangle is similar to a 1×2 rectangle, since $\frac{6}{3} = \frac{2}{1}$. We have drawn below the two rectangles, with millimeters for units.

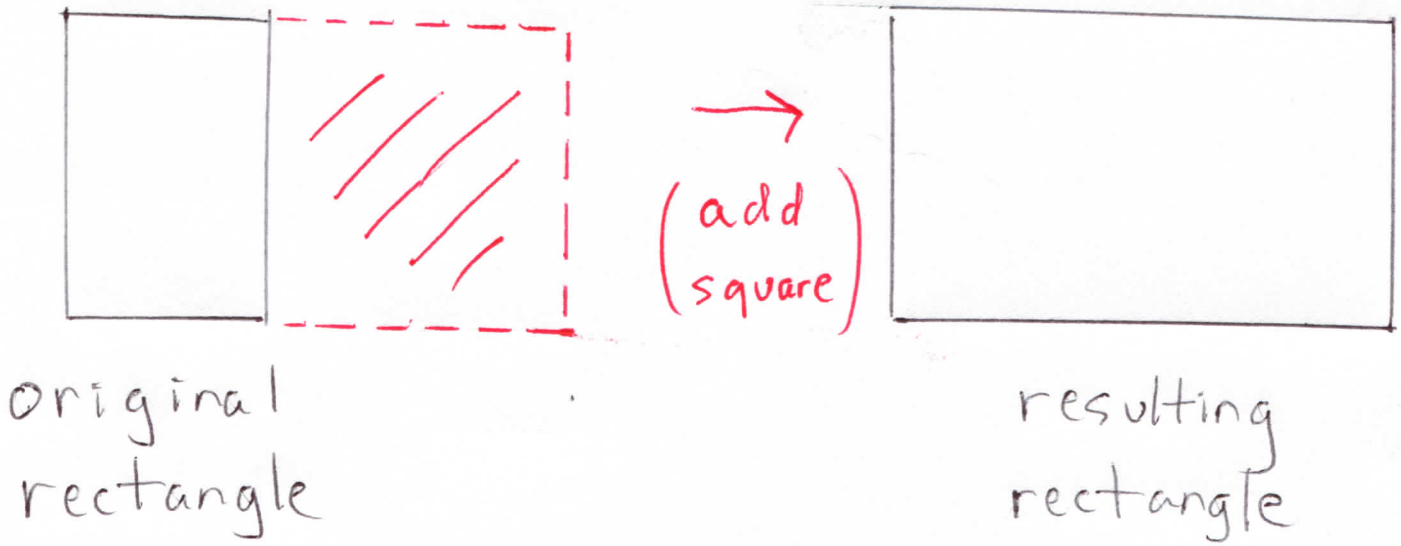


Intuitively, similar rectangles have the same shape; one is a magnification of the other. By zooming in to or out from a fixed rectangle, the naked eye sees similar rectangles.

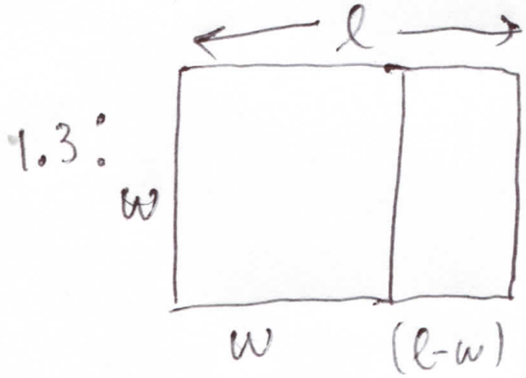
Definition 1.3. A **golden rectangle** is a rectangle with the following property. If the largest possible square is removed from one side of the original rectangle, the resulting smaller rectangle is similar to the original rectangle.



Equivalent Definition of Golden Rectangle 1.4. A **golden rectangle** is a rectangle with the following property. If a square whose sides are the same measure as the length of the original rectangle is pasted onto the original rectangle, the resulting larger rectangle is similar to the original rectangle.

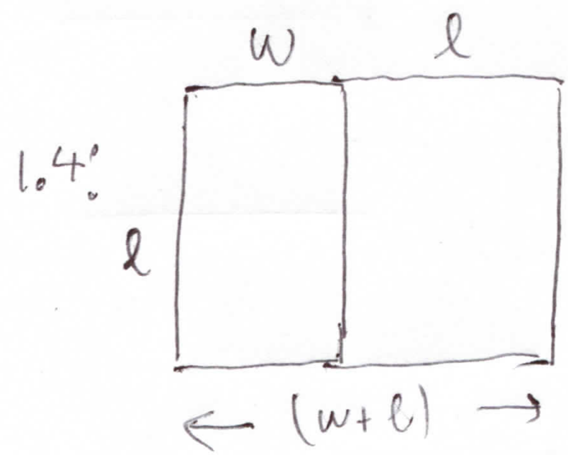


Here is a quick argument for the equivalence of 1.3 and 1.4.



$$\frac{l}{w} = \frac{w}{(l-w)} \leftrightarrow$$

$$l^2 - wl - w^2 = 0$$

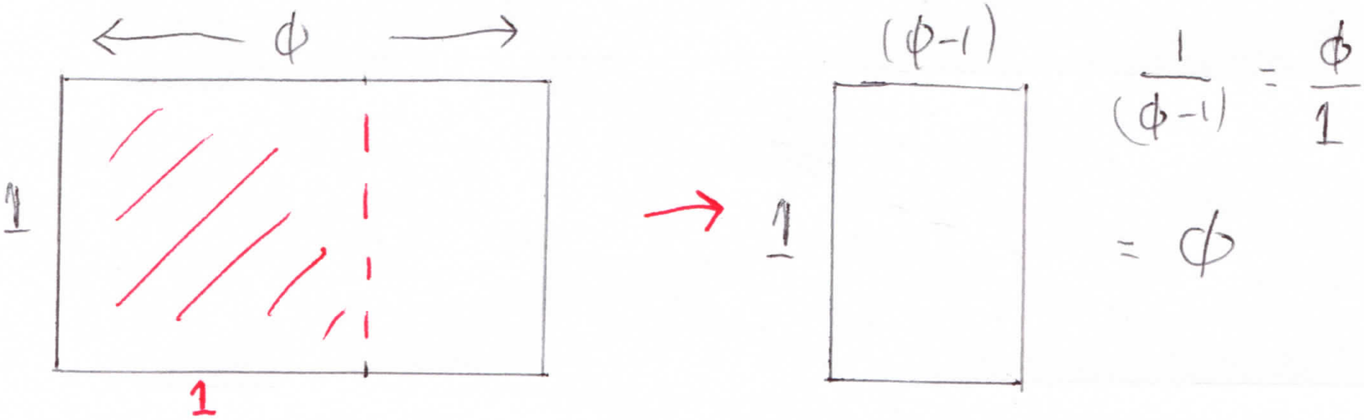


$$\frac{l}{w} = \frac{(w+l)}{l} \leftrightarrow$$

$$l^2 - wl - w^2 = 0$$

Definition 1.5. The **golden ratio**, denoted ϕ (spelled "phi" and pronounced "fee") is the length divided by the width in a golden rectangle.

Equivalently, if a golden rectangle has width one, then its length is ϕ.



We are assuming, without proof (to avoid mentioning the quadratic formula), that length divided by width is the same in all golden rectangles.

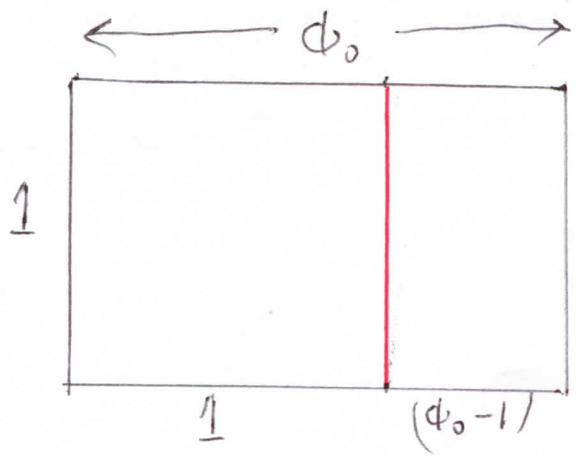
If the reader *is* familiar with the quadratic formula, said reader should apply it to the picture below Definition 1.5 to get the following number for ϕ.

Theorem 1.6. $\phi = \frac{1}{2}(1 + \sqrt{5})$.

Proof: In the following drawing, we must compare ratios of length to width, as in Definition 1.3; also see Definition 1.5. Denote by

$$\phi_0 \equiv \frac{1}{2}(1 + \sqrt{5})$$

our candidate for ϕ.



