CHAPTER II: LINEAR SYSTEMS and GAUSS-JORDAN ELIMINATION

If ontogeny really did recapitulate phylogeny, this would be the first chapter. But we will find matrix and vector terminology very useful.

With the linear systems of this chapter, <u>how</u> you solve is much more important than the solutions, since our techniques will reappear throughout this book, in settings sufficiently subtle to appreciate tangible predecessors.

SECTION II A: LINEAR SYSTEMS, DEFINITIONS AND TERMINOLOGY

DEFINITION 2.1. A **linear equation** has the form

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b,$$

where $b, a_1, a_2, \dots, a_n$ are real numbers, $x_1, x_2, \dots$ are variables.

DEFINITION 2.2

A **linear system**

is a finite set of linear equations, with each equation having the same variables.

Example 2.3.

$$(*)\quad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 2x_1 \qquad\quad - x_3 &= 0 \end{aligned}$$

is a linear system.

Notice that the second equation is assumed to have the variable x_2, since x_1 and x_3 are variables.

DEFINITION 2.4

More specifically, for n, m natural numbers, an **$(m \times n)$ (reads "m by n") linear system**

is m linear equations with n variables

$$
(2.5) \quad
\begin{aligned}
a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &= b_1 \\
a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n &= b_2 \\
&\vdots \\
a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n &= b_m
\end{aligned}
$$

Example 2.3 is a (2x3) linear system.

Please note how similar (2.5) is to Definition 1.1, of an (mxn) matrix.

DEFINITION 2.6

A **solution** of (2.5) is values of $x_1, x_2, \ldots, x_n$ that satisfy all equations.

Example 2.7

$x_1 = 0 = x_3$, $x_2 = 2$ is a solution of (*) in Example 2.3, but $x_1 = 0 = x_3 = x_2$ is not, even though it satisfies the second equation.

DEFINITION 2.8

A linear system is **consistent** if it has a solution; otherwise, it is **inconsistent**.

"Consistent" here is English usage, in the sense of "no contradiction."

For example,

$$x_1 = 1$$
$$x_1 = 2$$

is an inconsistent linear system, since the existence of a solution would imply that $1 = 2$, a contradiction, at least in our world.

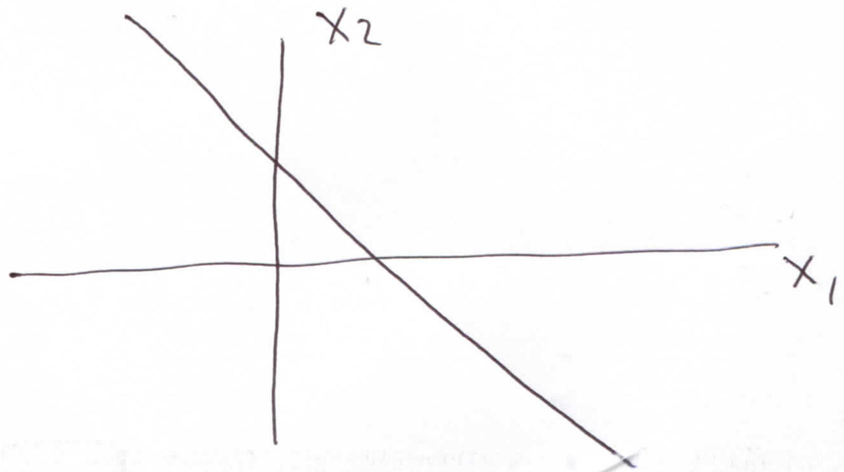
Examples 2.9.

Let's talk about linear systems that can be described in the Cartesian plane.

