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Geometric Sums
MATHematics MAGnification™

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GEOMETRIC SUMS MAGNIFICATION

This is one of a series of very short books on math, statistics, and physics called "Math Magnifications." The "magnification" refers to focusing on a particular topic that is pivotal in or emblematic of mathematics.

OUTLINE

A geometric sequence is formed by multiplying by a fixed number; for example, the sequence 3, 6, 12, 24, is geometric because each term is twice the prior term. A geometric sum means adding up the terms of a geometric sequence. Of particular interest is an infinite geometric sum: adding up all the terms of an infinite geometric sequence.

We will derive a general formula for a geometric sum, and, after indicating where it can go wrong, apply it to

1. One of Zeno's paradoxes;

2. Repeating decimal expansions;

3. A dropped ball allowed to bounce up and down infinitely often: how far does it travel and how long does it take to come to rest; and

4. The area of a popular fractal, the Koch Snowflake.

This magnification will expose the reader to calculus ideas and a smattering of physics and exotic geometry.

For this magnification, students should be comfortable with exponents and algebraic terms, such as

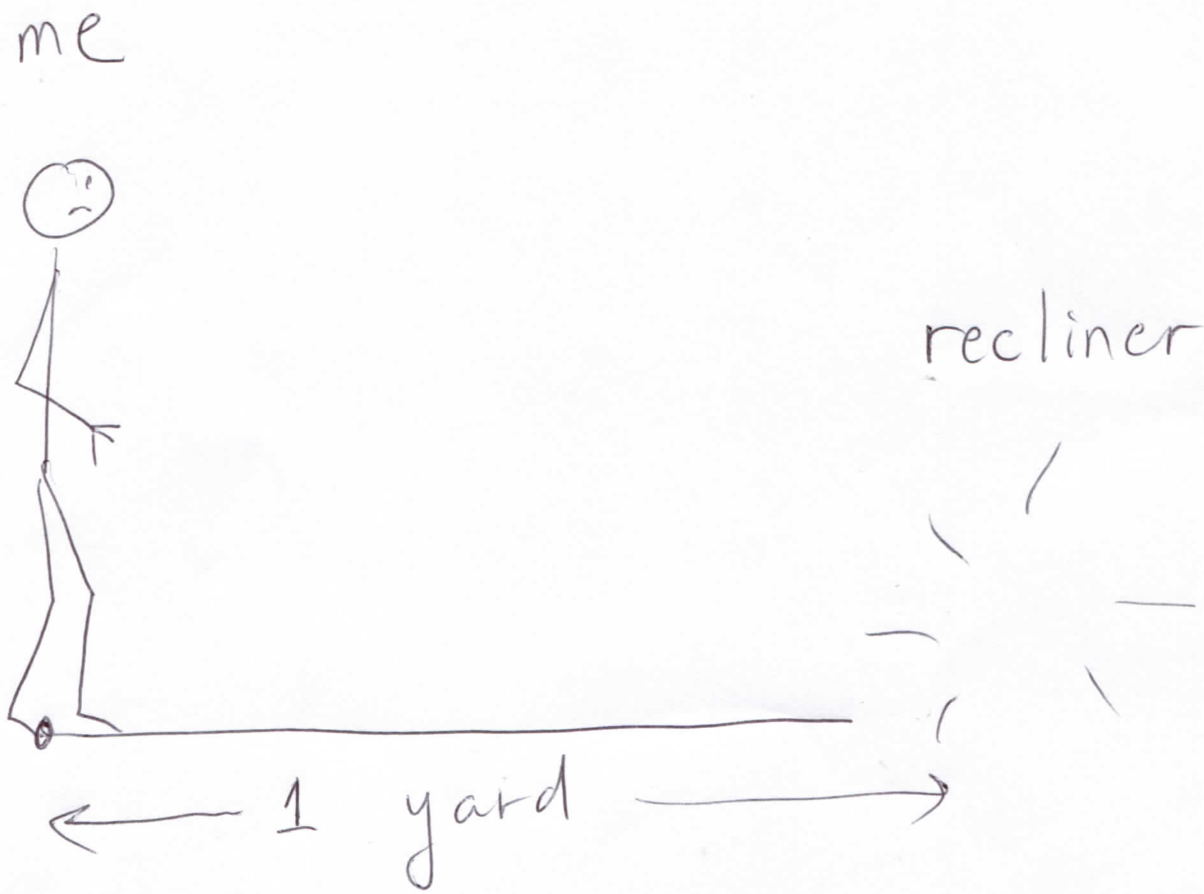
$$(1 + r + r^2 + r^3 + \cdots + r^n).$$

They should also be able to simplify fractions, especially fractions divided by fractions.

