CHAPTER IV: VECTOR SPACES. CONSTRUCTION and MAINTENANCE

This chapter presents the intellectual foundations needed to understand and apply matrices and vectors. This book will barely scratch the surface of applications, including solving Difference Equations and making more complete and precise statements about linear systems. Although all definitions are about sets of vectors, they will be seen to be equivalent to statements about matrices and linear systems. Rank will reappear often, as a tangible and easily calculated (especially by computers) unifying theme.

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SECTION IVA: LINEAR COMBINATIONS

Here is the pivotal idea for dealing with sets of vectors.

DEFINITION 4.1

If $x_1, x_2, \ldots, x_m$ are real numbers and $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are in $\mathbb{R}^n$, then

$(x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m)$ is a

linear combination

of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_m\}$.

This definition puts together ("combines") everything we know how to do with vectors: addition and scalar multiplication.

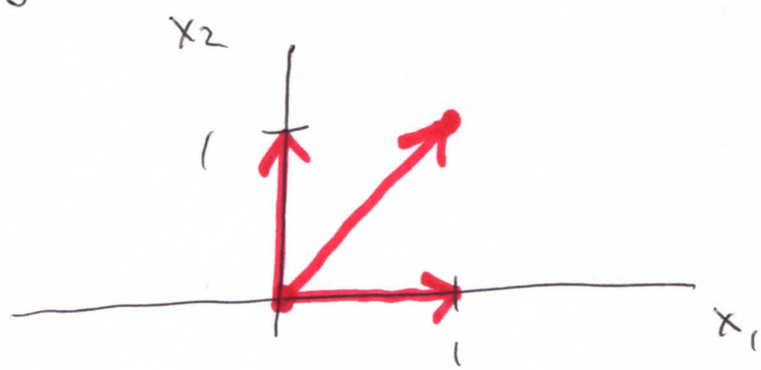
Example 4.2 $(1,2)$ is a linear combination of

$$\{(1,0), (1,1), (0,1)\},$$

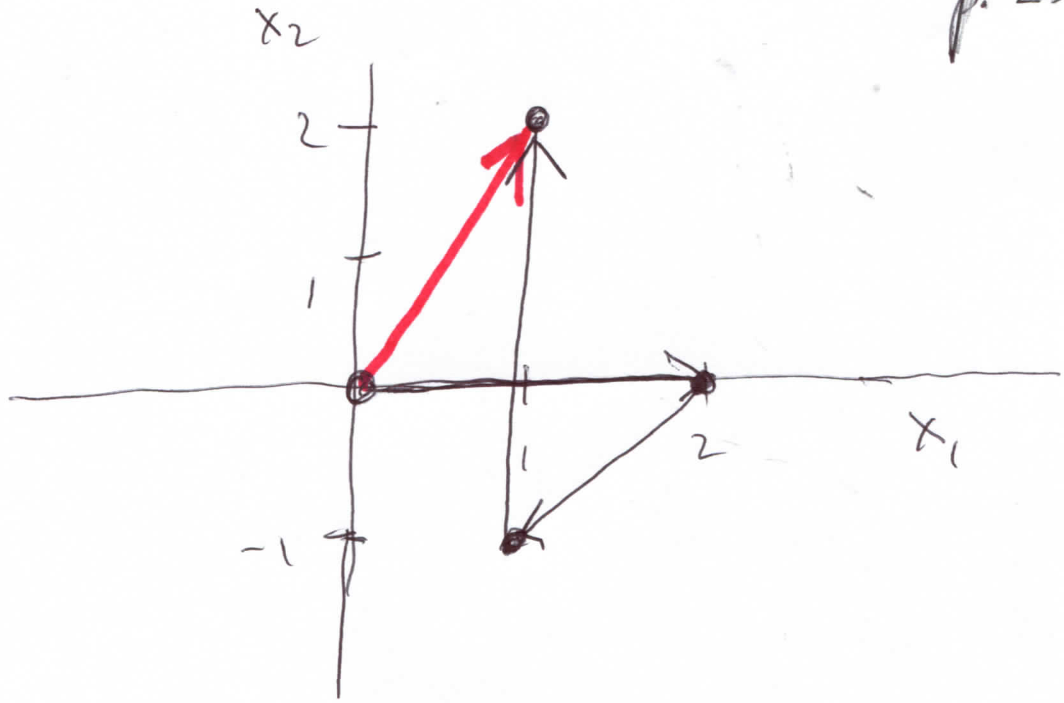
since

$$(1,2) = 2(1,0) - (1,1) + 3(0,1).$$

You should think of this in terms of travelling, with very limited steering options:



in our example, we wish to reach (the terminal point of) $(1,2)$, using only the directions drawn in red above, representing $(1,0)$, $(1,1)$ and $(0,1)$:



Notice that if we write all vectors as columns we get the <u>vector form</u> of a linear system

$$2\begin{bmatrix}1\\0\end{bmatrix} + (-1)\begin{bmatrix}1\\1\end{bmatrix} + 3\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$$

equivalent to (<u>matrix form</u>)

$$\begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\end{bmatrix}\begin{bmatrix}2\\(-1)\\3\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix};$$

that is,

$$(x_1, x_2, x_3) = (2, (-1), 3)$$

is a solution of

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

(vector form (2.16)), or

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

(matrix form (2.15)), or

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_2 + x_3 &= 2 \end{aligned}$$

(primitive form (2.5)).

Examples 4.3

(1) Is $(1,2,3)$ a linear combination of $\{(1,-1,0),(0,1,2),(1,0,2)\}$?

SOLUTION: We'd like to know if there are x_1, x_2, x_3 so that

$$x_1\begin{bmatrix}1\\-1\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\\2\end{bmatrix}+x_3\begin{bmatrix}1\\0\\2\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix};$$

in other words, is the linear system (vector form) we just wrote down consistent?

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Perform the "Gauss" of Gauss-Jordan

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 2 & 3 \end{array}\right] \rightarrow (R_2 + R_1)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 2 & 3 \end{array}\right] \rightarrow (R_3 - 2R_2)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & -3 \end{array}\right]$$ inconsistent, so

$\boxed{\text{no, not a linear combo}}$

(2) Same question for $(4, 1, 10)$. SOLUTION:

Again we have a linear system in vector form

$$x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 10 \end{bmatrix} \rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ -1 & 1 & 0 & 1 \\ 0 & 2 & 2 & 10 \end{array}\right] \rightarrow \cdots$$ (elementary operations as in (1))

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ (echelon form, no "$0 = 1$")

$\rightarrow$ consistent, so $\boxed{\text{yes, is a linear combo}}$

(3) Write (4, 1, 10) as a linear combination of $\{(1,-1,0), (0,1,2), (1,0,2)\}$.

SOLUTION: We must finish (do the "Jordan" of Gauss-Jordan) where we left off in (2).

By sheer luck, we are already in reduced echelon form, so we may put in variables

$$\begin{aligned} x_1 + x_3 &= 4 \\ x_2 + x_3 &= 5 \\ 0 &= 0 \end{aligned}$$

$\rightarrow \quad x_1 = 4 - x_3$

$x_2 = 5 - x_3$

x_3 arbitrary

describes all solutions, thus

$$(4-x_3)\begin{bmatrix}1\\-1\\0\end{bmatrix} + (5-x_3)\begin{bmatrix}0\\1\\2\end{bmatrix} + x_3\begin{bmatrix}1\\0\\2\end{bmatrix}$$

$$= \begin{bmatrix}4\\1\\10\end{bmatrix}, \quad \text{for any real } x_3.$$

Only one linear combo was asked for, so let's let $x_3 = 0$:

$$\boxed{(4,1,10) = 4(1,-1,0) + 5(0,1,2)}$$

In the following, let A be the matrix whose j^{th} column is the (column) vector $\vec{v}_j$, $1 \leq j \leq m$; that is,

$$A \equiv [\vec{v}_1 \, \vec{v}_2 \ldots \vec{v}_m].$$

Then, as in Examples 4.2 and 4.3, the vector form of a linear system relates linear systems to linear combinations as follows, where we have thrown in the equivalence with rank from 3.28.

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THEOREM 4.4

The following are equivalent; that is, if one is true, then the others are also true.

(1) $A\vec{x} = \vec{b}$ is consistent.

(2) $\vec{b}$ is a linear combination of $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\} = \{$columns of $A\}$.

(3) $\text{rank}[\vec{v}_1\ \vec{v}_2\ \ldots\ \vec{v}_m] = \text{rank}[\vec{v}_1\ \vec{v}_2\ \ldots\ \vec{v}_m\ \vec{b}]$.

REMARK 4.5

The equivalence of (2) and (3) in Theorem 4.4 reflects the informal image of rank as information; $\vec{b}$ has no new information if and only if $\vec{b}$ is a linear combination of the vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ already present.

Example 4.6

Is $(0,2,1,1)$ a linear combo of $\{(1,0,1,1), (2,1,2,3), (1,-1,0,1)\}$?