A (1×2) linear system

$$a_1 x_1 + a_2 x_2 = b$$

describes a line

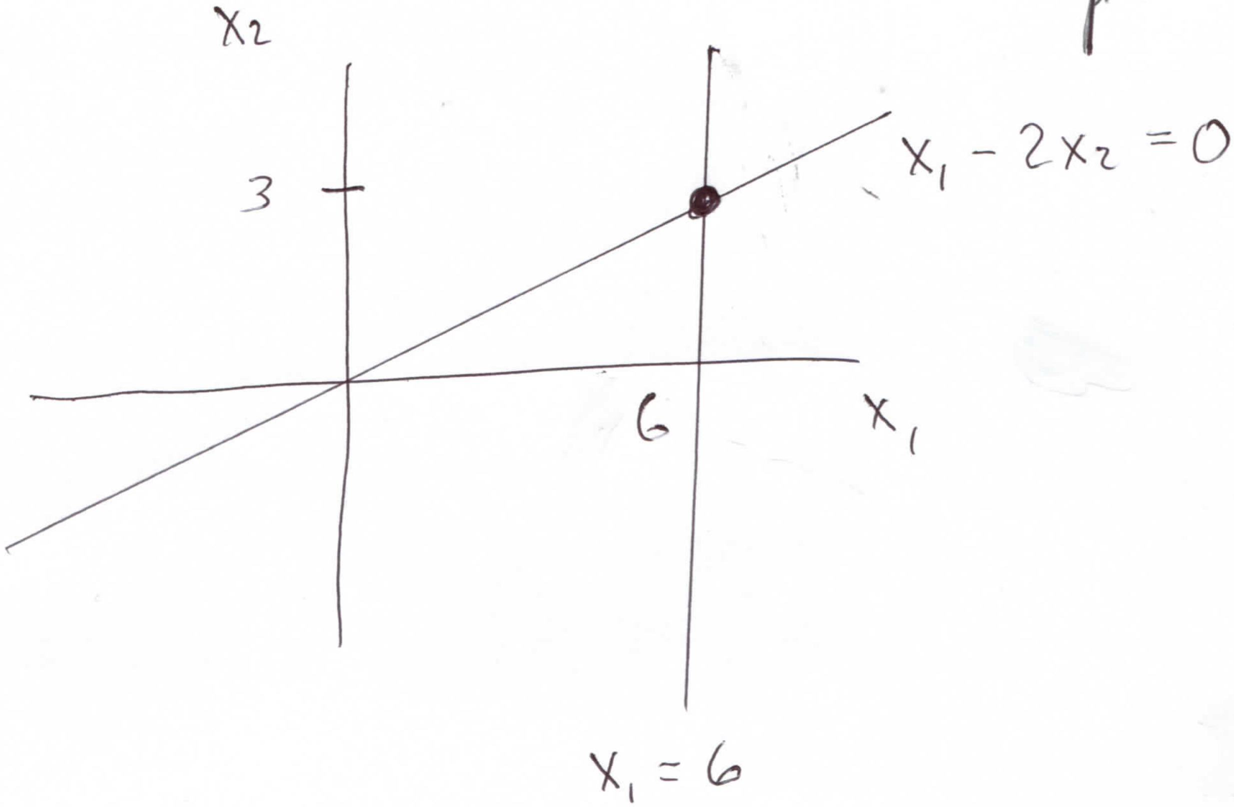


in the sense that a point (x_1, x_2) is on the line if and only if it's a solution of the linear system.

In the same sense, a solution of a (2×2) linear system corresponds to the intersection of two lines.

<u>Characteristic</u> <u>Example</u> 1.