One of Zeno's paradoxes, in an updated form, is the following tragedy. I'm standing a yard away from my recliner. I really want to sit in my recliner. To get to my recliner, I must travel half a yard, then one quarter of a yard (half of the remaining distance), then one eighth of a yard, ... That's a sum of infinitely many distances, which our intuition tells us must be infinity.

I am also concerned that each distance traveled takes some time, thus the time spent in traveling infinitely many distances is the sum of infinitely many times, also (as guided by intuition) infinity. This is saying I will never get there.

Zeno claimed to conclude that motion is an illusion. Distance is also not looking so coherent: is the one yard to my recliner actually ∞ yards?



Example 1. Distance to recliner. Let's look more closely at the problem. We are adding up powers of $\frac{1}{2}$:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4+\ldots;$$

the dots "..." indicate a process (in this case multiplying by $\frac{1}{2}$ and adding) that goes on forever.

When something eludes our understanding, it is good strategy to give it a name; it's a genial bluff aimed at the seeming intimidation of challenging ideas. Define

$$S\equiv\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4+\ldots.$$

Notice how similar

$$\frac{1}{2}S=\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4+\left(\frac{1}{2}\right)^5+\ldots$$

is to S; in particular, I can get nice cancellation by subtracting $\frac{1}{2}S$ from S:

$$\frac{1}{2}S=(1-\frac{1}{2})S=(1\times S)-\left(\frac{1}{2}\times S\right)=S-\frac{1}{2}S$$

$$=\left(\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4+\ldots\right)-\left(\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4+\left(\frac{1}{2}\right)^5+\ldots\right)=\frac{1}{2}.$$

The cancellation is more clear if we write our subtraction vertically, as follows.

$$S = \left(\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\ldots\right)$$

$$-\frac{1}{2}S = \left(\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\ldots\right)$$

$$\frac{1}{2}S = \frac{1}{2}$$

$$\rightarrow S = 1 \text{ (yard)}$$

Since $\frac{1}{2}S$ equals $\frac{1}{2}$, S must be one. This reassures us that the one yard to our recliner is not an illusion.

For motion we need, in addition to the distance we just established, coherent (meaning at least finite) time.

Example 2. Time to recliner. Let's assume that, for $k = 1, 2, 3, \ldots$, my time in travelling the distance of $\left(\frac{1}{2}\right)^k$ yards is $\left(\frac{2}{3}\right)^k$ seconds. Then our total time is

$$T \equiv \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^4 + \ldots.$$

Simplification will occur this time by multiplying T by $\frac{2}{3}$, then subtracting from T, creating cancellation as in Example 1:

$$\frac{1}{3}T = T - \frac{2}{3}T = \left(\frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^4 + \ldots\right) - \left(\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^4 + \left(\frac{2}{3}\right)^5 + \ldots\right) = \frac{2}{3},$$

so that T equals 2 seconds:

$$T = \left(\frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \ldots\right)$$

$$-\frac{2}{3}T = \left(\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \ldots\right)$$

$$\frac{1}{3}T = 2/3$$

$$\rightarrow T = 2 \text{ (seconds)}$$

Notice that, in the motion described in the previous calculation, my travel is getting arbitrarily slow: for $k = 1, 2, 3, \ldots$, since I'm traveling the distance of $\left(\frac{1}{2}\right)^k$ yards in $\left(\frac{2}{3}\right)^k$ seconds, that's a speed of $\frac{(\frac{1}{2})^k}{(\frac{2}{3})^k} = \left(\frac{3}{4}\right)^k$ yards per second. Despite my philosophical concerns slowing me down, I still make it to my recliner.

Example 3. Find the sum $\left(1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \ldots\right)$.

Call the sum S. Then

$$\frac{2}{3}S = S - \frac{1}{3}S = \left(1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \ldots\right) - \left(\frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \ldots\right) = 1,$$

thus $S = \frac{3}{2}$.

Definitions 4. For a and r real numbers, the sequence

$$a, ar, ar^2, ar^3, \ldots$$

is a **geometric sequence**. The number r is the **common ratio** of the sequence, the ratio of consecutive terms.

The sum

$$a + ar + ar^2 + ar^3 + \ldots$$

of a geometric sequence is called a **geometric sum** or **geometric series**.

In Example 1, $a = r = \frac{1}{2}$, in Example 2, $a = r = \frac{2}{3}$, and in Example 3, $a = 1$ and $r = \frac{1}{3}$.

