SECTION

III C:

NULL SPACE and RANGE SPACE

DEFINITION 3.35

The **null space** of a matrix A is

$$\mathcal{n}(A) \equiv \{\vec{x} \mid A\vec{x} = \vec{0}\}$$

If A is $(m \times n)$, then $\vec{0}$ is the m-vector consisting entirely of zeroes and $\mathcal{N}(A)$ is a set of n-vectors.

Example 3.36

The null space of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ is

$$\mathcal{N}\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}\right) =$$

$$\left\{ (x_1, x_2) \;\middle|\; \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$= \left\{ (x_1, x_2) \;\middle|\; \begin{matrix} x_1 + 2x_2 = 0 \\ 3x_1 + 4x_2 = 0 \\ 5x_1 + 6x_2 = 0 \end{matrix} \right\}$$

Notice that null space is always the set of solutions of a homogeneous system.

In a predator-prey Difference Equation (see Examples 1.25(4))

$$\vec{x}_{k+1} = A\vec{x}_k \quad (k = 0, 1, 2, \ldots),$$

$\mathcal{N}(A)$ contains the states (distributions of predator and prey) that precede extinction.

Examples 3.37

Let $A \equiv \begin{bmatrix} 1 & 0 & 2 \\ 3 & 0 & 6 \end{bmatrix}$

(1) Is $(2, 7, -1)$ in $\mathcal{N}(A)$?

SOLUTION: Multiply

$$A\begin{bmatrix} 2 \\ 7 \\ -1 \end{bmatrix} = \begin{bmatrix} 2+0-2 \\ 6+0-6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \equiv \vec{0}$$

YES

(2) Characterize $\mathcal{N}(A)$.

SOLUTION: This is the set of solutions of $A\vec{x} = \vec{0}$:

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$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 3 & 0 & 6 & 0 \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow x_1 \qquad + 2x_3 = 0 \quad \rightarrow$$

$$\boxed{\{(-2x_3, x_2, x_3) \mid x_2, x_3 \text{ arbitrary}\} = \mathcal{N}(A)}$$

Theorem 3.18 implies that a solution of the linear system $A\vec{x} = \vec{b}$ is unique if and only if $\mathcal{N}(A)$ is trivial; that is, equals $\{\vec{0}\}$. This deserves a name.

DEFINITION 3.38

A matrix A is **singular** if $A\vec{x} = \vec{0}$ has a nontrivial solution; otherwise it is **nonsingular**.

3.33 (1) and Theorem 3.18 indicate how tho is addressing uniqueness of solutions of linear systems. Let's put these into a single theorem.

THEOREM 3.39

Suppose $A\vec{x}_0 = \vec{b}$. Then the following are equivalent; that is, if one of them is true, then the other two are true.

(1) $\vec{x}_0$ is the only solution of $A\vec{x} = \vec{b}$.

(2) A is nonsingular.

(3) rank(A) = (number of columns of A).

Examples 3.40

(1) $\begin{bmatrix} 1 & -1 \\ 2 & -2 \\ -1 & 1 \end{bmatrix}$ is singular,

since its rank is one but it has two columns.

OR, we could have demonstrated singularity by noting that

$$\begin{bmatrix} 1 & -1 \\ 2 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(2) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 7 & 10 \end{bmatrix}$ is nonsingular, since its rank is three

and the number of columns is three

This guarantees that the linear system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ x_2 &= 0 \\ x_2 + x_3 &= 0 \\ 7x_2 + 10x_3 &= 0 \end{aligned}$$

has only the trivial solution $0 = x_1 = x_2 = x_3$, and,

whenever

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= b_1 \\ x_2 &= b_2 \\ x_2 + x_3 &= b_3 \\ 7x_2 + 10x_3 &= b_4 \end{aligned}$$

has a solution, it will be unique.

(3) Can a (5×6) matrix be nonsingular?

SOLUTION: The rank of the matrix is ≤ 5 and the matrix has 6 columns, so NO since the rank cannot equal the number of columns (see Theorem 3.39)

(4) Can a (6×5) matrix be nonsingular?

SOLUTION: All we know about rank is that it's ≤ 5; we now have 5 columns, so, to be nonsingular, we need a (6×5) matrix of rank 5. The easiest way to get this is to add a row of zeroes to I_5:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is a nonsingular (6×5) matrix, so YES

DEFINITIONS 3.41

A linear system $A\vec{x} = \vec{b}$ might have a specified coefficient matrix A, but an uncertain or changing vector $\vec{b}$; e.g., new measurements of $\vec{b}$ might be occurring. Since $\mathcal{N}(A)$, the null space of A, worried about $\vec{x}$ in $A\vec{x} = \vec{b}$, at least when $\vec{b}$ is $\vec{0}$, a sort of complement to $\mathcal{N}(A)$ is the

range space of A

(3.42)

$$R(A) \equiv \left\{ \vec{b} \;\middle|\; \begin{array}{l} A\vec{x} = \vec{b} \text{ is} \\ \text{consistent} \end{array} \right\}$$