$$\begin{aligned} x_1 - 2x_2 &= 0 \\ x_1 &= 6 \end{aligned}$$



The intersection of the two lines gives the $\boxed{\text{UNIQUE SOLUTION}}$

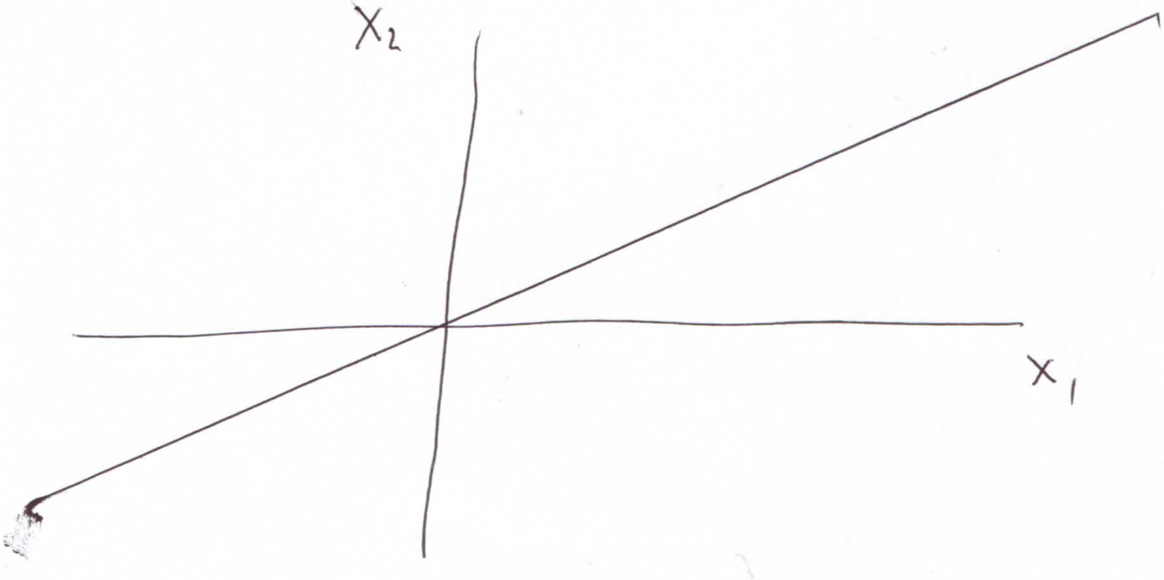
$(x_1, x_2) = (6, 3)$

Characteristic Example 2

$$x_1 - 2x_2 = 0$$
$$3x_1 - 6x_2 = 0$$

These two equations are coincident; that is, they describe the same line.

It's hard to draw both lines without looking like a single line



Professional wrestling describes this best; visualize a full-body slam of one line on top of the other.

This linear system has $\boxed{\text{INFINITELY MANY SOLUTIONS}}$; namely, any point on the repeated line I drew, or any point of the form $(2x_2, x_2)$, for x_2 real.

Characteristic Example 3

$$x_1 - 2x_2 = 0$$
$$3x_1 - 6x_2 = 3$$

We're changed only one number in the previous example, but the damage is severe.