We may argue as in Examples 1 through 3, to get a general formula for a geometric series. Denoting

$$S \equiv a + ar + ar^2 + ar^3 + \ldots,$$

multiply by the common ratio and subtract:

$$(1 - r)S = (a + ar + ar^2 + ar^3 + \ldots) - (ar + ar^2 + ar^3 + +ar^4 \ldots) = a;$$

written longhand,

$$\begin{array}{rcl} S & = & (a + ar + ar^2 + \ldots) \\ -rS & = & (ar + ar^2 + \ldots) \\ \hline (1-r)S & = & a \rightarrow S = \dfrac{a}{(1-r)} \end{array}$$

(see Formula 5)

Formula 5 (not always true; see Theorem 7). For $r \neq 1$, $\left(a + ar + ar^2 + ar^3 + \dots\right) = \frac{a}{1-r}$.

Examples 6. (a) $1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots = \frac{1}{1-\frac{1}{4}} = \frac{1}{\frac{3}{4}} = \frac{4}{3}$. ($a = 1, r = \frac{1}{4} = \frac{\frac{1}{4}}{1} = \frac{\frac{1}{16}}{\frac{1}{4}} = \dots$, the ratio of consecutive terms.)

(b) $4 + 3 + \frac{9}{4} + \frac{27}{16} + \frac{81}{64} + \cdots = \frac{4}{1-\frac{3}{4}} = \frac{4}{\frac{1}{4}} = 16$. ($a = 4, r = \frac{3}{4}$.)

(c) $3 - \frac{3}{2} + \frac{3}{4} - \frac{3}{8} + \frac{3}{16} - \cdots = \frac{3}{1-(-\frac{1}{2})} = \frac{3}{\frac{3}{2}} = 2$. ($a = 3, r = -\frac{1}{2}$.)

(d) $1 + 2 + 4 + 8 + \cdots = \frac{1}{1-2} = -1$?? ($a = 1, r = 2$.) SOMETHING is surely wrong: we're adding up positive numbers and getting a negative number.

Theorem 7. Conditions for Formula 5. Formula 5 is true if and only if $|r| < 1$. We will argue for this in Remarks 11(a).

Example 8. Repeating decimals. In a decimal expansion of a number, a horizontal line over a set of consecutive integers denotes infinite repetition. For example,

$$3.2\overline{7} \equiv 3.2777\dots, \quad 12.012\overline{84} \equiv 12.012848484\dots, \quad 0.0\overline{273} \equiv 0.0273273273\dots.$$

A repeating decimal always includes a geometric sum; e.g.,

$$\begin{aligned} 17.138\overline{43} \equiv 17.138434343\cdots &= 17.138 + (0.00043 + 0.0000043 + 0.000000043 + \cdots) \\ &= 17.138 + \left(\frac{43}{10^5} + \frac{43}{10^7} + \frac{43}{10^9} + \dots\right); \end{aligned}$$

the expression in parentheses is a geometric sum with $a = \frac{43}{10^5}, r = \frac{1}{100}$ (see Definitions 4).

This means that we could use Formula 5 (see Theorem 7) to write a repeating decimal as a fraction. But we will illustrate now an easier technique for repeating decimals.

In each of (a)–(d) below, write the repeating decimal as a fraction, that is, a ratio of integers.

(a) $0.\overline{1} = 0.111\dots$.
Call the decimal S. Then $10S = 1.111\dots$, so that

$$9S = 10S - S = 1:$$

$$\begin{aligned} 10S &= 1.111\dots \\ -\ S &= 0.111\dots \\ \hline 9S &= 1 \rightarrow S = \tfrac{1}{9} \end{aligned}$$

Thus $S = \frac{1}{9}$.
Notice that $0.\overline{2} = \frac{2}{9}, 0.\overline{3} = \frac{3}{9} = \frac{1}{3}, \ldots$, and, strangest of all, $0.\overline{9} = 0.999\cdots = \frac{9}{9} = 1$.

(b) $1.2\overline{47} \equiv 1.2474747\ldots$.
Call the decimal S. Then $100S = 124.747474\ldots$, so

$$99S = 100S - S = 123.5:$$

$$\begin{array}{rl} 100S = & 124.747474\ldots \\ -S = & 1.247474\ldots \\ \hline 99S = & 123.5 \rightarrow \end{array}$$