SOLUTION: Get both ranks from Theorem 4.4(3) simultaneously:

$$\operatorname{rank}\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 1 & 2 & 0 & 1 \\ 1 & 3 & 1 & 1 \end{array}\right] = \begin{pmatrix} R_3 - R_1 \\ R_4 - R_1 \end{pmatrix}$$

$$\operatorname{rank}\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \end{array}\right] = (R_4 - R_2)$$

$$\operatorname{rank}\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right] = \begin{pmatrix} R_4 + R_3, \\ \text{then } -R_3 \end{pmatrix}$$

$$\operatorname{rank}\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since $\operatorname{rank}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$

and $\operatorname{rank}\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 1 & 2 & 0 & 1 \\ 1 & 3 & 1 & 1 \end{bmatrix}$ are equal

(both equal 3), Theorem 4.4 implies that $(0,2,1,1)$ is a linear combination of $\{(1,0,1,1), (2,1,2,3), (1,-1,0,1)\}$.

We also showed that

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$ is consistent.

REMARK 4.7

Elementary (row) operations are another example of linear combinations, this time of rows of a matrix.

SECTION IVB: VECTOR SPACES and SPAN

We would like to inhabit a world where vector space activities are safe; that is, they don't propel us out of our world.

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DEFINITION 4.8

If n is a natural number, a subset V of $\mathbb{R}^n$ is a

vector subspace of $\mathbb{R}^n$ or (finite-dimensional) **vector space** for short

if any linear combination of vectors in V is also in V.

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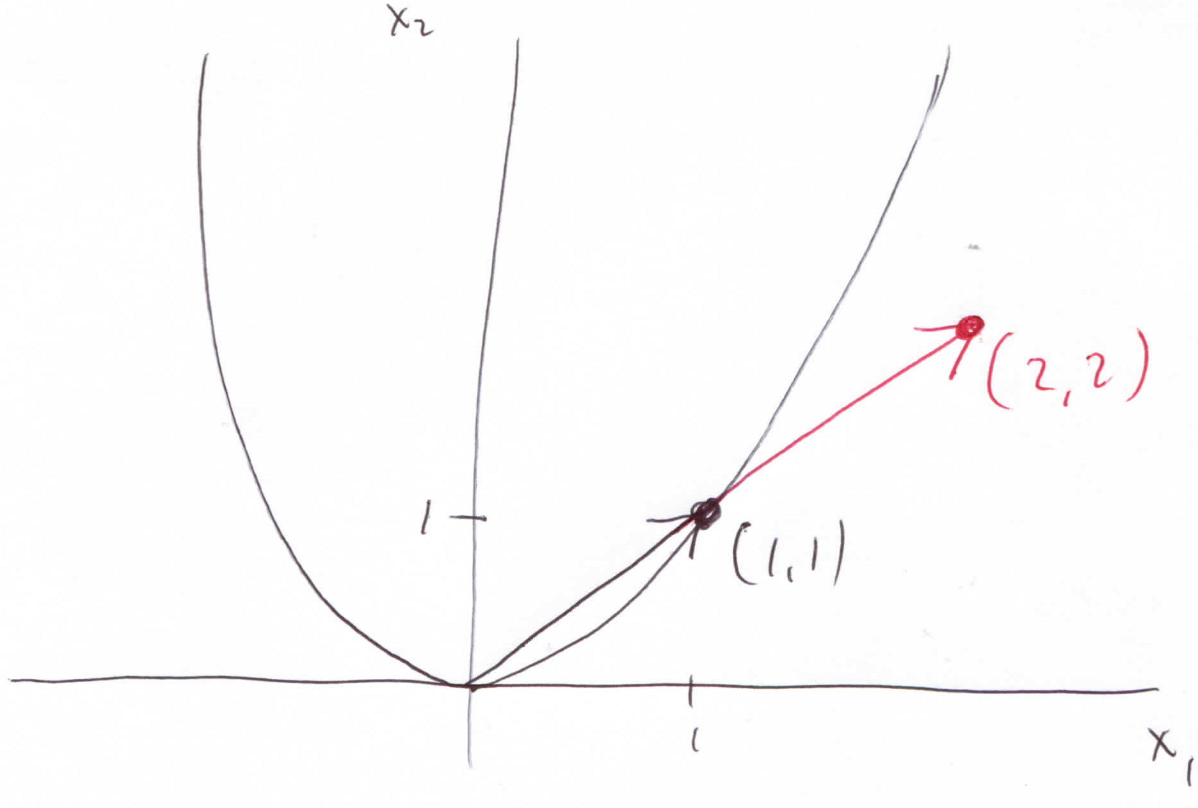
Example 4.9

The parabola $\{(x, x^2) \mid x \text{ is real}\}$ is not a vector space.