IF there were a solution, then we'd have

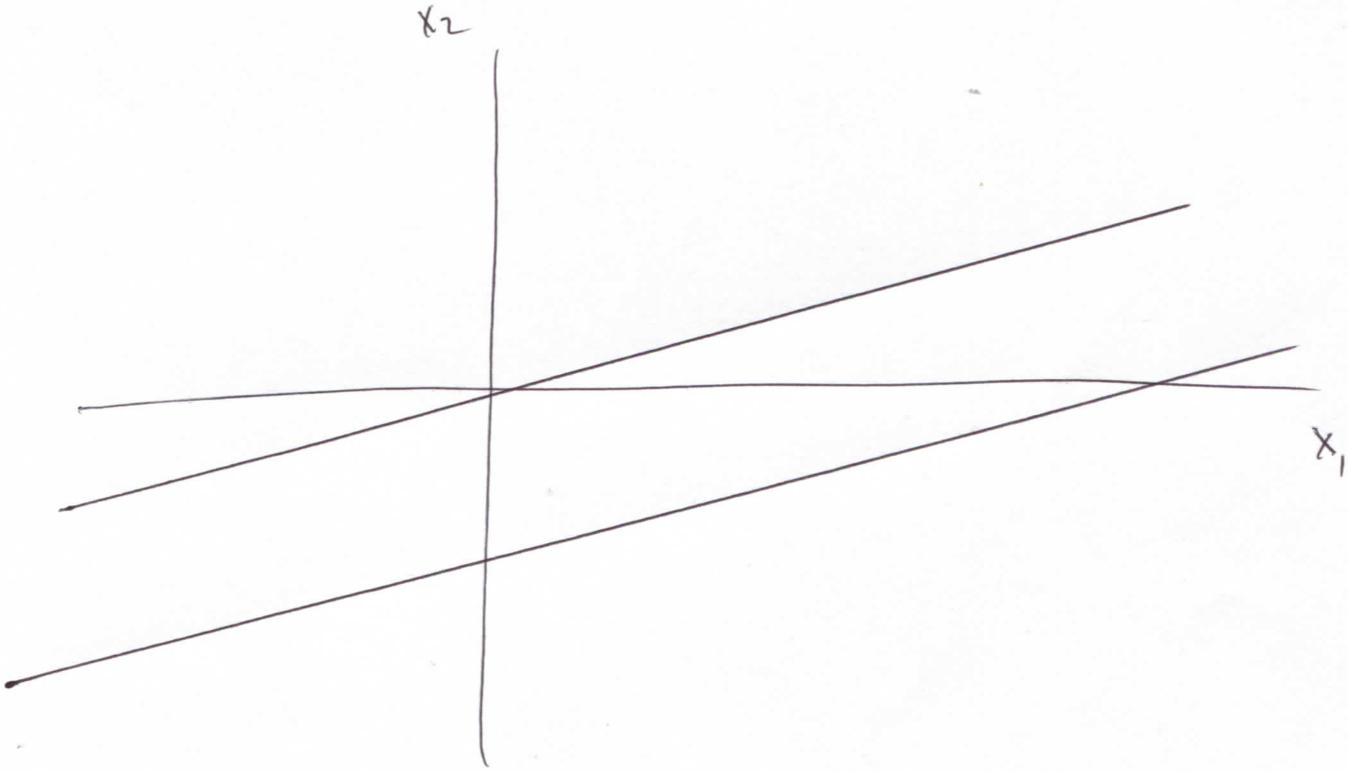
$$\begin{aligned} x_1 - 2x_2 &= 0 \\ x_1 - 2x_2 &= 1 \end{aligned} \longrightarrow 0 = 1,$$

a contradiction, making

this linear system
inconsistent; that is,

$\boxed{\text{NO SOLUTIONS}}$

The two equations describe
parallel, noncoincident lines.



REMARK 2.10

The characteristic examples of Examples 2.9 are emblematic in their red-rectangled counting of solutions: <u>any</u> linear system has either 0, 1, or ∞ solutions.

DEFINITION 2.11

Two linear systems are **equivalent** if they have the same set of solutions.

Examples 2.12

Consider the following four linear systems.

(1) $\begin{aligned} x_1 - x_2 &= 1 \\ x_2 &= 5 \end{aligned}$

(2) $\begin{aligned} x_1 - x_2 &= 1 \\ 2x_1 - 2x_2 &= 2 \end{aligned}$

(3) $\begin{aligned} x_1 \phantom{{}+x_2} &= 6 \\ x_2 &= 5 \end{aligned}$

(4) $\begin{aligned} x_1 - x_2 &= 1 \\ 2x_1 - x_2 &= 7 \end{aligned}$

(1) and (2) are <u>not</u> equivalent, even though they share a solution $((x_1, x_2) = (6,5))$, because $(x_1, x_2) = (1,0)$ is a solution of (2), but not of (1).

We will soon (later in this chapter) be able to show that (1), (3), & (4) are equivalent; specifically, each of them has the unique solution $(x_1, x_2) = (6,5)$.

DEFINITIONS 2.13

When a linear system has more than one solution, we will describe {all solutions} with free variables:
a set of variables is a set of

free variables

if each variable could be any real number, independent of the values of the other variables; in other words, there is no relationship between the variables.