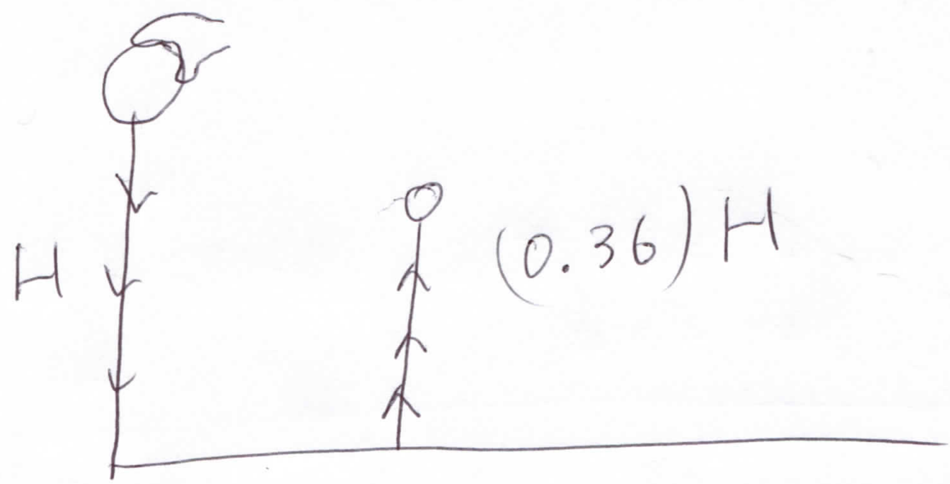
$$S = \frac{1,235}{990}$$

giving us $99S = 123.5$, so that $S = \frac{123.5}{99} = \frac{1235}{990}$.

(c) (left to reader) $3.\overline{21}$. (ANSWER: $\frac{318}{99}$)

(d) (left to reader) $0.05\overline{123}$. (ANSWER: $\frac{5,118}{99,900}$; HINT: writing S for the decimal, look at $999S = (1,000S - S)$)

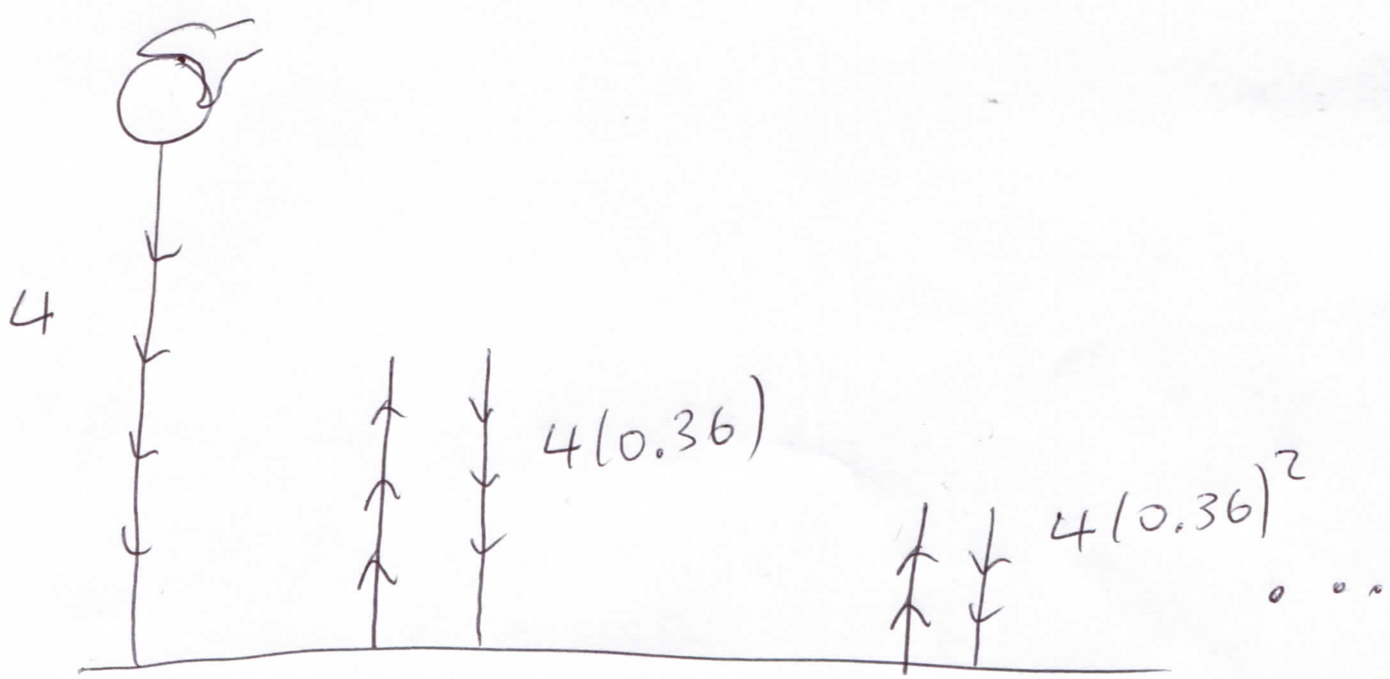
Example 9. Dropped ball, known elasticity. When a particular ball is dropped from a height H, it rebounds to a height of 0.36H, 36% of its original height.