Then

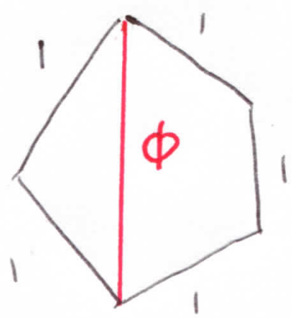
$$\frac{1}{(\phi_0 - 1)} = \frac{1}{\frac{1}{2}(\sqrt{5} - 1)} = \frac{2(1 + \sqrt{5})}{(\sqrt{5} - 1)(1 + \sqrt{5})} = \frac{2(1 + \sqrt{5})}{(5 - 1)} = \frac{1}{2}(1 + \sqrt{5}) = \phi_0 = \frac{\phi_0}{1}.$$

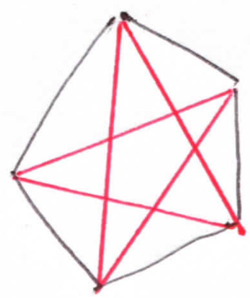
Comparing to Definition 1.5, this shows that $\phi = \phi_0$, as desired. □

Remarks 1.7. The golden ratio ϕ appears in surprisingly many places. We will mention here an appearance of particular interest to Pythagoreans, in a regular (equal sides and equal angles) pentagon and the pentagram enclosed (see drawing below; the pentagram is drawn in red in the second drawing).

For convenience, let P be a regular pentagon each side of which measures one. Then the distance between nonconsecutive vertices of P is ϕ; see drawing below.

See the Appendix for a proof (Theorem APP.2) of this result about ϕ, and more assertions about ϕ and the pentagram drawn in red below.





Definitions 1.8. 1.3, 1.4, and 1.5 are special cases of *golden sections.* A **golden section** or **golden mean** is two numbers such that the ratio of the larger number to the smaller number equals the ratio of the sum of the two numbers to the larger number. See [1, page 64].

In symbols, this would be numbers x, y such that

$$\frac{y}{x} = \frac{x+y}{y} \quad (1.9).$$

When x is width and y is length, this is Definition 1.4, of a golden rectangle; see the drawings of 1.4, with $x = w$ and $y = \ell$.

Here is a golden section that is a one-dimensional analogue of a golden rectangle. Given a line segment from a point P to a point Q, find a point R on the line segment so that, writing, for arbitrary points A, B, AB for the distance from A to B,

$$\frac{PR}{RQ} = \frac{PQ}{PR}.$$

Letting $y \equiv PR$ and $x \equiv RQ$, this becomes (1.9), so that $\frac{PR}{RQ} = \phi$.

CHAPTER II: Fibonacci numbers (recursive definition).

Definition 2.1. The **Fibonacci numbers** $F_0, F_1, F_2, F_3, \ldots$ are defined recursively as follows.

$$F_0 = 0, F_1 = 1, \ldots F_{n+2} = F_{n+1} + F_n, \text{ for } n = 0, 1, 2, 3, \ldots \quad (2.1).$$

Thus

$$F_2 = F_1 + F_0 = 1 + 0 = 1, \; F_3 = F_2 + F_1 = 1 + 1 = 2, \; F_4 = F_3 + F_2 = 2 + 1 = 3, \; \ldots.$$

See Examples 2.3(a) for more.

The *recursive* part of Definition 2.1 is

$$F_{n+2} = F_{n+1} + F_n \quad (2.2),$$

defining each term as a fixed function of prior terms; in words, each term is the sum of the previous two terms.

These numbers arise from a population model. Imagine an organism that lives forever, takes a day after birth to mature, then has one offspring every day. Then it can be shown (see [2, 7.24 and 7.25, pages 629–634]) that, for $n = 0, 1, 2, 3, \ldots$,

$$F_n = \text{ number of organisms in our kitchen sink } n \text{ days from now,}$$

if we start with nothing in our sink today and have one newborn organism placed in our sink tomorrow.

The recursive definition (2.2) may now be stated as

(population now) = (population yesterday) + (population day before yesterday);

the first term in the sum is yesterday's population surviving to today, and the second term in the sum is each of the now-mature organisms giving birth.

Examples 2.3. (a) Find F_{10}.

(b) Suppose $F_{112} = 123,456$ and $F_{113} = 199,750$ (they don't; but hypothetically *suppose* they do). Find F_{114}.

(c) Again hypothetically suppose $F_{230} = 1,000,398$ and $F_{232} = 2,620,821$ (they don't). Find F_{231} and F_{234}.

Solutions. (a) With our recursive definition (2.2), we can calculate F_n only for n advancing consecutively: we have already calculated F_3 and F_4 above, so proceed patiently further:

$$F_5 = F_4 + F_3 = 3 + 2 = 5; \quad F_6 = F_5 + F_4 = 5 + 3 = 8; \; F_7 = F_6 + F_5 = 8 + 5 = 13;$$

$$F_8 = F_7 + F_6 = 13 + 8 = 21; \quad F_9 = F_8 + F_7 = 21 + 13 = 34;$$

and, *finally,*

$$F_{10} = F_9 + F_8 = 34 + 21 = 55.$$

(b) The recursive definition (2.2) tells us to add:

$$F_{114} = F_{113} + F_{112} = 199,750 + 123,456 = 323,206.$$

(c) Again apply (2.2):

$$2,620,821 = F_{232} = F_{231} + F_{230} = F_{231} + 1,000,398,$$

so that a spot of algebra tells us that

$$F_{231} = 2,620,821 - 1,000,398 = 1,620,423.$$

For F_{234}, using (2.2), we need

$$F_{233} = F_{232} + F_{231} = 2,620,821 + 1,620,423 = 4,241,244;$$

now

$$F_{234} = F_{233} + F_{232} = 4,241,244 + 2,620,821 = 6,862,065.$$

Remarks 2.4. In Examples 2.3(a) we saw the inefficiency of the recursive definition 2.1. Using this definition, if we wanted $F_{1,000,000}$, we would have to use (2.2) to first get F_2, then F_3, then $F_4, \ldots$; we are talking about 999,999 acts of arithmetic. Even if we only cared about $F_{1,000,000}$, we would still have no choice but to also get F_2 through $F_{999,999}$.

To motivate the sort of expression for F_n that we'd like to have, let's consider a simpler population model: a population in our kitchen sink doubles every day. Let

$$P_n = \text{ number of organisms in our kitchen sink } n \text{ days from now,}$$

if we start with one organism placed in our sink today.

The recursive definition of $P_0, P_1, P_2, P_3, \ldots$ is

$$P_{n+1} = 2P_n \ \ (n = 0, 1, 2, 3, \ldots) \ \ (2.5),$$

$P_0 = 1$.

Notice now that

$$P_0 = 1 = 2^0, \ P_1 = 2 = 2^1, \ P_2 = 4 = 2^2, \ P_3 = 8 = 2^3, \ldots;$$

we might believe that, for *any* $n = 0, 1, 2, 3, 4, \ldots,$

$$P_n = 2^n \ \ (2.6).$$

The formula (2.6) is an *explicit* representation, or *closed form* for P_n. If, as with the second sentence of Remarks 2.4, we wanted $P_{1,000,000}$, with (2.6) we can immediately write down

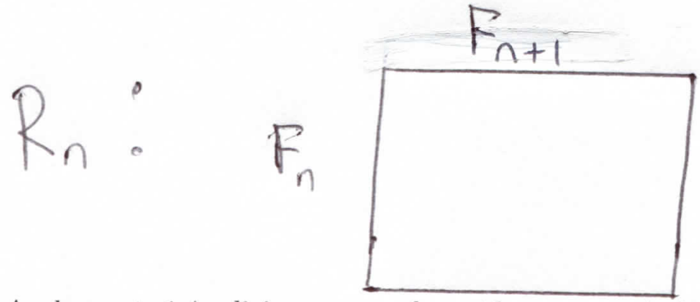
$$P_{1,000,000} = 2^{1,000,000},$$

without worrying about the unwanted, intermediate populations $P_1, P_2, P_3, \ldots P_{999,999}$.