More explicitly, if A is an (mxn) matrix, then R(A) is the following set of m-vectors

(3.43)

$$R(A) \equiv \left\{ A\vec{x} \;\middle|\; \vec{x} \text{ is in } \mathbb{R}^n \right\}.$$

Recall that, as with Difference Equations, we think of matrix multiplication as doing something to a vector:

$$\vec{x} \xrightarrow{A} A\vec{x}$$

For example, if $A \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

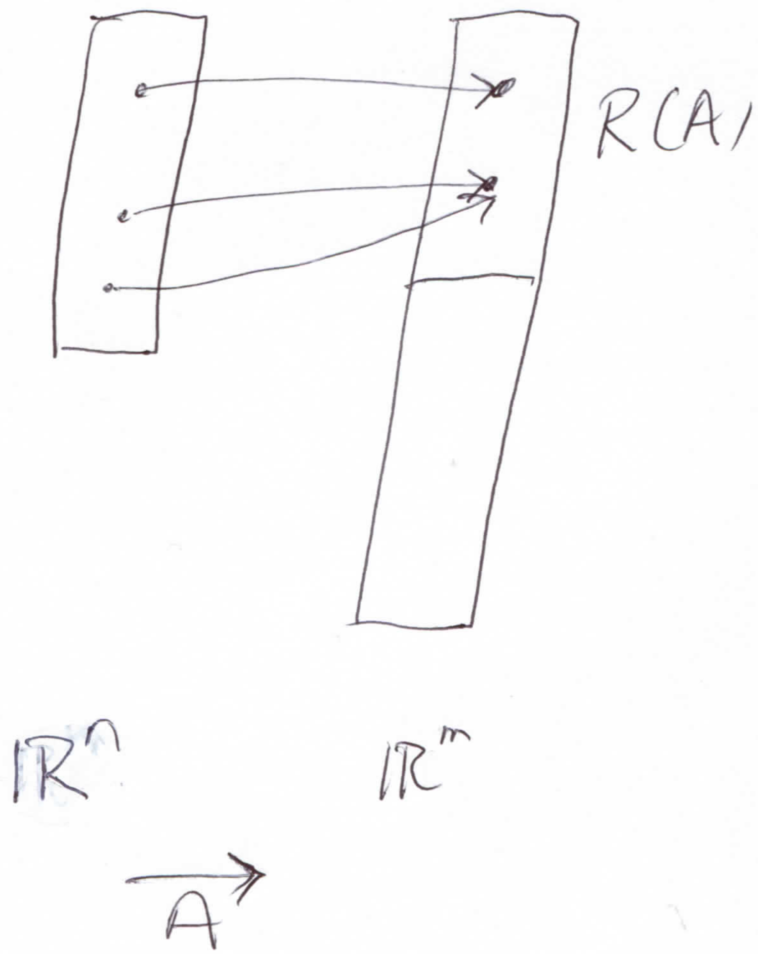
then

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{bmatrix};$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \xrightarrow{A} \begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{bmatrix}$$

A squashes the last component, then tacks on another zero.

$R(A)$ is the totality of all results of applying A to vectors in $\mathbb{R}^n$:



In the example above,

$$R(A) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{bmatrix} \middle| \; x_1, x_2 \text{ are real} \right\},$$

the set of all 4-vectors whose last two components are zero.

In terms of the Difference Equation

$$\vec{x}_{k+1} = A\vec{x}_k \quad (k = 0, 1, 2, \ldots)$$

$R(A)$ describes what happens to {possible states} as time passes.

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Examples 3.44

Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 0 & 6 \end{bmatrix}$,

as in Examples 3.37.

(1) Is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in $R(A)$?

SOLUTION: We need to know if $A\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is consistent.

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 3 & 0 & 6 & 1 \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

not consistent, so NO

(2) Characterize $R(A)$.

SOLUTION: We want $\vec{b}$ for which $A\vec{x} = \vec{b}$ is consistent.

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & b_1 \\ 3 & 0 & 6 & b_2 \end{array}\right] \xrightarrow{R_2 - 3R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & b_1 \\ 0 & 0 & 0 & b_2 - 3b_1 \end{array}\right] \longrightarrow$$

$b_2 - 3b_1 = 0$;

$$R(A) = \left\{ \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \;\middle|\; b_2 - 3b_1 = 0 \right\}$$

$$= \left\{ \begin{bmatrix} b_1 \\ 3b_1 \end{bmatrix} \;\middle|\; b_1 \text{ is arbitrary} \right\}$$

PLEASE NOTE that

$R(A)$ has 1 free variable,

$N(A)$ has 2 free variables

and $(2+1) = 3 =$

number of columns.