Suppose this ball is dropped from a height of 4 feet.

(a) How far does the ball travel?

(b) How long before it stops bouncing?



(distance, in feet)

(a) By Formula 5 (see Theorem 7), with $a = 8(0.36), r = 0.36$, our distance is

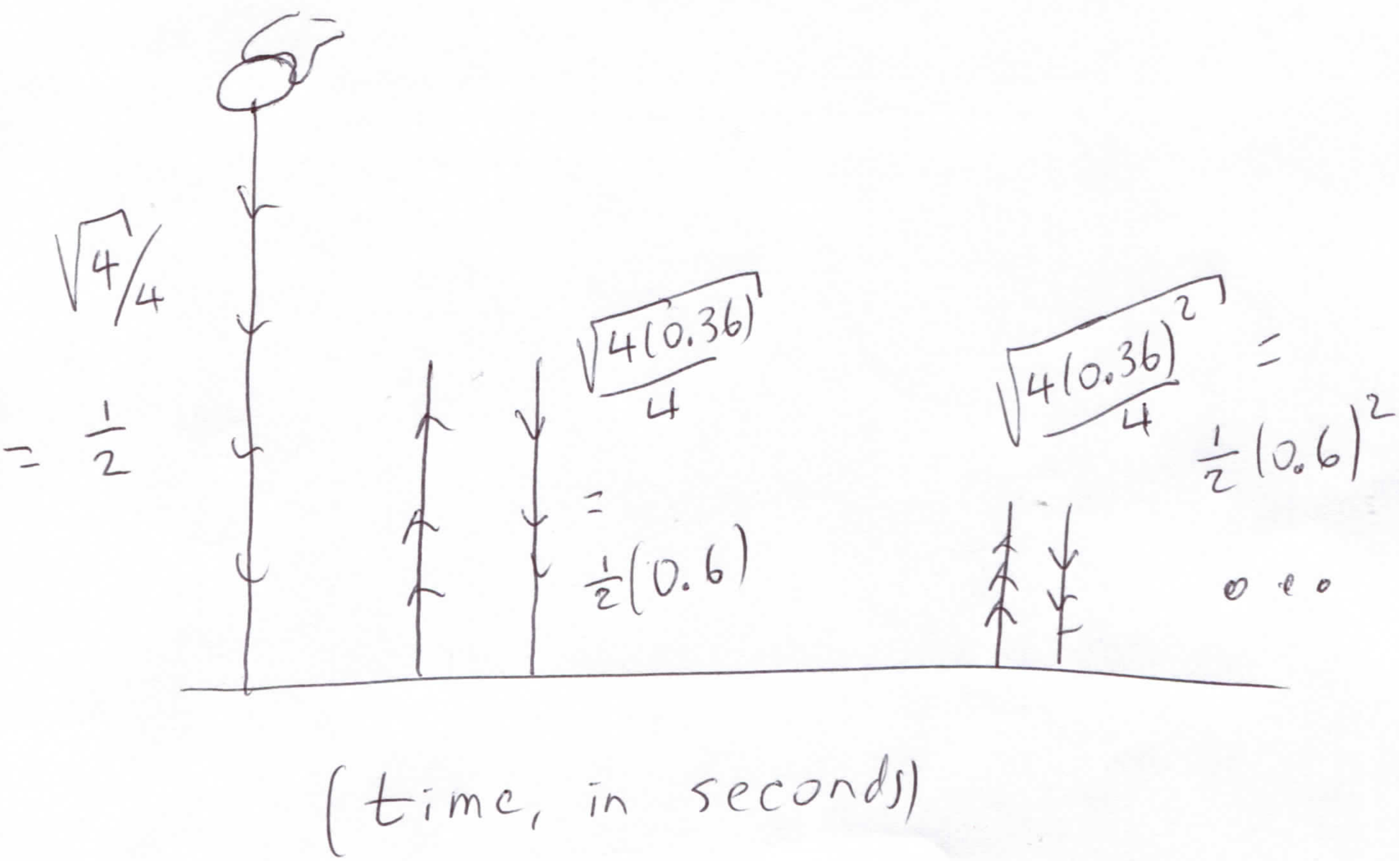
$$4 + \left[2 \times 4 \times (0.36) + 2 \times 4 \times (0.36)^2 + 2 \times 4 \times (0.36)^3 + \dots\right]$$

$$= 4 + \left[\frac{2 \times 4 \times (0.36)}{1 - 0.36}\right] = 4 + \frac{8(0.36)}{0.64} = 4 + \frac{0.36}{0.08} = 4 + \frac{36}{8} = 4 + \frac{9}{2} = \frac{17}{2} = 8.5 \text{ (feet)}.$$