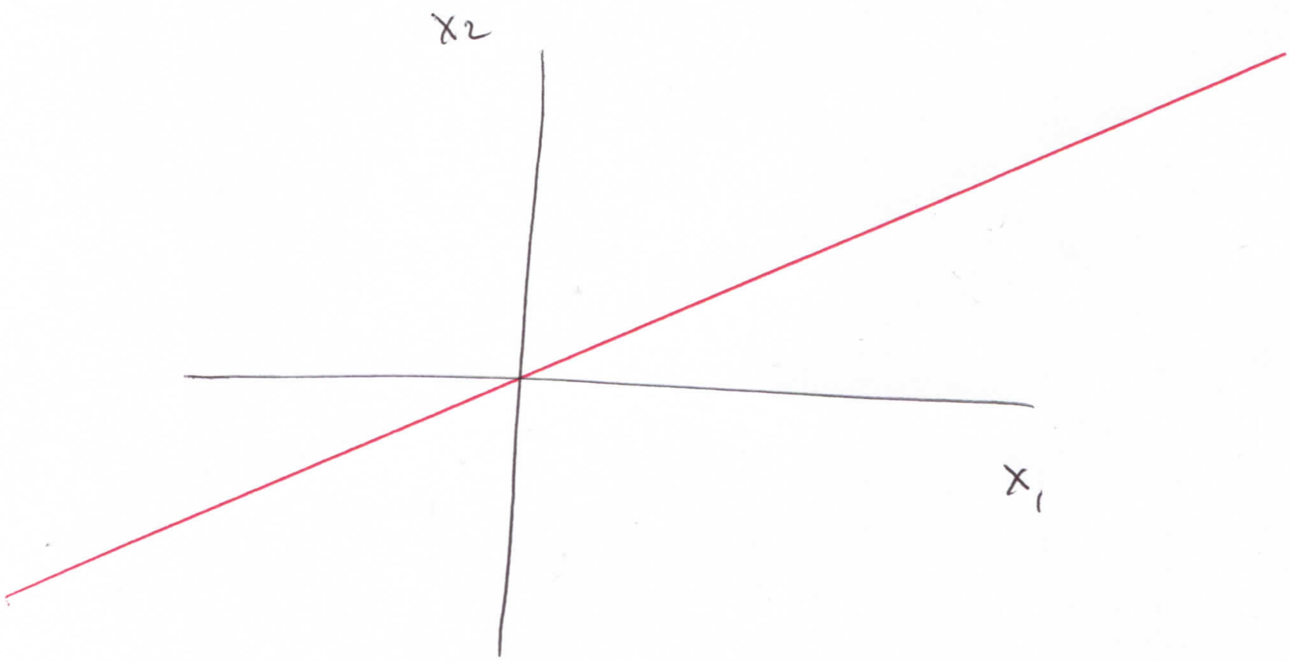
For example, $(1,1)$ is in the set but $2(1,1) = (2,2)$ is not.



Being a vector space is a very restrictive condition.

PROPOSITION 4.10

The only vector subspaces of $\mathbb{R}^2$ are $\{\vec{0}\}$, $\mathbb{R}^2$, and straight lines through $\vec{0}$



The previous section focussed on particular linear combinations. Given a set of vectors, we would like now to define taking <u>all possible</u> linear combinations of said set.

DEFINITIONS 4.11

If S is a set of n-vectors, then

<u>span(S)</u> (**"span of S"**)

$\equiv$ {all linear combinations of finite sets of vectors contained in S}

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If V is a vector space and $\mathrm{span}(S) = V$, we say that