Let's use our terminology from Chapter I to get intuitively suggestive and less unwieldly version of the linear system (2.5).

DEFINITIONS 2.14

For real numbers a_{ij}, x_j, b_i, $1 \le i \le m$, $1 \le j \le n$, denote

$$A \equiv \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \left(\begin{matrix} \text{as in} \\ 1.1 \end{matrix} \right)$$

$\vec{A}_j \equiv j^{\text{th}}$ column of A,

$$\vec{x} \equiv \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \vec{b} \equiv \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Then the **matrix form**

of (2.5) is

$$(2.15) \quad A\vec{x} = \vec{b}$$

and the **vector form**

of (2.5) is

(2.16)

$$x_1\vec{A}_1 + x_2\vec{A}_2 + \ldots + x_n\vec{A}_n = \vec{b}$$

$A \equiv$ the **coefficient**

matrix of (2.5) and

$[A\ \vec{b}] \equiv$ the **augmented**

matrix of (2.5).

Examples 2.17

(1) Find the coefficient matrix, augmented matrix, matrix form and vector form, of the linear system

$$\begin{aligned} x_1 \qquad - x_3 &= 0 \\ 2x_1 + x_2 \qquad &= 1 \end{aligned}$$

SOLUTION:

coefficient matrix $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$,

augmented matrix $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$,

matrix form $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

p. 98

vector form

$$x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(2) Suppose $\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$

Find a solution of

$$\begin{aligned} x_1 - x_2 &= 2 \\ x_1 + x_2 &= 4 \\ x_1 \quad &= 3 \end{aligned}$$

SOLUTION! $(x_1, x_2) = (3, 1)$

(3) Suppose

$$2\begin{bmatrix}1\\0\\-1\end{bmatrix} - \begin{bmatrix}0\\1\\2\end{bmatrix} + 3\begin{bmatrix}1\\2\\0\end{bmatrix} = \begin{bmatrix}5\\5\\-4\end{bmatrix}$$

Find a solution of

$$(*)\quad \begin{aligned} x_1 \qquad + x_3 &= 5 \\ x_2 + 2x_3 &= 5 \\ -x_1 + 2x_2 \qquad &= -4 \end{aligned}$$

Also find $\begin{bmatrix}1 & 0 & 1\\0 & 1 & 2\\-1 & 2 & 0\end{bmatrix}\begin{bmatrix}2\\-1\\3\end{bmatrix}$

SOLUTION: $(x_1, x_2, x_3) = (2, -1, 3)$ (vector form)

is a solution of $(*)$

$$\underset{A}{\begin{bmatrix}1 & 0 & 1\\0 & 1 & 2\\-1 & 2 & 0\end{bmatrix}}\underset{\vec{x}}{\begin{bmatrix}2\\-1\\3\end{bmatrix}} = \underset{\vec{b}}{\begin{bmatrix}5\\5\\-4\end{bmatrix}}$$

(matrix form)

SECTION IIB: GAUSS-JORDAN ELIMINATION

In the language we've just created, we can describe our **GOAL** in attacking a linear system:

Change to an equivalent system that's easy to solve.

This leads to two

QUESTIONS:

1. How to change to equivalent systems?

2. What do we mean by "easy"?

Here's a motivational example.

Example 2.18

$$\begin{aligned} x_1 - 2x_2 &= 0 \\ 3x_1 - 4x_2 &= 2 \end{aligned}$$

Add $(-3) \times$ (1st equation)

to the 2nd equation

$$\begin{aligned} x_1 - 2x_2 &= 0 \\ 2x_2 &= 2 \end{aligned}$$ (pretty easy)

Now do $\left(\frac{1}{2}\right) \times$ (2nd equation)

$$\begin{aligned} x_1 - 2x_2 &= 0 \\ x_2 &= 1 \end{aligned}$$ (better)