(b) We need the following physics factoid (see Remarks 11(b))): If an object is dropped from a height of H feet, then the time it takes to hit the ground is

$$t = \frac{\sqrt{\mathrm{H}}}{4} \text{ seconds.}$$

Use this to change the distance drawing on the previous page into a time drawing:



Again use Formula 5, but now with $a = \left(\frac{2\sqrt{4}}{4}\right) 0.6 = 0.6, r = \sqrt{0.36} = 0.6$:

$$\frac{1}{2} + \left[2 \times \frac{1}{2} \times (0.6) + 2 \times \frac{1}{2} \times (0.6)^2 + 2 \times \frac{1}{2} \times (0.6)^3 + \dots\right]$$

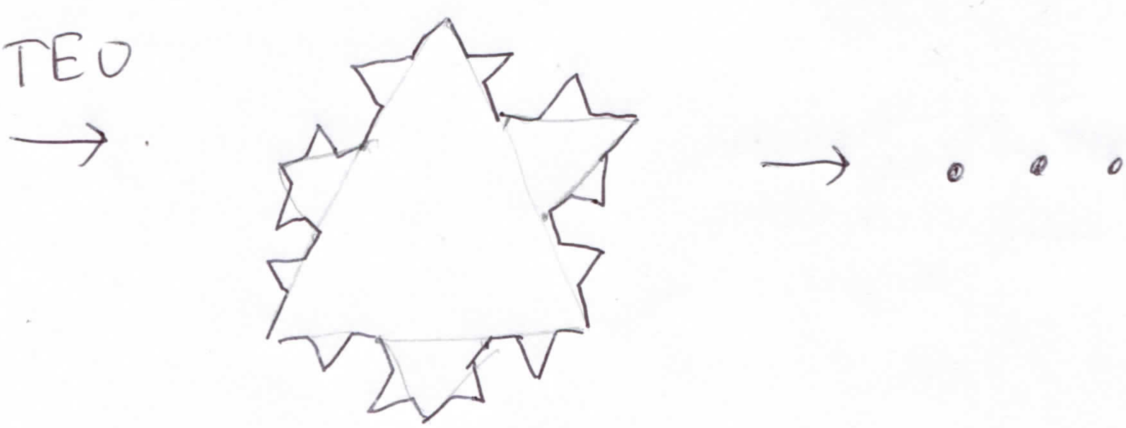
$$= \frac{1}{2} + \left[\frac{2 \times \frac{1}{2} \times (0.6)}{1 - 0.6}\right] = \frac{1}{2} + \frac{0.6}{0.4} = \frac{1}{2} + \frac{3}{2} = 2 \text{ (seconds)}.$$

Example 10. Koch snowflake. The Koch snowflake is an example of a *fractal.* Very informally, a fractal looks the same when magnified.

This fractal behavior is in contrast to more familiar macroscopic curves or areas; for example, your view of the coastline as you land in a helicopter: complexity and irregularity get smoothed and flattened as you descend.

Define the **Triangular Eructation Operation (TEO)** on any polygon: Each side eructates an equilateral triangle whose base has midpoint in the center of the side and length equal to one third of the length of the side.

The Koch snowflake is constructed by beginning with an equilateral triangle then applying the TEO infinitely often.



Notice that each application of the TEO multiplies the perimeter by $\frac{4}{3}$. Thus the Koch snowflake has infinite perimeter. We shall see, however, that the Koch snowflake has finite area.

Beginning with an equilateral triangle of area one, for simplicity, I'd like the area of the resulting Koch snowflake. We will focus on the area added by each application of the TEO, then put together all those additions.

Notice, as we progress through the TEOs, that each application of the TEO multiplies the number of sides by 4, while the triangles added have sides multiplied by $\frac{1}{3}$, hence area multiplied by $\frac{1}{9} = \left(\frac{1}{3}\right)^2$. This suggests (after the initial triangle of area one) that the total area added will be a geometric sum, with common ratio $r = \frac{4}{9}$.

Each term expressing area added will be the prior number of sides times the area of the triangle added on each side:

number of applications of TEO	number of sides	area of each triangle added	total area added
0	3	1	1
1	(3×4)	$\frac{1}{9}$	$(3) \times \frac{1}{9}$
2	(3×4^2)	$(\frac{1}{9})^2$	$(3 \times 4) \times (\frac{1}{9})^2$
3	(3×4^3)	$(\frac{1}{9})^3$	$(3 \times 4^2) \times (\frac{1}{9})^3$
.	.	.	.
.	.	.	.
.	.	.	.