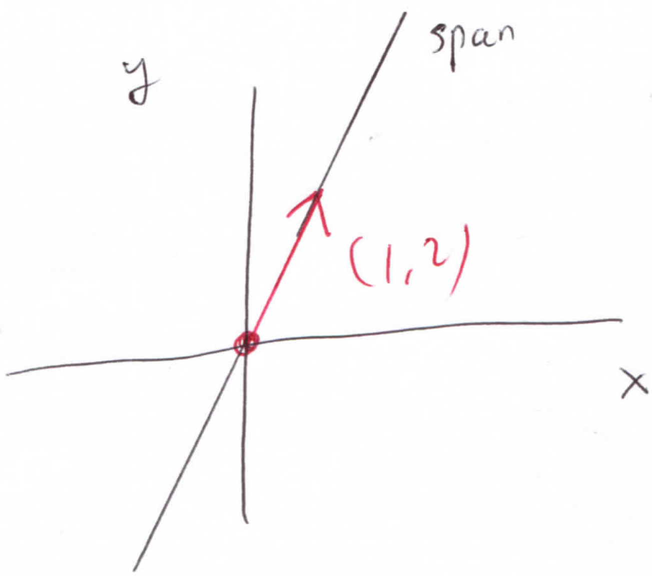
<u>V is spanned by S</u>

or

<u>S spans V.</u>

Examples 4.12

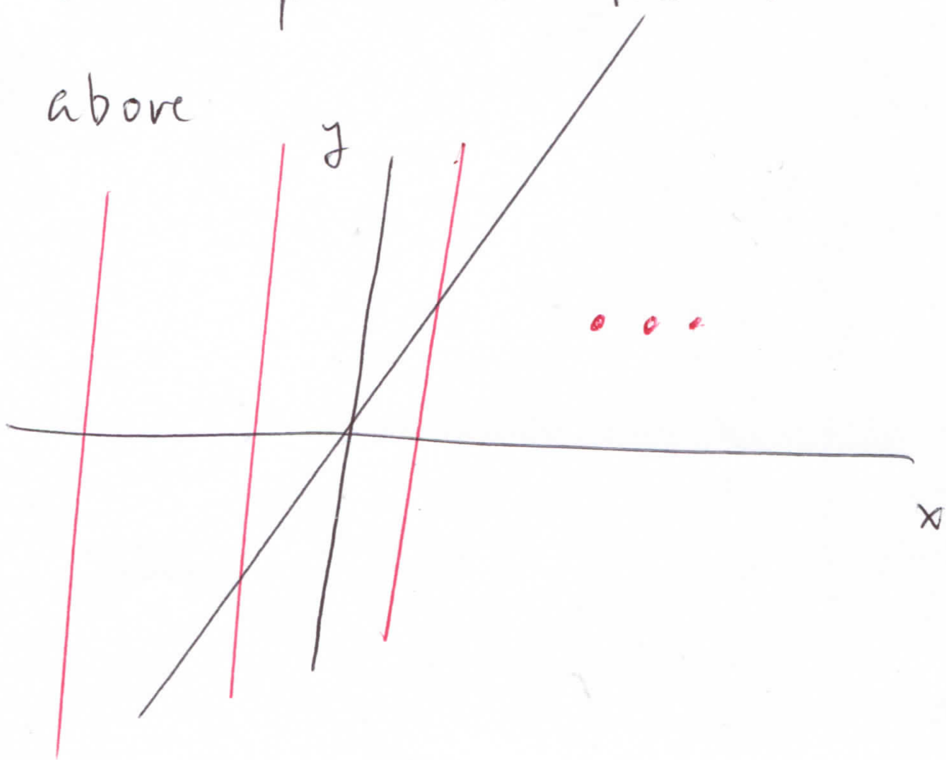
(1) $\mathrm{span}(\{(1,2)\}) =$
$\{x(1,2) \mid x \text{ is real}\} =$
$\{(x, 2x) \mid x \text{ is real}\}$, the graph of $y = 2x$ in $\mathbb{R}^2$



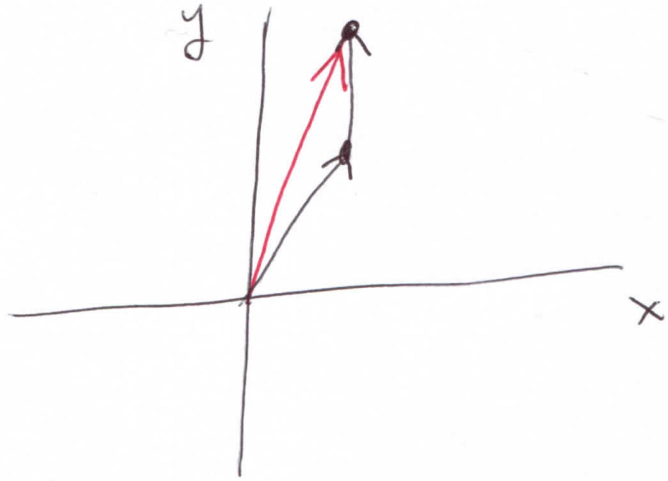
(2) $\mathrm{span}(\{(1,2),\ (0,1)\}) =$

$\{x_1(1,2) + x_2(0,1) \mid x_1,\ x_2 \text{ real}\}$;

this is adding vertical lines to each point on the line in (1) above



Any point in $\mathbb{R}^2$ can be reached this way



thus it appears geometrically that our span is (all of) $\mathbb{R}^2$.

We show this algebraically by solving, for arbitrary $\vec{b} \equiv (b_1, b_2)$,

$$x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 0 & b_1 \\ 2 & 1 & b_2 \end{array}\right] \longrightarrow (R_2 - 2R_1)$$

$$\left[\begin{array}{cc|c} 1 & 0 & b_1 \\ 0 & 1 & b_2 - 2b_1 \end{array}\right] \rightarrow \begin{array}{l} x_1 = b_1 \\ x_2 = b_2 - 2b_1 \end{array};$$

for any $\vec{b}$,

$$b_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (b_2 - 2b_1) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{b},$$

thus $\text{span}(\{(1,2), (0,1)\}) = \mathbb{R}^2$.