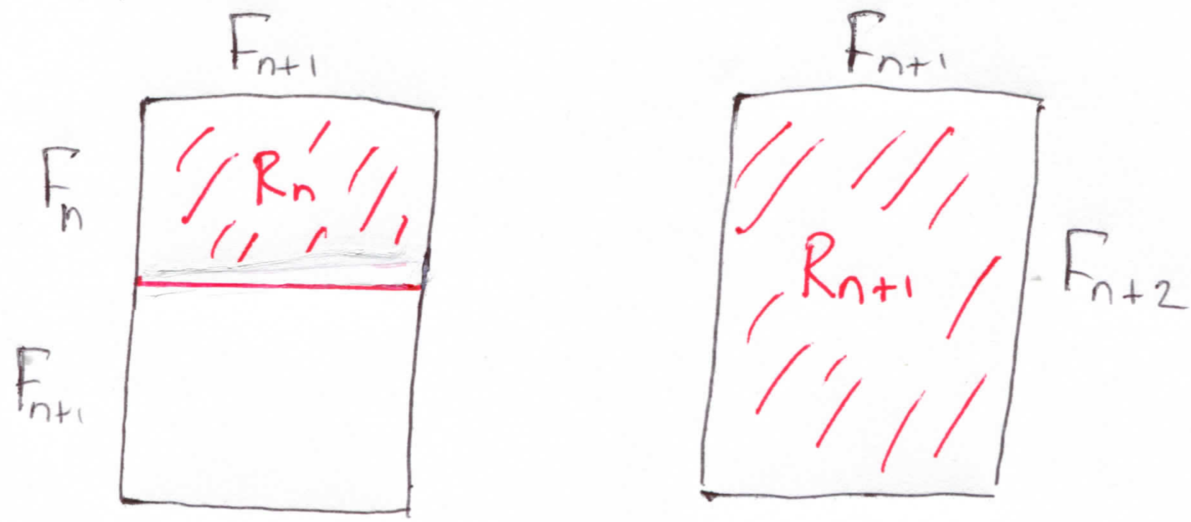
We'd like a closed form for the Fibonacci numbers, analogous to (2.6). This will be put off until Chapter IV. Because one of the consequences of the closed form we'll obtain in Chapter IV will be quite surprising connections between Fibonacci numbers and the golden ratio (Definition 1.5), we will first (in Chapter III) present some hints about these connections.

CHAPTER III:
Some clues about the relationship between Fibonacci numbers and the golden ratio.

Construction 3.1. For $n = 0, 1, 2, \ldots$, form a rectangle, denoted R_n, with sides F_n and F_{n+1}, where $F_0, F_1, F_2, F_3 \ldots$ are the Fibonacci numbers of Definition 2.1.



Analogous to 1.4, adjoin a square whose sides measure the length of R_n. We now have a rectangle of width F_{n+1} and length $(F_{n+1} + F_n) = F_{n+2}$, by the recursive definition of the Fibonacci numbers (2.2).



We now see that, for $n = 0, 1, 2, 3, \ldots$, R_{n+1} is formed from R_n by pasting on the square of width and length F_{n+1}, analogous to 1.4. On the graph paper on the page after next, we have drawn in R_0, R_1, R_2, R_3, and R_4, and have requested that you, the reader, draw in R_5 and R_6; the subsequent page has R_5 and R_6 drawn in.

If, for some n, R_n were a golden rectangle, then R_{n+1} would be similar to R_n, R_{n+2} would be similar to R_{n+1}, hence similar to R_n; in fact, for any $k \geq n$, R_k would be similar to R_n. By our definition of R_k and F_k, said similarity would be equivalent to the ratios of each Fibonacci number to its immediate predecessor becoming equal:

$$\frac{F_{k+1}}{F_k} = \frac{F_{n+1}}{F_n}, \text{ for } k = (n+1), (n+2), (n+3), \ldots \quad (3.2).$$

We don't quite get R_n to be a golden rectangle, for any n. But staring at the drawings of R_n on the next page and the page after that, you might notice, as n gets larger, that R_{n+1} is looking *almost* similar to R_n; that is, R_n is "looking" "almost" "like" a golden rectangle.

We can avoid the quotation marks, and the ambiguities they conceal, by invoking (3.2). On the page after the next two graph paper pages, we have put a worksheet of ratios of each Fibonacci number to its (immediate) predecessor. We have filled in the first few rows, and ask the reader to fill in the remaining ones; the page after the page just described has all the rows filled in. Keep in mind, as you peruse the column of ratios, that the golden ratio ϕ of Definition 1.5 equals $\frac{1}{2}(1+\sqrt{5})$, whose decimal expansion is $1.618033989\ldots$.

The last-mentioned two pages have an additional column, that approximates each Fibonacci number with ϕ times its (immediate) predecessor. Here is the motivation.

If, in the immortal words of a recent paragraph, R_n is "looking" "almost" "like" a golden rectangle, then (3.2) suggests that $\frac{F_{n+1}}{F_n}$, the ratio of length to width in R_n, should be getting close to ϕ, the ratio of length to width in a golden rectangle. In symbols, we expect, at least for large n,

$$\frac{F_{n+1}}{F_n} \sim \phi \quad (3.3),$$

where $\sim$ means "approximately equal to." Multiplying out the denominator in (3.3) gives

$$F_{n+1} \sim \phi \times F_n \quad (3.4)$$

an approximation we hope to motivate with the last column of the worksheet appearing after the two pages of graph paper.

Example 3.5. If $F_{37} = 57,000$ (it doesn't, but hypothetically pretend it does), use (3.4) to

(a) approximate F_{38};

(b) approximate F_{36}; and

(c) approximate F_{39}.

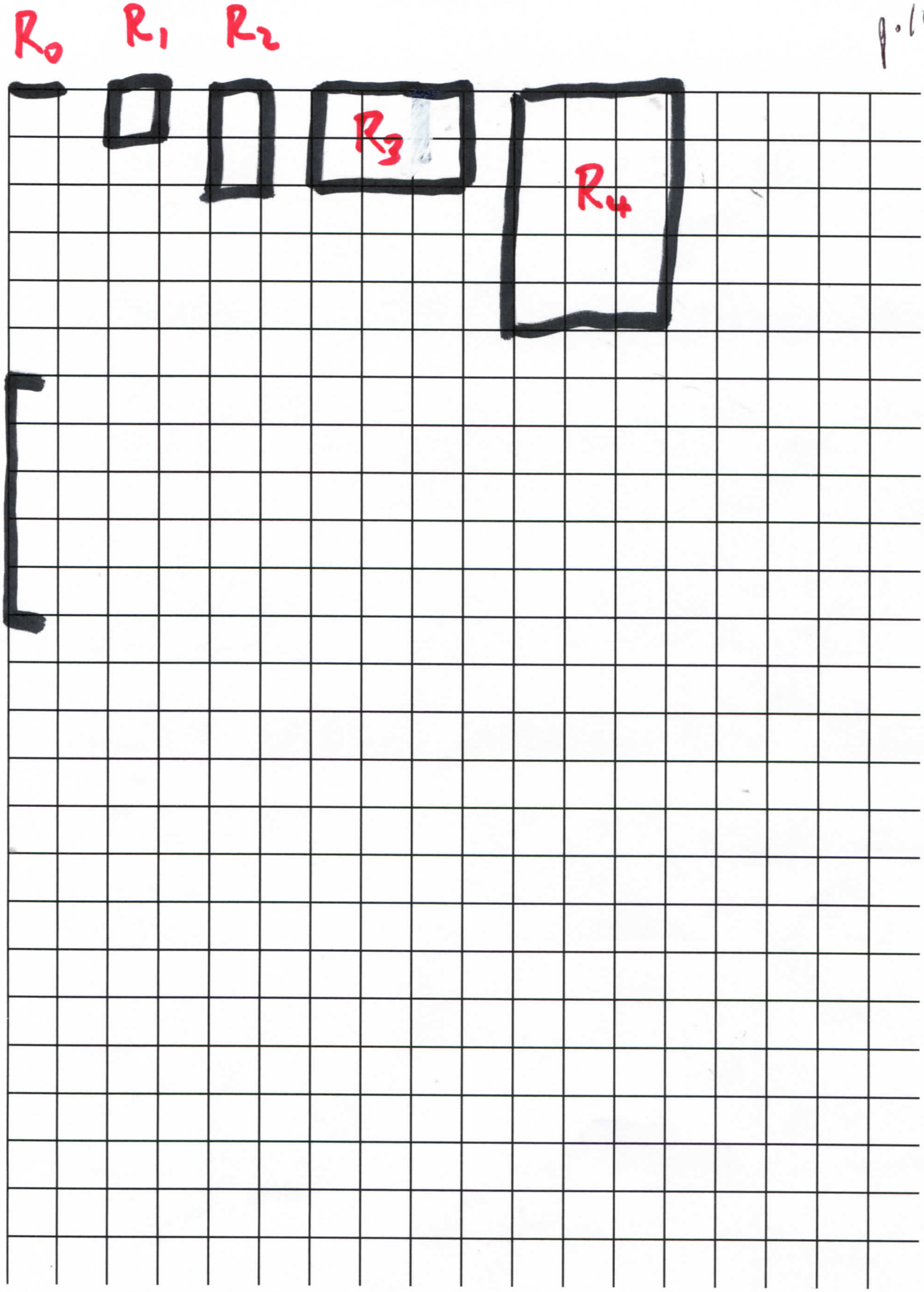
Round all answers to the nearest integer.