The area of the Koch snowflake is the sum of the areas in the rightmost column:

$$1 + \left[(3) \times \frac{1}{9} + (3 \times 4) \times (\frac{1}{9})^2 + (3 \times 4^2) \times (\frac{1}{9})^3 + (3 \times 4^3) \times (\frac{1}{9})^4 + \dots\right]$$

$$= 1 + \left[(3 \times \frac{1}{9}) + (3 \times \frac{1}{9})(4 \times \frac{1}{9}) + (3 \times \frac{1}{9})(4 \times \frac{1}{9})^2 + (3 \times \frac{1}{9})(4 \times \frac{1}{9})^3 + \dots\right]$$

$$= 1 + \left[\frac{(3 \times \frac{1}{9})}{1 - (4 \times \frac{1}{9})}\right] = 1 + \frac{3}{9-4} = 1 + \frac{3}{5} = \frac{8}{5},$$

by Formula 5.

In general, the area of a Koch snowflake is $\frac{8}{5}$ times the area of the initial triangle. The perimeter is always infinity.

Stranger things can happen. The *Sierpinski gasket* is a fractal with zero area and infinite perimeter.

Remarks 11. (a) To make sense out of Formula 5, we must begin with something we *do* understand, namely, for any positive integer n, the finite sum

$$S_n \equiv \left(a + ar + ar^2 + \cdots + ar^{n-1}\right),$$

the sum of the first n terms of the geometric series.

As with the infinite sum, cancellation causes

$$(1-r)S_n = S_n - rS_n = a(1-r^n),$$

so that

$$S_n = \frac{a(1-r^n)}{(1-r)}.$$

The infinite sum of Theorem 5 *by definition* is a number that those finite sums S_n get arbitrarily close to as n gets arbitrarily large; in other words, as we add up more and more terms, we want the finite sums to get close to $\frac{a}{(1-r)}$. (In the language of calculus, $\frac{a}{(1-r)}$ is the *limit*, as n goes to infinity, of S_n, or S_n *converges* to $\frac{a}{(1-r)}$.)

Looking at our expression for S_n, we need r^n to shrink arbitrarily close to zero as n gets large; this happens precisely when $|r| < 1$.

(b) It can be shown, with a spot of calculus, that, if an object is dropped from a height h_0, with initial velocity v_0, then the height above the ground, in feet, is

$$h(t) = h_0 + v_0 t - 16t^2,$$

where t is the time, in seconds, after the object is dropped, at least until after the object hits the ground (the only physics information needed, when armed with calculus, is the acceleration due to gravity of 32 feet per second squared).

In Example 9, $v_0 = 0$ and h_0 is called H, so $h(t) = \mathrm{H} - 16t^2$.

Setting $h(t) = 0$ and solving for t gives $t = \frac{\sqrt{\mathrm{H}}}{4}$.

(c) Another paradox of Zeno leads to the general idea of infinite sums, at least of nonnegative terms.

Zeno considered a turtle s_1 feet in front of him, moving in the same direction. Zeno wishes to catch up to the turtle. To do this, Zeno must first reach the spot where the turtle began; that is, he must travel s_1 feet. Unfortunately, the turtle kept moving out of Zeno's grasp; say the turtle is now s_2 feet in front of Zeno. Identically, Zeno must also travel s_2 feet, only to find the turtle a distance, call it s_3, in front of him. Continuing in this way, we get an infinite sequence of positive numbers $s_1, s_2, s_3, s_4, \ldots$ such that Zeno must travel

$$(s_1 + s_2 + s_3 + s_4 + \ldots) \text{ feet}$$

to catch up to the turtle. With the intuition (apparently false) that an infinite sum of positive numbers must be infinite, the conclusion is that Zeno will never catch the turtle.

HOMEWORK

1. Write each of the following as a ratio of integers.

(a) $2.0\overline{4} \equiv 2.0444\ldots.$

(b) $1.\overline{72} \equiv 1.727272\ldots.$

(c) $0.01\overline{234} \equiv 0.01234234234\ldots.$

2. Write each of the following as a ratio of integers

(a) $2 + \frac{2}{3} + \frac{2}{9} + \ldots.$

(b) $1 + \frac{3}{8} + \frac{9}{64} + \frac{27}{512} + \ldots.$

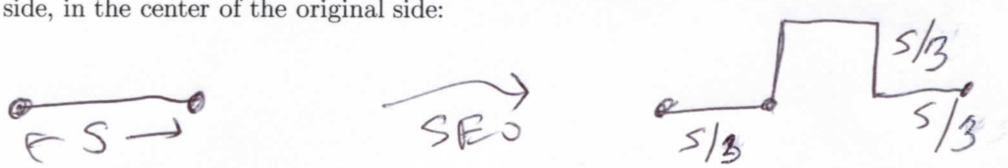
(c) $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \ldots.$

3. A ball has elasticity 25%; that is, when dropped, it rebounds to 25% of its initial height. Suppose this ball is dropped from a height of 9 feet.