INTUITION 4.13 (might be false)

For any $\vec{v}_1, \vec{v}_2$ in $\mathbb{R}^2$,

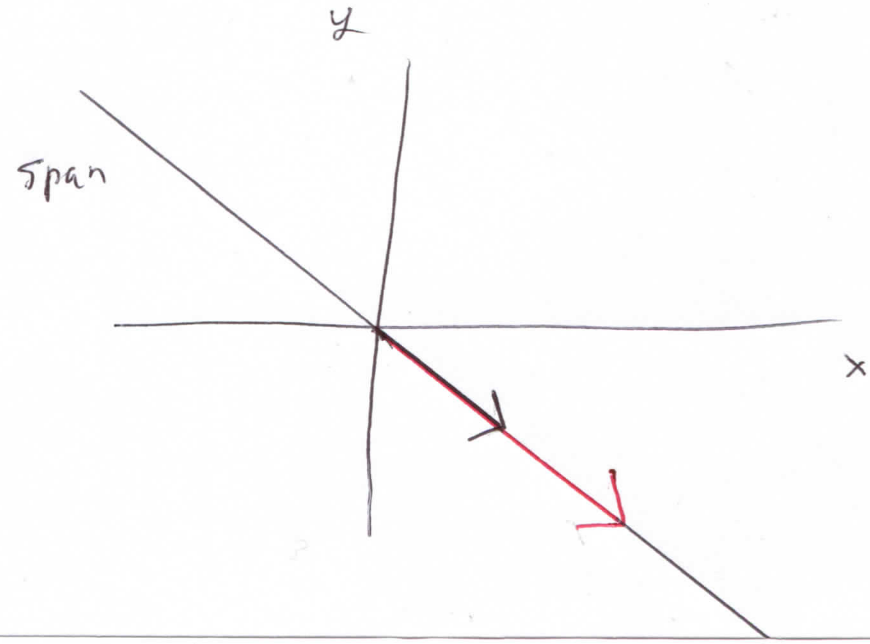
$\text{span}(\{\vec{v}_1, \vec{v}_2\}) = \mathbb{R}^2$

PROPOSITION 4.14

Intuition 4.13 is false, in general.

Proof: Take $\vec{v}_1 \equiv (1,-1)$, $\vec{v}_2 \equiv (2,-2)$. Since $\vec{v}_2$ is a multiple of $\vec{v}_1$,

$\text{span}(\{\vec{v}_1, \vec{v}_2\}) = \text{span}(\{\vec{v}_1\})$
$=$ line through $(0,0)$ and $(1,-1)$



To make 4.13 correct, we need an extra condition on $\{\vec{v}_1, \vec{v}_2\}$, being linearly independent (see next section and Theorem 4.56).

By a different name, we have seen span in Chapter III. For A an $(m \times n)$ matrix, the range space of A we defined as

$$R(A) \equiv \left\{ A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \;\middle|\; x_1, x_2, \ldots \text{ real} \right\}$$

If $A = [\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_n]$,

the matrix whose j^{th} column

is $\vec{v}_j$, $1 \le j \le n$, then

$$R(A) = \left\{ x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_n\vec{v}_m \ \middle| \ x_1, x_2, \dots \text{ are real} \right\}$$

(see vector form versus matrix form of a linear system).

Here's the relationship between range space and span.

Theorem 4.15

$$R([\vec{v}_1 \ \vec{v}_2 \ \dots]) = \operatorname{span}(\{\vec{v}_1, \vec{v}_2, \dots\})$$

for any vectors $\vec{v}_1, \vec{v}_2, \dots$

Example 4.16

$$R\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}\right) =$$

$$\left\{ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \;\middle|\; x_1, x_2, x_3 \text{ real} \right\} =$$

$$\left\{ x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \;\middle|\; \begin{array}{l} x_1, x_2, x_3 \\ \text{real} \end{array} \right\}$$

$$= \operatorname{span}\left(\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\} \right)$$

$$= \operatorname{span}\left(\left\{ (1,0,0,-1), (0,1,1,1), (1,1,1,0) \right\} \right)$$

POPULAR VECTOR SPACES 4.17

We will look at vector spaces of the following form, where A is an $(m \times n)$ matrix.

(1) $\mathcal{N}(A) \equiv$ null space of A $\equiv \{ \vec{x} \text{ in } \mathbb{R}^n \mid A\vec{x} = \vec{0} \}$ (see Chapter III).