Add $2 \times$ (2nd equation)

to the 1st equation

$$\begin{aligned} x_1 \quad &= 2 \\ x_2 &= 1 \end{aligned}$$ (really easy)

We DON'T NEED the variables until the end. Do the same operations as the previous page to the augmented matrix:

$$(2.19)\quad \left\{\begin{array}{l} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 3 & -4 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 2 & 2 \end{array}\right] \rightarrow \\ \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]; \end{array}\right.$$

that last matrix is the augmented matrix for

$$\begin{aligned} x_1 \quad\quad &= 2 \\ x_2 &= 1 \end{aligned}$$

DEFINITION 2.20

A matrix is in

Echelon Form if

(1) In each row, the first nonzero term is to the right of the first nonzero term in the row above.

(2) The first nonzero term in each row is 1.

Examples 2.21

The second-to-last matrix in (2.19) is in echelon form; the matrix preceding it is not, because the first nonzero term in the second row is 2 rather than 1. The first matrix in (2.19) fails to be echelon because the "3" should be "0."

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 6 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$, and

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ are in echelon form.

$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ is <u>not</u> in echelon form, even though interchanging rows would make it echelon.

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ fails to be echelon, because of the last 1

$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

and $\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ are in echelon form.

DEFINITION 2.22

A matrix is in

Reduced Echelon Form

if it is in Echelon Form **AND**

(3) The first nonzero term in a row is the only nonzero term in its column

Examples 2.23

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

is in echelon, but not reduced echelon form, because of

$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ is in reduced echelon form. Notice that the "2" in the last column does no harm, since it is in a column that does not contain a row's first nonzero term.

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

are in reduced echelon form.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are in echelon, but not reduced echelon, form.

DEFINITION 2.24

Our preferred method for getting equivalent linear systems consists of

Elementary Operations,

meaning the following, that may be done on any matrix.

$R_i \leftrightarrow R_j$ interchange i^{th} & j^{th} rows

kR_i multiply i^{th} row by nonzero k

$R_i + kR_j$ add k times j^{th} row to i^{th} row $(i \neq j)$

Examples 2.25

$$\begin{bmatrix} 1 & 0 & -2 \\ 3 & 7 & 0 \end{bmatrix} \xrightarrow{R_2 - 3R_1} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 7 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 6 \\ 1 & 7 \\ 0 & 5 \end{bmatrix} \xrightarrow{\frac{1}{3}R_1} \begin{bmatrix} 1 & 2 \\ 1 & 7 \\ 0 & 5 \end{bmatrix}$$

DEFINITION 2.26

Two matrices are **row equivalent** if one can be changed to the other with elementary operations.

THEOREM 2.27

If $[A \ \vec{b}]$ is row equivalent to $[A' \ \vec{b}']$, then the linear system

$A\vec{x} = \vec{b}$ and $A'\vec{x} = \vec{b}'$

are equivalent.

Example 2.28 (see Example 2.25)

$$\begin{aligned} x_1 \phantom{{}+7x_2} &= -2 \\ 3x_1 + 7x_2 &= 0 \end{aligned}$$ is equivalent to

$$\begin{aligned} x_1 \phantom{{}+7x_2} &= -2 \\ 7x_2 &= 6 \end{aligned}$$

Here is how you should solve linear systems.

2.29 GAUSS-JORDAN ELIMINATION

(1) Use elementary operations to change the augmented matrix to echelon form.

(2) The linear system is inconsistent if and only if "$0 = 1$" appears.

(3) If consistent, use elementary operations to change the augmented matrix to reduced echelon form.

(4) Put the variables back and solve; use free variables if we have more than one solution.

Examples 2.30

Solve; that is, find all solutions or assert inconsistency, with Gauss-Jordan elimination.

At each step, we will circle in red numbers slated for elimination in the next step.