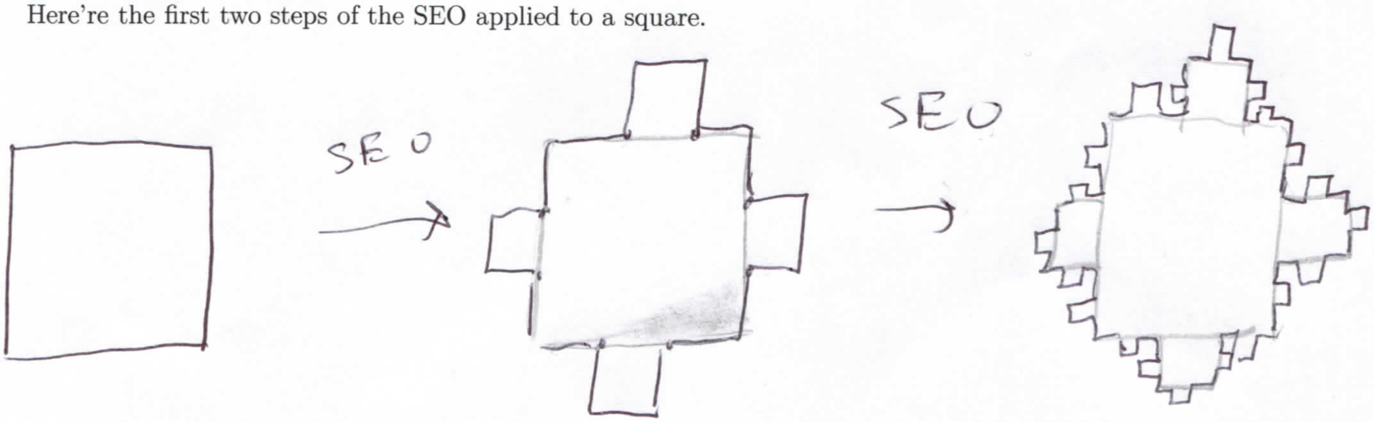
(a) How far does the ball travel?

(b) How long before it stops bouncing?

4. Construct a new fractal, using squares instead of triangles as in Example 10, by starting with a square of area one and applying the **Square Eructation Operation (SEO)** infinitely often, where the SEO does the following to each side of a polygon, eructating a square of side $\frac{1}{3}$ of the original side, in the center of the original side:



Here're the first two steps of the SEO applied to a square.



$\longrightarrow \; \cdot \; \cdot \; \cdot$

Find the area of the resulting fractal.

HINTS for HOMEWORK

1. See Example 8.

2. Use Formula 5; take the ratio of consecutive terms for r.

3. See Example 9.

(a) $9 + 2 \times 9 \times (0.25) + 2 \times 9 \times (0.25)^2 + 2 \times 9 \times (0.25)^3 + \ldots$

(b) $\frac{3}{4} + 2 \times \frac{3}{4} \times (0.5) + 2 \times \frac{3}{4} \times (0.5)^2 + 2 \times \frac{3}{4} \times (0.5)^3 + \ldots$

4. See Example 10, especially the table describing the growth of area as we apply the TEO; here is the analogous table for the SEO.

number of applications of SEO	number of sides	area of each square added	total area added
0	4	1	1
1	(4×5)	$\frac{1}{9}$	$(4) \times \frac{1}{9}$
2	(4×5^2)	$(\frac{1}{9})^2$	$(4 \times 5) \times (\frac{1}{9})^2$
3	(4×5^3)	$(\frac{1}{9})^3$	$(4 \times 5^2) \times (\frac{1}{9})^3$
.	.	.	.
.	.	.	.
.	.	.	.

ANSWERS to HOMEWORK

1. (a) $\frac{184}{90}$.

(b) $\frac{171}{99}$.

(c) $\frac{1,233}{99,900}$.

2. (a) 3.

(b) $\frac{8}{5}$.

(c) $\frac{3}{4}$.

3. (a) 15 feet.

(b) $\frac{9}{4}$ seconds.

4. 2.