(2) $\text{span}\{ \vec{v}_1, \vec{v}_2, \ldots \}$, for a fixed set of vectors $\{ \vec{v}_1, \vec{v}_2, \ldots \}$. By Theorem 4.15, this includes $\mathcal{R}(A)$ from Chapter III.

Two other (seemingly) special cases of (2):

<u>row</u> <u>space</u> of A

$\equiv$ span{ rows of A }

<u>column</u> <u>space</u> of A

$\equiv$ span{ columns of A }.

By Theorem 4.15, column space is another name for range space.

GOALS 4.18

(a) Describe subspaces efficiently (as the span of as small a set as possible).

(b) Define dimension of subspaces.

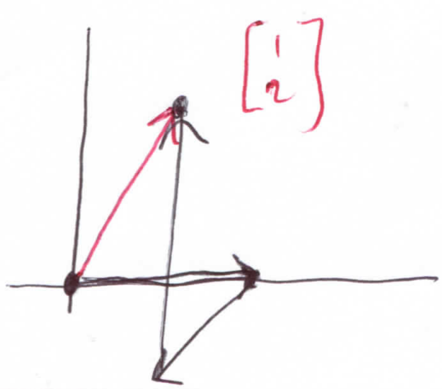
(a) and (b) are intimately related: dimension will be the number of vectors in a smallest spanning set as in (a).

SECTION IVC

LINEAR DEPENDENCE

Please go back to Example 4.2, especially the travelling picture on page 234, representing the linear combination

$$2\begin{bmatrix}1\\0\end{bmatrix}+(-1)\begin{bmatrix}1\\1\end{bmatrix}+3\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}.$$

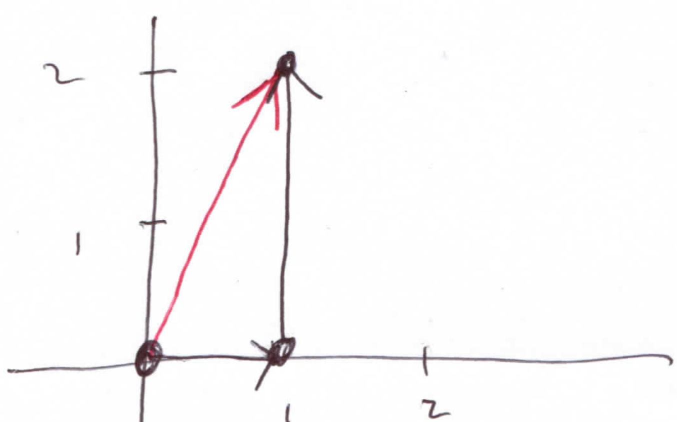


As instructions for getting from $(0,0)$ to $(1,2)$, that linear combination of

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

leaves much to be desired. Much better instructions would be

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$



writing (1,2) as a linear combination of

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

The gratuitous complication in our initial linear combination was possible, because we didn't need the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The reason we don't need $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, when taking linear combinations of $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$, is that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is itself a linear combination of the other two vectors.

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DEFINITION 4.19

A set of vectors is

linearly dependent

if one vector is a linear combination of the others. Otherwise, the set is

linearly independent

Example 4.20

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

is linearly dependent, since

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The intuition of linear dependence is that you have more vectors than you need.

Theorem 4.4 implies the following rank characterization.

THEOREM 4.21

$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly dependent if and only if $\operatorname{rank}\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} < n$.

Examples 4.22

(1) $\operatorname{rank}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = 2 < 3$,

thus $\{(1,0), (1,1), (0,1)\}$ is linearly dependent.

$\{(1,0), (1,1)\}$ is linearly independent, since

$$\operatorname{rank}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = 2 = \left(\begin{array}{c}\text{number of} \\ \text{vectors}\end{array}\right)$$

(2) Is $\{(1,1,2), (1,-1,0), (-1,0,-1)\}$ linearly dependent?

SOLUTION:

$$\text{rank}\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix} = \begin{pmatrix} R_2 - R_1 \\ R_3 - 2R_1 \end{pmatrix}$$

$$\text{rank}\begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 1 \\ 0 & -2 & 1 \end{bmatrix} = \begin{pmatrix} R_3 - R_2, \text{ then} \\ -\frac{1}{2} R_2 \end{pmatrix}$$

$$\text{rank}\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} = 2 < 3 = \begin{pmatrix} \text{number} \\ \text{of vectors} \end{pmatrix}$$

so yes, linearly dependent

Here is an equivalent definition of linearly dependent.