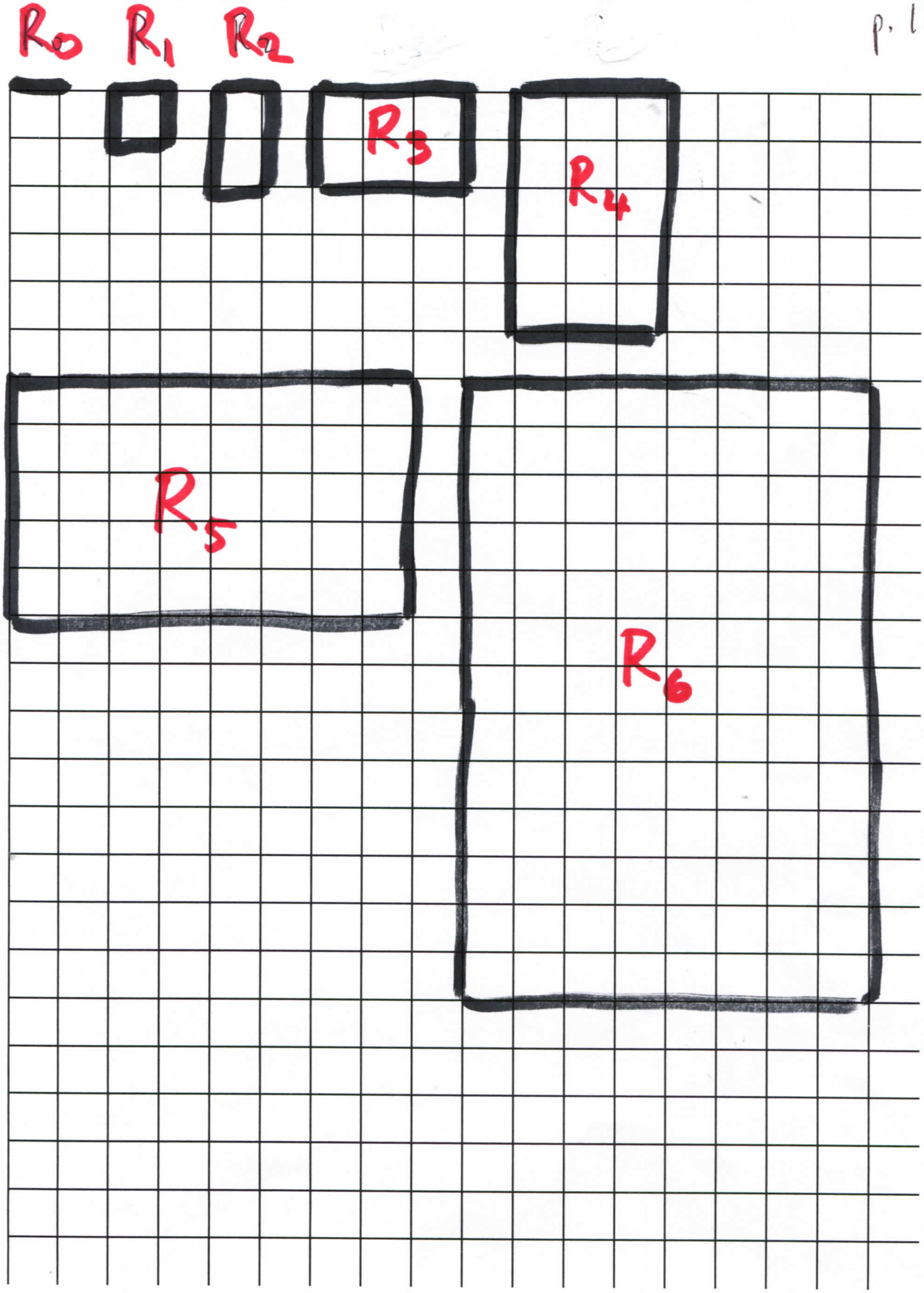
Solutions. (a) $F_{38} \sim \phi \times F_{37} = \frac{1}{2}(1+\sqrt{5}) \times 57,000 \sim 92,228$.

(b) $\phi \times F_{36} \sim F_{37}$, thus $F_{36} \sim \frac{F_{37}}{\phi} = \frac{57,000}{\frac{1}{2}(1+\sqrt{5})} \sim 35,228$.

(c) $F_{39} \sim \phi \times F_{38} \sim \frac{1}{2}(1+\sqrt{5}) \times 92,228 \sim 149,228$.

Remark 3.6. (3.4) is like (2.2) in that it defines (approximately) F_n *recursively*, that is, in terms of F_k for $k < n$. The advantage of (3.4) over (2.2) is that (2.2) requires the prior two values of F_k to get F_n, while (3.4) requires only the prior value of F_k to get (approximately) F_n. In terms of the population model the Fibonacci numbers represent (see Definition 2.1), (2.2) requires the populations both yesterday and the day before yesterday, while (3.4) requires only the population yesterday. The disadvantage of (3.4) over (2.2) is that (3.4) is only an approximation.


R0
R1
R2
R3
R4


R0
R1
R2
R3
R4
R5
R6

In the following, round to five decimal places in the right-most column and to seven decimal places in the column immediately to the left of the right-most column.

Recall that $F_0 = 0$, and $F_1 = 1$, and

$$\phi = \frac{1}{2}(1+\sqrt{5}) \sim 1.6180340$$

(don't use a decimal approximation of ϕ when multiplying by ϕ; use $\phi = \frac{1}{2}(1+\sqrt{5})$).

The reader should fill in all the empty entries in the table below; said reader may check the next page for correctness.

TABLE 3.7, to be filled in

N	Fibonacci number $F_N =$ sum of previous two Fibonacci numbers	ratio of Fibonacci number to immediate predecessor	golden ratio times immediate predecessor
2	$F_2 = 1 = 1+0$	$\frac{1}{1} = 1$	$\phi \times 1 \sim 1.61803$
3	$F_3 = 2 = 1+1$	$\frac{2}{1} = 2$	$\phi \times 1 \sim 1.61803$
4	$F_4 = 3 = 2+1$	$\frac{3}{2} = 1.5$	$\phi \times 2 \sim 3.23607$
5	$F_5 = 5 = 3+2$	$\frac{5}{3} \sim 1.6666667$	$\phi \times 3 \sim 4.85410$
6	$F_6 = 8 = 5+3$	$\frac{8}{5} = 1.6$	$\phi \times 5 \sim 8.09017$
7	$F_7 = 13 = 8+5$	$\frac{13}{8} = 1.625$	$\phi \times 8 \sim 12.94427$
8	$F_8 = 21 = 13+8$	$\frac{21}{13} \sim 1.6153846$	$\phi \times 13 \sim 21.03444$
9	$F_9 = 34 = 21+13$		
10	$F_{10} = 55 = 34+21$		
11	$F_{11} =$		
12	$F_{12} =$		
13	$F_{13} =$		
14	$F_{14} =$		
15	$F_{15} =$		
16	$F_{16} =$		
17	$F_{17} =$		
18	$F_{18} =$		

In the following, we will round to five decimal places in the right-most column and to seven decimal places in the column immediately to the left of the right-most column.

Recall that $F_0 = 0$, and $F_1 = 1$, and

$$\phi = \frac{1}{2}(1+\sqrt{5}) \sim 1.6180340$$

(don't use a decimal approximation of ϕ when multiplying by ϕ; use $\phi = \frac{1}{2}(1+\sqrt{5})$).