Compare to 3.28(b).

SECTION

III D:

INVERTIBILITY

and

INVERSES

We'd like to return to our initial historical motivation, where

$$(*) \quad \begin{aligned} x_1 + 2x_2 &= 3 \\ 4x_1 + 5x_2 &= 6 \end{aligned}$$

is transmogrified into

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

This LOOKS like $2x = 5$, which we know how to solve;

$$x = \frac{5}{2} = (2^{-1}) \cdot 5.$$

Could we similarly solve $(*)$:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 6 \end{bmatrix} ?$$

This looks great, except we have no idea what

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}^{-1}$$

means. Let's return to one variable. "2^{-1}" is the number such that

$$2 \cdot 2^{-1} = 1 = 2^{-1} \cdot 2$$

Recall that "1" is the multiplicative identity

$$1 \cdot x = x = x \cdot 1,$$

for any real x.

In the multivariable world, replace 1 with the identity matrix I (Definitions 3.5) and we will have a coherent definition of inverse.

DEFINITIONS 3.45

A matrix A is **invertible** if there's a matrix B so that

$$AB = I = BA.$$

B is then $\equiv$ the **inverse** of A, denoted $B \equiv A^{-1}$

Note that $AB = BA$ forces A to be a square matrix.

Example 3.46

Calculation tells us that

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix};$$

this means that $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ is invertible, with

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

This can be used to instantly solve

$$x_1 + x_2 = 1$$
$$x_1 + 2x_2 = 3$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

If we were solving

$$x_1 + x_2 = 0$$
$$x_1 + 2x_2 = -1 ,$$

then $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$$\rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} ;$$

for <u>any</u> $\vec{b}$,

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} =$$

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 2b_1 - b_2 \\ -b_1 + b_2 \end{bmatrix}.$$

Much more generally, if we have our hands on **the** inverse (IF it exists) of the coefficient matrix A, then we can solve

$$A\vec{x} = \vec{b}$$

immediately, for <u>any</u> $\vec{b}$ that's thrown at us.

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THEOREM 3.47

$A\vec{x} = \vec{b}$ if and only if

$\vec{x} = A^{-1}\vec{b}$ (if A is invertible)

REMARKS 3.48

Note the relationship between A and A^{-1} in terms of what they do to vectors:

$$\begin{tikzcd} \vec{x} \arrow[r, bend left, "A"] & A\vec{x} \arrow[l, bend left, "A^{-1}"] \end{tikzcd}$$

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A^{-1} <u>undoes</u> whatever A did.

From Example 3.46:

$$\begin{tikzcd} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \arrow[r, bend left, "\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}"] & \begin{bmatrix} 1 \\ 3 \end{bmatrix} \arrow[l, bend left, "\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1}"] \end{tikzcd}$$

$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ undoes what $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ did:

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix};$$

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Invertibility is good news for actions that you regret. Suppose A represents a heinous insult and you've applied A to a vector representing your best friend; that is, you've insulted your best friend.

__If__ A is invertible, then A^{-1} represents some sequence of actions — apology, bribe, whatever it takes — that returns your relationship to what it was, pre-insult.

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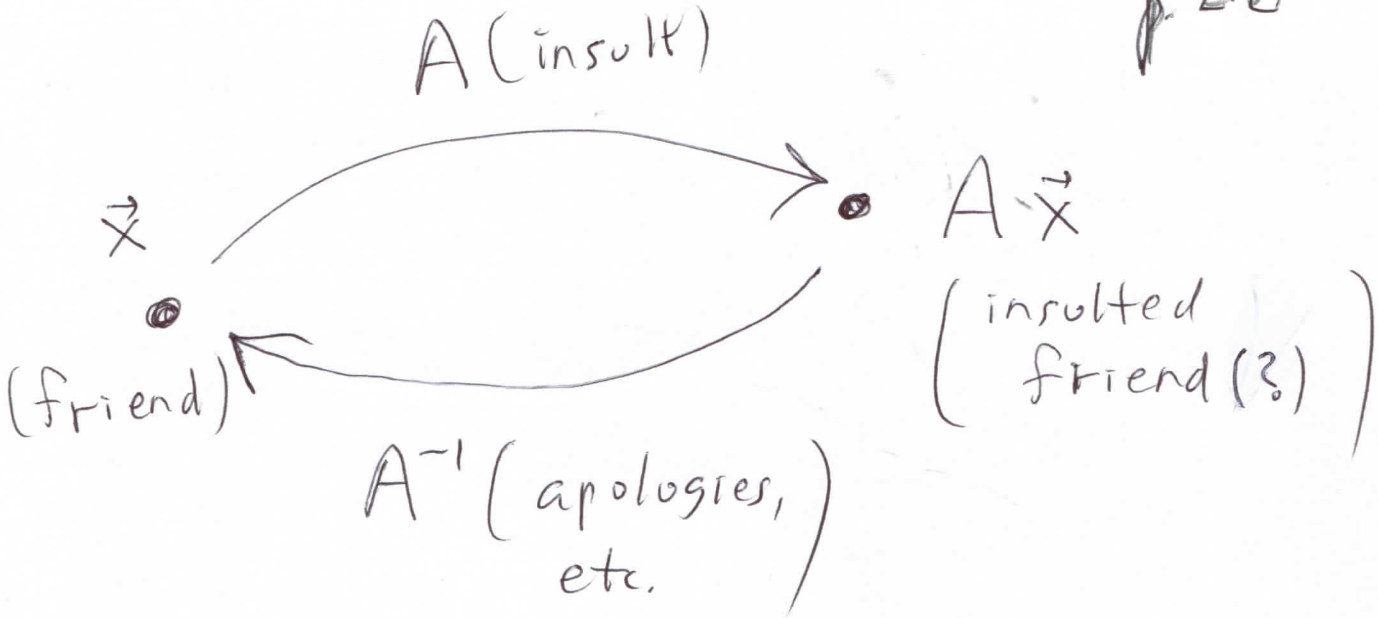