(a) $x_1 + x_2 - x_3 = 2$

$x_1 - x_2 = 1$

$2x_1 - x_3 = 1$

The first step is always pulling out the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & -1 & 0 & 1 \\ 2 & 0 & -1 & 1 \end{array}\right] \longrightarrow \begin{pmatrix} R_2 - R_1 \\ R_3 - 2R_1 \end{pmatrix}$$

USE R_1 to remove

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -2 & 1 & -1 \\ 0 & -2 & 1 & -3 \end{array}\right] \longrightarrow (R_3 - R_2)$$

delete with R_2

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

This is $0 = -2$, so we could stop here;

legally, for echelon form we need

$$\begin{pmatrix} -\frac{1}{2}R_2 \\ -\frac{1}{2}R_3 \end{pmatrix} \to \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & -1/2 & 1/2 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

"$0 = 1$" $\to$ $\boxed{\text{inconsistent (no solution)}}$;

our matrix in echelon form is the augmented matrix for

$$\begin{aligned} x_1 + x_2 - x_3 &= 2 \\ x_2 - \tfrac{1}{2}x_3 &= 1/2 \\ 0 &= 1 \end{aligned},$$

the last equation giving us a contradiction.

(b)
$$\begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ -2x_1 + x_2 - x_3 &= 0 \\ 3x_1 - x_2 + x_3 &= 4 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ -2 & 1 & -1 & 0 \\ 3 & -1 & 1 & 4 \end{array}\right] \rightarrow \begin{pmatrix} R_2 + 2R_1 \\ R_3 - 3R_1 \end{pmatrix}$$

disappear with R_1

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & -1 & 3 & 2 \\ 0 & 2 & -5 & 1 \end{array}\right] \rightarrow \left(R_3 + 2R_2 \right)$$

use R_2 to kill

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & -1 & 3 & 2 \\ 0 & 0 & 1 & 5 \end{array}\right] \xrightarrow{(-R_2)} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

END OF STEP (1)

STEP (2): system is consistent (has a solution) MEANS more work.

STRATEGY FOR STEP(1): clean up, from left to right

STRATEGY FOR STEP(3): clean up, from right to left.

Sanction with R_3

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 1 & 5 \end{array}\right] \rightarrow \begin{pmatrix} R_1 - 2R_3 \\ R_2 + 3R_3 \end{pmatrix}$$

p. 118

get rid of, with R_2

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & -9 \\ 0 & 1 & 0 & 13 \\ 0 & 0 & 1 & 5 \end{array}\right] \rightarrow (R_1 + R_2)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 13 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

END OF STEP (3)

This is the augmented matrix

for

$$\begin{aligned} x_1 &= 4 \\ x_2 &= 13 \\ x_3 &= 5 \end{aligned}$$

or $\boxed{(x_1, x_2, x_3) = (4, 13, 5)}$

(unique solution)

(c)
$$\begin{aligned} x_2 - x_3 &= 1 \\ x_1 + x_2 &= 0 \\ 2x_1 + 3x_2 + x_3 &= 5 \end{aligned} \quad \rightarrow$$

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 1 \\ 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 5 \end{array}\right] \rightarrow \left(\begin{array}{l} R_2 + (?)R_1 \\ R_3 + (?)R_1 \end{array} ??\right)$$

make them go?

NEED nonzero in upper left corner; e.g., could do

$$(R_1 \leftrightarrow R_2) \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 2 & 3 & 1 & 5 \end{array}\right] \rightarrow \left(\begin{array}{l} R_3 \\ -2R_1 \end{array}\right)$$

destroy with R_1

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & 5 \end{array}\right] \rightarrow (R_3 - R_2)$$

exterminate with R_2

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 4 \end{array}\right] \rightarrow \left(\tfrac{1}{2}R_3\right)$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \rightarrow \left(R_2 + R_3\right)$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \rightarrow \left(R_1 - R_2\right)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \rightarrow \begin{array}{l} x_1 = -3 \\ x_2 = 3 \\ x_3 = 2 \end{array}$$