TABLE 3.7, completed

N	Fibonacci number $F_N =$ sum of previous two Fibonacci numbers	ratio of Fibonacci number to immediate predecessor	golden ratio times immediate predecessor
2	$F_2 = 1 = 1 + 0$	$\frac{1}{1} = 1$	$\phi \times 1 \sim 1.61803$
3	$F_3 = 2 = 1 + 1$	$\frac{2}{1} = 2$	$\phi \times 1 \sim 1.61803$
4	$F_4 = 3 = 2 + 1$	$\frac{3}{2} = 1.5$	$\phi \times 2 \sim 3.23607$
5	$F_5 = 5 = 3 + 2$	$\frac{5}{3} \sim 1.6666667$	$\phi \times 3 \sim 4.85410$
6	$F_6 = 8 = 5 + 3$	$\frac{8}{5} = 1.6$	$\phi \times 5 \sim 8.09017$
7	$F_7 = 13 = 8 + 5$	$\frac{13}{8} = 1.625$	$\phi \times 8 \sim 12.94427$
8	$F_8 = 21 = 13 + 8$	$\frac{21}{13} \sim 1.6153846$	$\phi \times 13 \sim 21.03444$
9	$F_9 = 34 = 21 + 13$	$\frac{34}{21} \sim 1.6190476$	$\phi \times 21 \sim 33.97871$
10	$F_{10} = 55 = 34 + 21$	$\frac{55}{34} \sim 1.6176471$	$\phi \times 34 \sim 55.01316$
11	$F_{11} = 89 = 55 + 34$	$\frac{89}{55} \sim 1.6181818$	$\phi \times 55 \sim 88.99187$
12	$F_{12} = 144 = 89 + 55$	$\frac{144}{89} \sim 1.6179775$	$\phi \times 89 \sim 144.00503$
13	$F_{13} = 233 = 144 + 89$	$\frac{233}{144} \sim 1.6180556$	$\phi \times 144 \sim 232.99689$
14	$F_{14} = 377 = 233 + 144$	$\frac{377}{233} \sim 1.6180258$	$\phi \times 233 \sim 377.00192$
15	$F_{15} = 610 = 377 + 233$	$\frac{610}{377} \sim 1.6180371$	$\phi \times 377 \sim 609.99881$
16	$F_{16} = 987 = 610 + 377$	$\frac{987}{610} \sim 1.6180328$	$\phi \times 610 \sim 987.00073$
17	$F_{17} = 1,597 = 987 + 610$	$\frac{1597}{987} \sim 1.6180344$	$\phi \times 987 \sim 1596.99955$
18	$F_{18} = 2,584 = 1,597 + 987$	$\frac{2584}{1597} \sim 1.6180338$	$\phi \times 1597 \sim 2584.00028$

CHAPTER IV: Linear algebra produces explicit closed form of Fibonacci numbers.

We will only outline the linear algebra (vectors and matrices) required to deal with the Fibonacci numbers; we wish to make no assumptions or requirements about linear algebra knowledge for this magnification. See [2] for an exposition of linear algebra.

Definitions and Terminology 4.1. Let $F_0, F_1, F_2, \ldots$ be the Fibonacci numbers of Definition 2.1.

The linear algebra approach is to put together consecutive Fibonacci numbers

$$\vec{x}_k \equiv \begin{bmatrix} F_k \\ F_{k+1} \end{bmatrix} \quad k = 0, 1, 2, \ldots \quad (4.2)$$

The object $\vec{x}_k$ is called a *vector*. Note that $\vec{x}_k$ contains both the width and length of the rectangle R_k formed from Fibonacci numbers in Chapter III.

Let's look more closely at the sequence of vectors $\{\vec{x}_k\}_{k=0}^{\infty}$ defined by (4.2):

$$\vec{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \vec{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ \ldots \vec{x}_{n+1} = \begin{bmatrix} F_{n+1} \\ F_{n+2} \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_n + F_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} \equiv \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \vec{x}_n,$$

$n = 0, 1, 2, 3, \ldots$; in the expression for $\vec{x}_{n+1}$, we have pulled out the coefficients of F_n and F_{n+1} to form what is called a *matrix*. If we denote this matrix by

$$A \equiv \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},$$

(4.2) has the form

$$\vec{x}_{n+1} = A\vec{x}_n \quad (4.2 \text{ rewritten}),$$

a recursive definition of the sequence of vectors $\{\vec{x}_n\}_{n=0}^{\infty}$. This is only a restating of (2.2), but it *looks* very much like the recursion $P_{n+1} = 2P_n$ in Remarks 2.4. The same reasoning as in Remarks 2.4 suggests that

$$\vec{x}_n = A^n \vec{x}_0 = A^n \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (4.3),$$

where, as with numbers, A^n denotes A multiplied by itself n times.

$$A^n = \underbrace{(AAA \cdots A)}_{n \text{ terms}}$$

Needed Factoids 4.4. We need to use, without proof, the following from [2, 8.17, pages 676–682].

Define

$$\vec{y_1} \equiv \begin{bmatrix} (1-\sqrt{5}) \\ -2 \end{bmatrix}, \quad \vec{y_2} \equiv \begin{bmatrix} (1+\sqrt{5}) \\ -2 \end{bmatrix}, \quad \lambda_1 \equiv \frac{1}{2}(1+\sqrt{5}), \quad \lambda_2 \equiv \frac{1}{2}(1-\sqrt{5}), \quad \alpha_1 \equiv \frac{-1}{5-\sqrt{5}}, \quad \alpha_2 \equiv \frac{-1}{5+\sqrt{5}}.$$

Then

$$A\vec{y_1} = \lambda_1 \vec{y_1}, \quad A\vec{y_2} = \lambda_2 \vec{y_2}, \quad (4.5)$$

and

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \alpha_1 \vec{y_1} + \alpha_2 \vec{y_2} \quad (4.6).$$

In the language of linear algebra, $\vec{y_1}$ and $\vec{y_2}$ are *eigenvectors*, with corresponding *eigenvalues* λ_1 and λ_2. Still speaking this language, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a *linear combination* of $\vec{y_1}$ and $\vec{y_2}$.

(4.2), (4.3), (4.6), and (4.5), in that order, now imply that, for $N = 0, 1, 2, 3, \ldots$,

$$\begin{bmatrix} F_N \\ F_{N+1} \end{bmatrix} = A^N \begin{bmatrix} 0 \\ 1 \end{bmatrix} = A^N (\alpha_1 \vec{y_1} + \alpha_2 \vec{y_2}) = \alpha_1 A^N \vec{y_1} + \alpha_2 A^N \vec{y_2} = \alpha_1 \lambda_1^N \vec{y_1} + \alpha_2 \lambda_2^N \vec{y_2},$$

so that

$$F_N = \alpha_1 \lambda_1^N (1-\sqrt{5}) + \alpha_2 \lambda_2^N (1+\sqrt{5}) = \cdots = \frac{1}{\sqrt{5}} \left[(\frac{1}{2}(1+\sqrt{5}))^N - (\frac{1}{2}(1-\sqrt{5}))^N \right],$$

where "…" here means unpleasant calculation; see [2, page 681]

Since $\frac{1}{\sqrt{5}} |\frac{1}{2}(1-\sqrt{5})|^N < \frac{1}{2}$ and we know F_N is an integer, we can combine what we just calculated for F_N to get the following two closed forms for F_N.

Theorem 4.7. For $N = 0, 1, 2, 3, \ldots$,

(a) $F_N = \frac{1}{\sqrt{5}} \left[(\frac{1}{2}(1+\sqrt{5}))^N - (\frac{1}{2}(1-\sqrt{5}))^N \right]$,

and

(b) F_N is the integer closest to $\frac{1}{\sqrt{5}} \left[(\frac{1}{2}(1+\sqrt{5})) \right]^N$.

Example 4.8. (a) $F_{12} = \frac{1}{\sqrt{5}} \left[(\frac{1}{2}(1+\sqrt{5}))^{12} - (\frac{1}{2}(1-\sqrt{5}))^{12} \right] = 144.$

It is surprising that these combinations of rational numbers and $\sqrt{5}$ come out to an integer, as Definition 2.1 guarantees.

If we wanted to save calculations, we could've calculated

$$\frac{1}{\sqrt{5}} \left[(\frac{1}{2}(1+\sqrt{5})) \right]^{12} = 144.0013889\ldots,$$

and then observed that 144 is the closest integer.

Recall that, using the recursive definition of the Fibonacci number F_{12}, as in (2.2), would've required

$F_0 = 0, F_1 = 1 \to F_2 = 1+0 = 1, F_3 = 1+1 = 2, F_4 = 2+1 = 3, F_5 = 3+2 = 5, F_6 = 5+3 = 8, F_7 = 8+5 = 13,$
$F_8 = 13+8 = 21, F_9 = 21+13 = 34, F_{10} = 34+21 = 55, F_{11} = 55+34 = 89, F_{12} = 89+55 = 144.$

(b)

$$\frac{1}{\sqrt{5}} \left[(\frac{1}{2}(1+\sqrt{5})) \right]^{17} = 1596.999875\ldots,$$

thus $F_{17} = 1597$.

Staring at Theorem 4.7(a) tells us that

$$F_N < \frac{1}{\sqrt{5}}\left[(\frac{1}{2}(1+\sqrt{5}))\right]^N$$

when N is even and

$$F_N > \frac{1}{\sqrt{5}}\left[(\frac{1}{2}(1+\sqrt{5}))\right]^N$$

when N is odd, because $(1-\sqrt{5}) < 0$, hence $(1-\sqrt{5})^N$ is positive when N is even and is negative when N is odd.