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PROPOSITION 4.23

A set of vectors

$\{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \}$

is linearly dependent if and only if there is a nontrivial linear combination that equals $\vec{0}$; that is, $(x_1, x_2, \ldots, x_n) \neq (0, 0, \ldots)$

so that

$$(x_1 \vec{v}_1 + x_2 \vec{v}_2 + \cdots + x_n \vec{v}_n) = \vec{0}$$

Example 4.24 $\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

thus (again) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is linearly dependent.

Notice that the final line of Proposition 4.23 is the vector form of the homogeneous linear system, in matrix form,

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \vec{0}$$

The existence of a nontrivial solution defines $[\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_n]$ being a <u>singular</u> matrix (Def. 3.38).

Theorems 3.39 and 4.21 imply the following.

THEOREM 4.25

For A a matrix, the following are equivalent; that is, if one is true, the others are true.

(1) A is singular

(2) rank $A <$ number of columns of A

(3) {columns of A} are linearly dependent.

Examples 4.26

(1) In Examples 4.22(2) we showed, by calculating

$$\operatorname{rank}\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix} = 2,$$

that $\{(1,1,2), (1,-1,0), (-1,0,-1)\}$ is linearly dependent.

This also shows that $\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix}$ is singular; that is,

$$(*)\quad \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

for some nontrivial $\vec{x} \equiv (x_1, x_2, x_3)$

Let's check this by solving (*):

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 2 & 0 & -1 & 0 \end{array}\right] \to \begin{pmatrix} R_2 - R_1 \\ R_3 - 2R_1 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & 1 & 0 \end{array}\right] \to \begin{pmatrix} R_3 - R_2, \text{ then} \\ -\frac{1}{2} R_2 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to (R_1 - R_2)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \begin{aligned} x_1 \quad - \tfrac{1}{2} x_3 &= 0 \\ x_2 - \tfrac{1}{2} x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

$$\to \{\text{solutions}\} = \left\{ \begin{bmatrix} \frac{1}{2} x_3 \\ \frac{1}{2} x_3 \\ x_3 \end{bmatrix} \;\middle|\; x_3 \text{ arbitrary} \right\};$$

in particular, we can get nontrivial solutions; e.g., letting $x_3 \equiv 2 \rightarrow$

$(x_1, x_2, x_3) = (1, 1, 2)$ is a nontrivial solution of $(*)$:

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \vec{0} \rightarrow$$

$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix}$ is singular;

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} = \vec{0} \rightarrow$$

$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} \right\}$ is linearly dependent.

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(2) Suppose $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Is $\{(1,0,1,2), (1,1,0,1), (1,-1,2,3)\}$ linearly independent?

NO, we know that $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \\ 2 & 1 & 3 \end{bmatrix}$ is singular, so

{columns} are linearly dependent (Theorem 4.25).

We can write down the nontrivial linear combo that equals $\vec{0}$!

$$2\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} + (-1)\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(3) Believe that

$$\operatorname{rank}\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = 3.$$

(a) Is $\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$ singular?

(b) Is $\{(1,-1,2,0), (1,0,1,2), (1,1,0,1)\}$ linearly dependent?

SOLUTION: (a) & (b) are equivalent, by Theorem 4.25; (2) of Theorem 4.25 tells us the answer to both (a) and (b) is

no

(4) Can a set of 6 5-vectors be linearly independent?

SOLUTION: Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_6\}$ be a set of 6 5-vectors.

Then $A \equiv [\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_6]$ is a (5×6) matrix, which we have shown in Examples 3.40 (3) (with rank) cannot be nonsingular; by Theorem 4.25, the answer to our question is no

(5) Can a set of 5 6-vectors be linearly independent?

SOLUTION: Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_5\}$ be a set of 5 6-vectors;

$$A \equiv [\vec{v}_1 \; \vec{v}_2 \; \cdots \; \vec{v}_5]$$

is then a (6×5) matrix, which we have shown in Examples 3.40(4) can be nonsingular; the example given there was

$$A \equiv \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$ which has columns

$$\left\{ \vec{v}_1 \equiv \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 \equiv \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_3 \equiv \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_4 \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_5 \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

forming a linearly independent set of 5 6-vectors.

ANSWER: SURE

The same reasoning used in Examples 4.26(4) implies the following corollary of Theorem 4.25

COROLLARY 4.27

If $n > m$, then any set of n m-vectors is linearly dependent.

Proof: Denote by

$$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$$

a set of n m-vectors.

Then

$$A \equiv \left[\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_n \right]$$

is an $(m \times n)$ matrix; since

$$\operatorname{rank}(A) \leq m < n = \left(\begin{array}{c} \text{number of} \\ \text{columns} \\ \text{of } A \end{array} \right),$$

Theorem 4.25 concludes the proof.