or $\boxed{(x_1, x_2, x_3) = (-3, 3, 2)}$

(d)
$$\begin{aligned} x_1 + x_2 - x_3 &= 2 \\ x_1 - x_2 \phantom{{}- x_3} &= 1 \\ 2x_1 \phantom{{}+ x_2} - x_3 &= 3 \end{aligned} \rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & -1 & 0 & 1 \\ 2 & 0 & -1 & 3 \end{array}\right] \rightarrow \left(\begin{array}{c} R_2 - R_1 \\ R_3 - 2R_1 \end{array}\right)$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -2 & 1 & -1 \\ 0 & -2 & 1 & -1 \end{array}\right] \rightarrow (R_3 - R_2)$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left(-\tfrac{1}{2} R_2\right)$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & -1/2 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(consistent; the last equation is "$0 = 0$", harmless)

$$\xrightarrow{(R_1 - R_2)} \left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 3/2 \\ 0 & 1 & -1/2 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right];$$

this is reduced echelon form, so we put variables back; that is, the linear system with the matrix above as augmented matrix is

$$\begin{aligned} x_1 \quad - \tfrac{1}{2} x_3 &= 3/2 \\ x_2 - \tfrac{1}{2} x_3 &= 1/2 \\ 0 &= 0 \end{aligned}$$

x_1s in 1st column

x_2s in 2nd column

x_3s in 3rd column

PREFERRED FORM of {solutions}:

$$\boxed{\begin{aligned} x_1 &= \tfrac{3}{2} + \tfrac{1}{2}x_3 \\ x_2 &= \tfrac{1}{2} + \tfrac{1}{2}x_3 \\ x_3 &\text{ arbitrary} \end{aligned}}$$

x_3 is a <u>free</u> or <u>independent</u> variable (see Definition 2.13)

Notice that (a) of Example 2.30 differs from (d) by only one number; in fact, if the "3" in (d) were replaced by <u>any</u> other number, we would have no solutions.

(e) $x_1 + x_2 - x_3 - x_4 = 0$

(a (1×4) linear system)

You could write {solutions} as

$$\boxed{\begin{array}{l} x_1 = -x_2 + x_3 + x_4 \\ x_2, x_3, x_4 \text{ arbitrary} \end{array}}$$

OR

$$\boxed{\begin{array}{l} x_2 = -x_1 + x_3 + x_4 \\ x_1, x_3, x_4 \text{ arbitrary} \end{array}} ,$$

etc.; here you have a choice of free variables, although it will always be a set of 3 free variables.

(8) $x_1 \qquad\qquad - x_5 = 0$

$x_3 = 5$

$x_4 - x_5 = 1$

We could have x_4 or x_5 be a free variable; let's make it x_5. Then

$$\boxed{\begin{aligned}&\{\text{solutions}\} = \{(x_1, x_2, x_3, x_4, x_5) \mid \\ &x_1 = x_5,\ x_3 = 5,\ x_4 = 1 + x_5, \\ &x_2 \text{ and } x_5 \text{ arbitrary}\}\end{aligned}}$$

Notice that x_3 cannot be a free variable; it must equal 5 (no freedom). x_2, since it is not mentioned, must be a free variable; no contraints.

REMARK 2.31

Elementary operations can be peformed with matrix multiplication.

For example,

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

($R_1 \leftrightarrow R_2$ applied to $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$)

$$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ c & d \end{bmatrix}$$

(kR_1 applied to $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$)

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} =$$

$$\begin{bmatrix} a & b \\ (c+ka) & (d+kb) \end{bmatrix}$$

($R_2 + kR_1$ applied to $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$)

See Section III D, especially Definitions 3.52.

Multiplying $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ on the <u>right</u> performs elementary <u>column</u> operations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

(1st & 2nd columns interchanged)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} ka & b \\ kc & d \end{bmatrix}$$

(1st column multiplied by k)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & (b+ka) \\ c & (d+kc) \end{bmatrix}$$

(k times 1st column added to 2nd column)