Remark 4.9. The dominant term, $\frac{1}{2}(1+\sqrt{5})$, in the closed forms for F_N, is the *golden ratio* (Definition 1.5).

Remark 4.10. The closed form in Theorem 4.7(a) was first obtained by deMoivre, in 1730, using *power series*, meaning functions defined by infinite sums of powers of x,

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots$$

Specifically, use the Fibonacci numbers for the coefficients of x^k:

$$f(x) = F_0 + F_1 x + F_2 x^2 + \dots,$$

then the recursive definition (2.2) implies (after some work) that

$$(x + x^2) f(x) = f(x) - x,$$

so that

$$f(x) = \frac{x}{(1 - x - x^2)}.$$

See [5, Section 9.6] and [1, pages 123–124] for more details and the remainder of the argument.

CHAPTER V: Fibonacci numbers and the golden ratio.

Theorem 4.7 will allow us to prove (3.3) (see Table 3.7 for motivation for (3.3)). Let's return to denoting the golden ratio (Definition 1.5) by ϕ; in 4.4 we were shocked to discover that, of the two eigenvalues we denoted λ_1 and λ_2, one of them (λ_1), turned out to be ϕ.

The closed form for Fibonacci numbers in Theorem 4.7(a) states explicitly the approximation

$$F_n \sim \frac{\phi^n}{\sqrt{5}} \quad (5.1),$$

since

$$|F_n - \frac{\phi^n}{\sqrt{5}}| = \frac{1}{\sqrt{5}}|\frac{1}{2}(1-\sqrt{5})|^n,$$

which, since $|\frac{1}{2}(1-\sqrt{5})| < 1$, gets arbitrarily small as n gets large.

We could use (5.1):

$$\frac{F_{n+1}}{F_n} \sim \frac{(\frac{\phi^{n+1}}{\sqrt{5}})}{(\frac{\phi^n}{\sqrt{5}})} = \phi,$$

or we could be more precise: denoting, as in Chapter IV, $\lambda_2 \equiv \frac{1}{2}(1-\sqrt{5})$, Theorem 4.7(a) tells us

$$\frac{F_{n+1}}{F_n} = \frac{\frac{1}{\sqrt{5}}(\phi^{n+1} - \lambda_2^{n+1})}{\frac{1}{\sqrt{5}}(\phi^n - \lambda_2^n)} = \frac{(\phi^{n+1} - \lambda_2^{n+1})}{(\phi^n - \lambda_2^n)} = \frac{(\phi - (\frac{\lambda_2}{\phi})^n \lambda_2)}{(1 - (\frac{\lambda_2}{\phi})^n)},$$

which gets arbitrarily close to ϕ as n gets large, since $|\frac{\lambda_2}{\phi}| < 1$.

In the language of calculus, (3.3) is saying that $\frac{F_{n+1}}{F_n}$ *converges* to ϕ, or

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi \quad (5.2),$$

short for "the limit, as n goes to ∞, of $\frac{F_{n+1}}{F_n}$, is ϕ".

In more conversational English, both (3.3) and (5.1) are describing *long term* or *asymptotic* behavior of $\{\frac{F_{n+1}}{F_n}\}_{n=0}^{\infty}$.

CHAPTER VI: Generalized Fibonacci numbers and the golden ratio.

Definition 6.1. A sequence of nonnegative numbers $\{G_k\}_{k=0}^{\infty}$ that satisfies the same recursion relation as (2.2):

$$G_{n+1} = G_{n+1} + G_n \quad n = 0, 1, 2, 3, \ldots$$

is sometimes called a **generalized Fibonacci sequence**; the numbers $G_0, G_1, G_2, \ldots$ are then **generalized Fibonacci numbers**.

The generalization over Fibonacci numbers is that G_0 and G_1 are not specified.

Our goal in this chapter is to generalize some of the results in Chapters IV and V to generalized Fibonacci numbers. Most of the arguments are the same as those in Chapters IV and V, thus we will only sketch them.

Definitions 6.2. As in (4.2), define, for $k = 0, 1, 2, 3, \ldots$,

$$\vec{w}_k \equiv \begin{bmatrix} G_k \\ G_{k+1} \end{bmatrix}.$$

Let $A, \vec{y}_1, \vec{y}_2, \lambda_1, \lambda_2$ be as in 4.1 and 4.4. We still have (4.5). In place of (4.3), we have

$$\vec{w}_n = A^n \vec{w}_0 = A^n \begin{bmatrix} G_0 \\ G_1 \end{bmatrix} \quad (6.3).$$

It is a fact, that we will not prove (see [2, Definition 6.32, pages 439–440]), that there are numbers β_1 and β_2 such that

$$\vec{w}_0 = \beta_1 \vec{y}_1 + \beta_2 \vec{y}_2 \quad (6.4).$$

Arguing as in Chapter IV prior to the statement of Theorem 4.7, we have the following analogue of Theorem 4.7, with β_1, β_2 replacing α_1, α_2 of (4.6). Recall that λ_1 is the golden ratio ϕ of Definition 1.5.

Theorem 6.5. For $N = 0, 1, 2, 3, \ldots$,

(a) $G_N = \beta_1 \lambda_1^N (1 - \sqrt{5}) + \beta_2 \lambda_2^N (1 + \sqrt{5})$,

and

(b) If G_0 and G_1 are integers, then G_N is the integer closest to $\beta_1 \lambda_1^N (1 - \sqrt{5})$.

We may now generalize (3.3), arguing as in Chapter V. For $n = 0, 1, 2, 3, \ldots$,

$$\frac{G_{n+1}}{G_n} = \frac{\beta_1 \lambda_1^{n+1}(1 - \sqrt{5}) + \beta_2 \lambda_2^{n+1}(1 + \sqrt{5})}{\beta_1 \lambda_1^n (1 - \sqrt{5}) + \beta_2 \lambda_2^n (1 + \sqrt{5})} = \frac{\beta_1 \lambda_1 (1 - \sqrt{5}) + \beta_2 \lambda_2 (\frac{\lambda_2}{\lambda_1})^n (1 + \sqrt{5})}{\beta_1 (1 - \sqrt{5}) + \beta_2 (\frac{\lambda_2}{\lambda_1})^n (1 + \sqrt{5})},$$

which gets arbitrarily close to $\lambda_1 = \phi$ as n gets large, since $|\frac{\lambda_2}{\lambda_1}| < 1$.

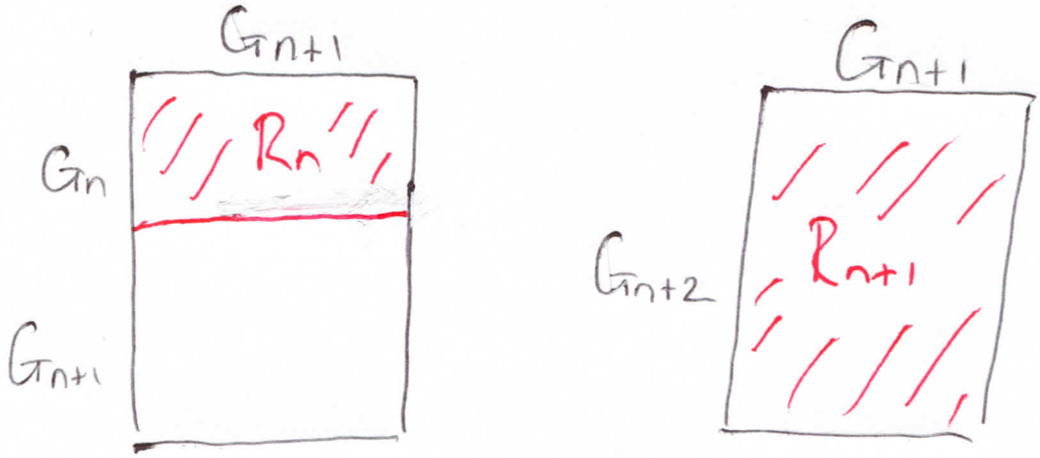
In the language of (3.3),

$$\frac{G_{n+1}}{G_n} \sim \phi \quad (6.6);$$

in the language of calculus, as in (5.2),

$$\lim_{n\to\infty} \frac{G_{n+1}}{G_n} = \phi.$$

Remarks 6.7. As in Construction 3.1, we may use a generalized Fibonacci sequence $G_0, G_1, G_2, \ldots$ to form rectangles R_n of length G_{n+1} and width G_n, $n = 0, 1, 2, \ldots$; the difference from 3.1 is that R_0 and R_1 could be arbitrary rectangles.



Stated informally, (6.6) is saying that, regardless of R_0 and R_1, the rectangles R_n are getting arbitrarily close to a golden rectangle, as n gets large.