3.49 HOW TO GET A^{-1}

This relies on Theorem 3.47:

$$A\vec{x} = \vec{b} \quad \text{if \& only if} \quad \vec{x} = A^{-1}\vec{b}$$

OR

$$A\vec{x} = I\vec{b} \quad \text{if \& only if} \quad I\vec{x} = A^{-1}\vec{b}$$

OR

$$[A \mid I] \xrightarrow{\text{Gauss-Jordan}} [I \mid A^{-1}]$$

Examples 3.50

(1) $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$?

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 - R_2} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right] \rightarrow$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

(2) $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}^{-1}$?

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right]$$

not invertible

We conclude that

$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ is not invertible,

for the following reasons.

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \rightarrow$$

$$\left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 2 & 2 & b_2 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 0 & b_2 - 2b_1 \end{array}\right],$$

thus we get a solution <u>only</u> when $(b_2 - 2b_1) = 0$. If

$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ were invertible,

Theorem 3.47 would imply that

$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ has a solution for <u>any</u> $\vec{b}$.

IN GENERAL, if

$$[A \mid I] \xrightarrow[\text{ops.}]{\text{elem.}} \left[\begin{array}{c|c} 0\,0 \ldots 0 & m \end{array}\right],$$

then A is not invertible.

(3) Get the inverse of

$$A \equiv \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3}$$

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$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{\quad} R_2 - R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_1 - R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right] \longrightarrow$$

$$A^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

REMARKS 3.51

Cancellation of matrix products is not always legal; that is, $AB = AC$ does not always imply $B = C$.

EXAMPLE: Take $A \equiv B \equiv \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $C \equiv \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Then $AB = AC = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, but $B \neq C$.

However, if A is invertible, then $AB = AC \rightarrow B = IB = A^{-1}AB = A^{-1}AC = C$.

DEFINITIONS 3.52

It is useful that elementary operations, as in Definition 2.24, may be performed by matrix multiplication.

An **elementary matrix** is the result of applying an elementary operation to the identity matrix.

Specifically, using the terminology of Definition 2.24:

$E_{R_i \leftrightarrow R_j}$ is the result of interchanging the i^{th} and j^{th} rows in the identity matrix;

E_{kR_i} is the result of multiplying the i^{th} row of the identity matrix with nonzero k, and

$E_{R_i+kR_j}$ is the result of adding k times the j^{th} row to the i^{th} row of the identity matrix $(i \neq j)$

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THEOREM 3.53

For any matrix A, and elementary operation, call it O,

$E_O A$ equals A after performing O

Let's demonstrate this for (2×2) elementary matrices:

$$(R_1 \leftrightarrow R_2) \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} d & e & f \\ a & b & c \end{bmatrix}$$

$$(kR_1) \quad \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} ka & kb & kc \\ d & e & f \end{bmatrix}$$

$$\begin{pmatrix} R_2 + \\ kR_1 \end{pmatrix} \quad \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} a & b & e \\ d & e & f \end{bmatrix} = \begin{bmatrix} a & b & c \\ (d+ka) & (e+kb) & (f+kc) \end{bmatrix}$$

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We brought up elementary matrices in this section because they have inverses that aid our intuition about inverses:

$$\left(E_{R_i \leftrightarrow R_j}\right)^{-1} = \left(E_{R_i \leftrightarrow R_j}\right)$$

$$\left(E_{kR_i}\right)^{-1} = \left(E_{\frac{1}{k}R_i}\right)$$

$$\left(E_{R_i + kR_j}\right)^{-1} = \left(E_{R_i - kR_j}\right)$$