CHAPTER VII: Some comments about the evolution of ideas.

Most answers to questions lead to many more questions and ideas. For example, the germ explanation of disease leads inevitably to the field of bacteriology. Answering the question of whether Euclid's parallel postulate follows from his other postulates (the answer is "yes and no"; the parallel postulate is *independent* of Euclid's other postulates) led to non-Euclidean geometry, including the mathematical model for general relativity.

Partly counterbalancing the explosion of natural questions and subject matters is the fact that ideas grow back on themselves. We have seen in this magnification the idea of the *golden ratio* from classical Greek mathematics, roughly, in its prime, from 500–200 BC, then we saw Fibonacci numbers arising in the 13^{th} century from a population model. These are seemingly much different ideas, in different cultures, studied by different people. It should be quite surprising that another quite different looking idea, linear algebra, from the 19^{th} century, relates the golden ratio and Fibonacci numbers, as outlined in Chapters IV through VI.

In physics, electricity and magnetism is another example of a pair of seemingly unrelated subjects, studied for centuries as different ideas by different people, until they were unified; in this case by Maxwell's Equations in the 19^{th} century.

Another phenomenon in the history of ideas is illustrated by the initial discovery of Theorem 4.7(a) using calculus (see Remark 4.10). More than 100 years later, a more elementary proof (using linear algebra) became possible. Over time, ideas not only grow back on themselves, but are simplified as they are better understood; like the clarity that results from the settling of silt in a river, fundamental ideas are sifted out from what turns out to be unnecessary complexity.

APPENDIX: the golden ratio in pentagrams.

The proofs in this section require some knowledge of complex numbers and trigonometry, as in [3]. All angles will be measured in radians.

Recall the drawing (in red) of a pentagram inside a regular pentagon, in Remarks 1.7. We will prove (Theorem APP.2) the following result from Remarks 1.7: in a regular polygon each side of which measures one, the distance between nonconsecutive vertices is ϕ, the golden ratio of Definition 1.5. This will be preceded (Theorem APP.1) by a surprising result relating ϕ to a certain cosine. We will conclude with some assertions without proof of more relationships between pentagrams, ϕ, and the angle $\frac{\pi}{5}$ about to appear in Theorem APP.1.

Connoisseurs of trigonometry notice that the number of angles with clean, closed expressions for cosines and sines are sparse; among angles strictly between 0 and $\frac{\pi}{2}$, $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$ are usually the only angles one memorizes sines and cosines of. Thus we should be surprised to find the following result for the cosine of $\frac{\pi}{5}$, where ϕ is the golden ratio of Definition 1.5. See [1, pages 73–74] for similar techniques with the angle $\frac{2\pi}{5}$.

Theorem APP.1. $\phi = 2\cos(\frac{\pi}{5})$.

Proof. Denote $\theta \equiv \frac{\pi}{5}, x \equiv \cos(\theta)$ and $y \equiv \sin(\theta)$. Then, denoting by "Im" the *imaginary part* (of a complex number),

$$0 = \text{Im}(-1) = \text{Im}(e^{i\pi}) = \text{Im}\left((e^{i\theta})^5\right) = \text{Im}\left((x+iy)^5\right)$$
$$= \text{Im}\left(x^5 + 5x^4(iy) + 10x^3(iy)^2 + 10x^2(iy)^3 + 5x(iy)^4 + (iy)^5\right) = \left(5x^4y - 10x^2y^3 + y^5\right),$$

which is equivalent to

$$0 = \left(5x^4 - 10x^2y^2 + y^4\right) = \left(5x^4 - 10x^2(1-x^2) + (1-x^2)^2\right) = 5x^4 - 10x^2 + 10x^4 + 1 - 2x^2 + x^4 = 16(x^2)^2 - 12x^2 + 1,$$

so that

$$x^2 = \frac{1}{32}\left(12 \pm \sqrt{144-64}\right) = \frac{1}{8}\left(3 \pm \sqrt{5}\right).$$

To see that x^2 cannot equal $\frac{1}{8}\left(3-\sqrt{5}\right)$, note that, since cosine is decreasing on $(0, \frac{\pi}{2})$,

$$0 < \frac{\pi}{6} < \theta < \frac{\pi}{4} < \frac{\pi}{2}$$

implies that

$$\frac{\sqrt{3}}{2} = \cos(\left(\frac{\pi}{6}\right) > \cos(\theta) > \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}},$$

so that

$$\frac{3}{4} > (\cos(\theta))^2 = x^2 > \frac{1}{2};$$

meanwhile,

$$\frac{1}{8}\left(3-\sqrt{5}\right) < \frac{1}{8}(3-2) = \frac{1}{8} < \frac{1}{2} < x^2.$$

Thus

$$x^2 = \frac{1}{8}\left(3+\sqrt{5}\right).$$

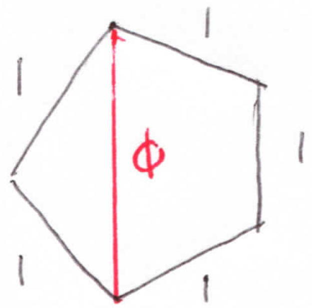
Compare this now to

$$\left(\frac{1}{2}\phi\right)^2 = \frac{1}{4}\left(\frac{1}{2}(1+\sqrt{5})\right)^2 = \frac{1}{16}\left(1+2\sqrt{5}+5\right) = \frac{1}{8}(3+\sqrt{5}).$$

We see that $x^2 = \left(\frac{1}{2}\phi\right)^2$; since both x and $\frac{1}{2}\phi$ are positive, this implies that $x = \frac{1}{2}\phi$, as desired. □

Now we will use Theorem APP.1 and trigonometry to prove the following pentagram result, from Remarks 1.7.

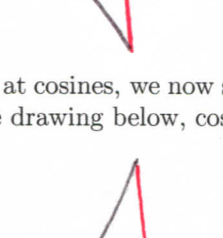
Theorem APP.2. In a regular pentagon whose sides measure one, the distance between any pair of nonconsecutive vertices is ϕ.



Proof: This will primarily be a sequence of drawings, beginning with the drawing below the statement of Theorem APP.2.

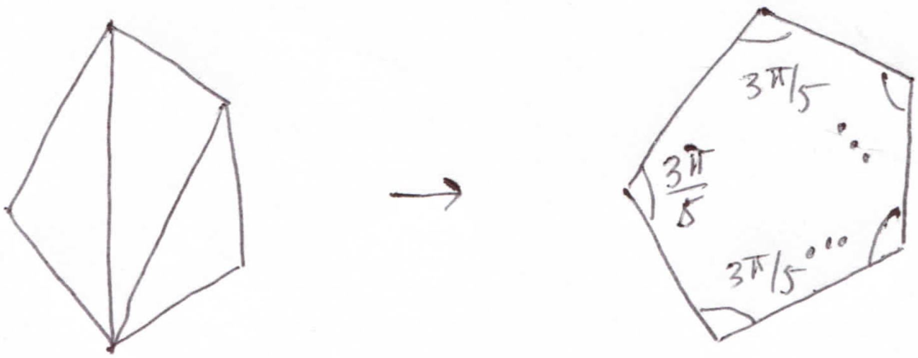
Draw a line from the left-most vertex to the red line between vertices, perpendicular to said red line.

By the Pythagorean theorem, the two right triangles formed have corresponding sides of equal measure; in the drawing below, $y_1^2 = 1 - x^2 = y_2^2$.



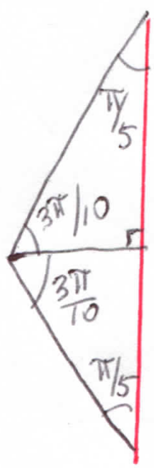
By looking at cosines, we now see that the two angles formed at the left-most vertex are of equal measure; in the drawing below, $\cos(\theta_1) = x = \cos(\theta_2)$.

For the shared measure of the angles θ_1 and θ_2, we need a separate argument. By drawing the pentagon as a union of three triangles, using the fact that the sum of the measures of interior angles in a triangle is π, combined with the fact that a regular polygon has equal angles, we see that each interior angle in a regular pentagon measures $\frac{3\pi}{5}$.

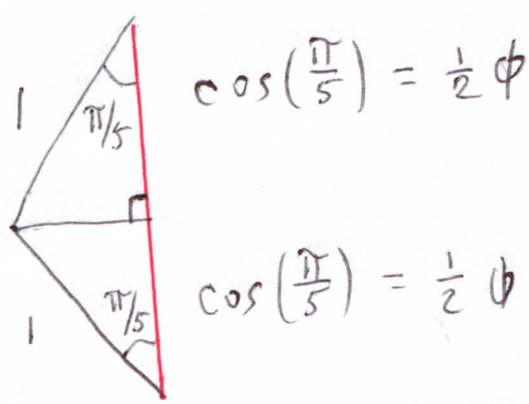


Thus we may fill in angles in our penultimate drawing: since θ_1 and θ_2 are of equal measure, they each measure $\frac{3\pi}{10}$, so that the remaining angle in each right triangle measures

$$\pi - (\frac{\pi}{2} + \frac{3\pi}{10}) = \frac{\pi}{5}.$$

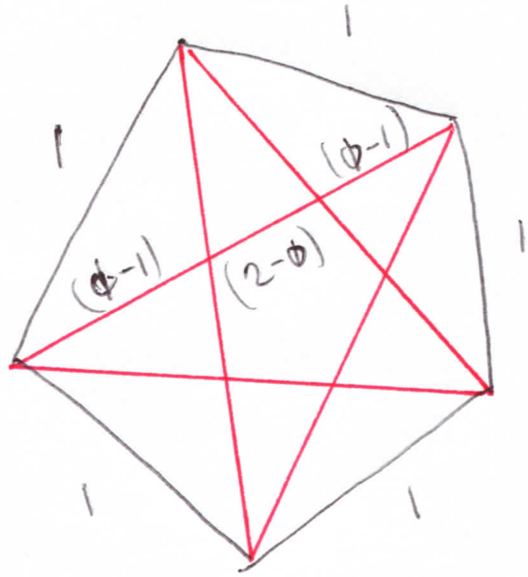


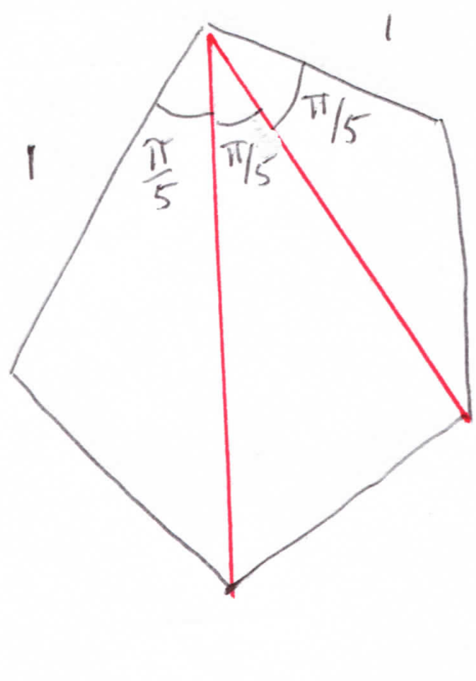
Finally, Theorem APP.1 allows us to fill in the measures of the two halves of the red line, so that they add up to ϕ. □



Some other results about ϕ, $\frac{\pi}{5}$, and pentagrams APP.3. We mention here, in pictures, two results about the pentagram in a pentagon, as in Remarks 1.7, without proof (some algebra and invocation of results about similar triangles would do the trick).

Each drawing below is of a pentagram in a regular pentagon, with each side of the pentagon measuring one.





HOMEWORK

1. Use only Definition 2.1 and Table 3.7 to get F_{19} and F_{20}.
HINT: Examples 2.3.

2. Use Theorem 4.7(b) to get F_{23}.
HINT: Example 4.8.

3. Suppose $F_{63} = 6,557,470,322,000,000,000,000$ and $F_{61} = 2,504,730,784,000,000,000,000$. Use only Definition 2.1 to get the following.

(a) F_{62}. (b) F_{64}. (c) F_{60}.
HINT: Examples 2.3.

4. Suppose $F_{30} = 832,040$. Use (3.3) to approximate each of the following.

(a) F_{31}. (b) F_{29}. (c) F_{32}.
HINT: Example 3.5.

5. Suppose $G_0 = 4, G_1 = 5$, and, for $n = 0, 1, 2, 3, \ldots$,

$$G_{n+2} = G_{n+1} + G_n.$$

(a) Get $G_2, G_3, G_4, \ldots, G_{10}$.
HINT: $G_2 = G_1 + G_0 = 5 + 4 = 9, G_3 = G_2 + G_1 = 9 + 5 = 14, \ldots$

(b) Use (6.6) to approximate (round each answer to three decimal places) $G_2, G_3, G_4, \ldots, G_{10}$.
HINT: For each $n = 2, 3, 4, \ldots, 9$, use G_n from (a) and (6.6) to approximate G_{n+1}.

(c) For $F_0, F_1, F_2, \ldots$ the Fibonacci numbers, get

$$(G_0 - F_0), (G_1 - F_1), (G_2 - F_2), \ldots (G_{10} - F_{10}).$$

Describe the sequence $(G_n - F_n), n = 0, 1, 2, 3 \ldots$, in the language of Chapter VI.

HOMEWORK ANSWERS

1. $F_{19} = F_{18} + F_{17} = 2,584 + 1,597 = 4,181$; $F_{20} = F_{19} + F_{18} = 4,181 + 2,584 = 6,765$.

2. $\frac{1}{\sqrt{5}}\left(\phi^{23}\right) \sim 28,656.99999$, so $F_{23} = 28,657$.

3. (a) $F_{62} = F_{63} - F_{61} = 4,052,739,538,000,000,000,000$.

(b) $F_{63} + F_{62} = 10,610,209,860,000,000,000,000$.

(c) $F_{62} - F_{61} = 1,548,008,754,000,000,000,000$.

4. (a) $\phi \times 832,040 \sim 1,346,269$.

(b) $\frac{832,040}{\phi} \sim 514,229$.

(c) $\phi \times F_{31} \sim 2,178,309$.

5. (a) We already have $G_0 = 4, G_1 = 5, G_2 = 9, G_3 = 14$. Continue:

$$G_4 = G_3 + G_2 = 14 + 9 = 23, G_5 = 23 + 14 = 37, G_6 = 37 + 23 = 60, G_7 = 60 + 37 = 97,$$
$$G_8 = 97 + 60 = 157, G_9 = 157 + 97 = 254, G_{10} = 254 + 157 = 411.$$

(b)

$$G_2 \sim (\phi \times G_1) = (\frac{1}{2}(1+\sqrt{5}) \times 5) \sim 8.090, \quad G_3 \sim (\phi \times G_2) = (\frac{1}{2}(1+\sqrt{5}) \times 9) \sim 14.562, \quad G_4 \sim (\phi \times G_3) \sim 22.652,$$
$$G_5 \sim (\phi \times 23) \sim 37.215, \quad G_6 \sim (\phi \times 37) \sim 59.867, \quad G_7 \sim (\phi \times 60) \sim 97.082, \quad G_8 \sim (\phi \times 97) \sim 156.949,$$
$$G_9 \sim (\phi \times 157) \sim 254.031, \quad G_{10} \sim (\phi \times 254) \sim 410.981.$$

Notice that, as n gets larger, the approximation of G_{n+1} in (b) is getting closer to the G_{n+1} in (a); that is,

$$|G_{n+1} - (\phi \times G_n)|$$

gets smaller as n gets larger.

(c)

$$(G_0 - F_0) = 4, (G_1 - F_1) = 4, (G_2 - F_2) = 8, (G_3 - F_3) = 12, (G_4 - F_4) = 20, (G_5 - F_5) = 32, (G_6 - F_6) = 52,$$
$$(G_7 - F_7) = 84, (G_8 - F_8) = 136, (G_9 - F_9) = 220, (G_{10} - F_{10}) = 356.$$

The sequence $(G_n - F_n), n = 0, 1, 2, 3\ldots$ is a generalized Fibonacci sequence. You might believe this by looking at $(G_n - F_n), n = 0, 1, 2, 3, \ldots$ above, or, for a proof, note that

$$(G_{n+2} - F_{n+2}) = ((G_{n+1} + G_n) - (F_{n+1} + F_n)) = (G_{n+1} - F_{n+1}) + (G_n - F_n)$$

for $n = 0, 1, 2, 3, \ldots$.

REFERENCES

1. W.S. Anglin and J. Lambek, "The Heritage of Thales," Springer, 1995.

2. Ralph deLaubenfels, "Linear Algebra," or E Pluribus Unum, http://teacherscholarinstitute.com/FreeMathBooksHighschool.html (2017).

3. Ralph deLaubenfels, "Trig to the Point," http://teacherscholarinstitute.com/Books/TrigToThePoint.pdf (2017).

4. John Saxon, "Algebra 1. An Incremental Development," Second Edition, Saxon Publishers, Inc., 1990.

5. J. Stillwell, "Mathematics and Its History," Springer-Verlag, 1989.