SECTION

IVD:

BASIS

We have introduced in this chapter the ideas of <u>span</u> and <u>linear dependence</u>. Here is a relationship between them.

THEOREM 4.28

A set of vectors is linearly dependent if and only if a vector may be removed without changing the span.

Example 4.29

$$\text{span}\left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \right\} = \mathbb{R}^2$$

$$= \text{span}\left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \right\},$$

thus $\left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \right\}$

is linearly dependent.

Theorem 4.28 expresses our intuition about linearly dependent sets as having <u>too</u> <u>many</u> vectors.

For our goal (4.18(a)) of writing a vector space as the span of as few vectors as possible, Theorem 4.28 suggests that we add linear independence to our requirement of spanning.

DEFINITION 4.30

A **basis** for a vector space V is a linearly independent set that span V.

Examples 4.31

$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$ (we leave it to the reader to separately verify spanning and linear independence).

$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is not a basis for anything, since it's linearly dependent.

$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ is not a basis for $\mathbb{R}^2$, because it does not span $\mathbb{R}^2$.

Our goal in this chapter is to construct bases for our favorite vector spaces (see 4.17).

In Examples 4.31,

we mentioned the basis

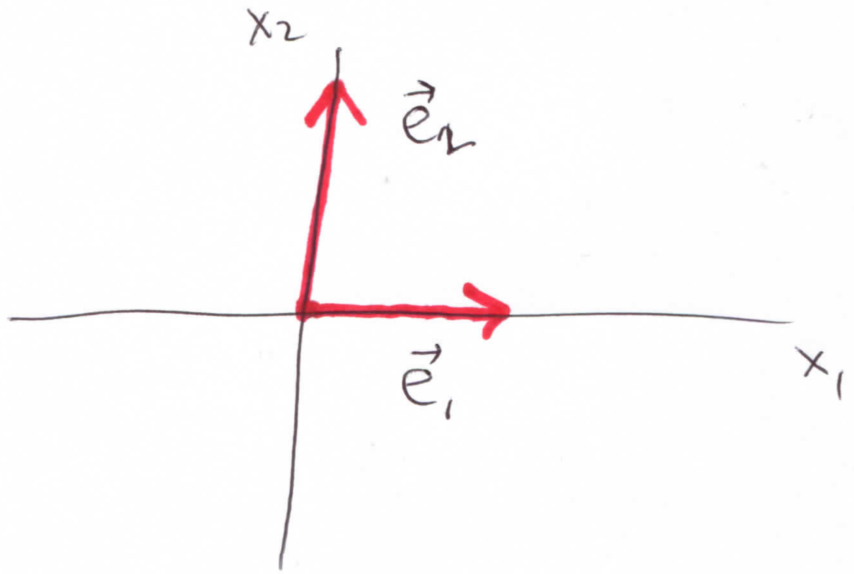
$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^2$.

This basis is called the

standard basis for $\mathbb{R}^2$,

with basis vectors denoted

$$\vec{e}_1 \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{e}_2 \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



DEFINITION 4.32

For any natural number n, the **standard basis** for $\mathbb{R}^n$ is $\{\vec{e}_1, \vec{e}_2, \vec{e}_3, \ldots, \vec{e}_n\}$, where $\vec{e}_1 \equiv (1, 0, 0, 0, \ldots)$, $\vec{e}_2 \equiv (0, 1, 0, 0, \ldots)$, etc.; for $1 \le i \le n$,

$\vec{\mathbf{e}}_i$ is the vector whose i^{th} component is 1, all other components are 0.

When we think of

$$\mathbb{R}^n \equiv \left\{ (x_1, x_2, \ldots, x_n) \;\middle|\; \begin{matrix} x_i \text{ real}, \\ 1 \le i \le n \end{matrix} \right\}$$

as defined by the <u>free</u> <u>variables</u> of the components $x_1, x_2, x_3, \ldots, x_n$, we will see how to extend the standard basis construction to any vector space defined by free variables, such as null spaces.

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4.33 CONSTRUCTION OF BASIS FOR NULL SPACE.

For A a matrix, let

$$V \equiv \mathcal{N}(A),$$

the null space of A.

To get a basis for V:

1. Use Gauss-Jordan to solve

$$A\vec{x} = \vec{0}$$

2. Denoting by

$y_1, y_2, \dots y_K$

the resulting free variables

in

$\{$ solutions of $A\vec{x} = \vec{0}\} \equiv \mathcal{N}(A)$,

let

$\vec{V}_1 \equiv$ the solution when
$y_1 = 1, \quad 0 = y_2 = y_3 = \dots = y_K$

$\vec{V}_2 \equiv$ the solution when
$y_2 = 1, \quad 0 = y_1 = y_3 = y_4 = \dots = y_K$

$\vdots$

Then $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is a basis for $\mathcal{N}(A)$.

Examples 4.34. In each of the following, get a basis for $\mathcal{N}(A)$.

(1) $A = \begin{bmatrix} 1 & -2 & -3 \end{bmatrix}$.

$$\mathcal{N}(A) = \left\{ (x_1, x_2, x_3) \;\middle|\; \begin{bmatrix} 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \right\}$$

The augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \end{array}\right]$$

is already in reduced echelon form, so leave variables in:

$$x_1 - 2x_2 - 3x_3 = 0 \rightarrow$$

$$x_1 = 2x_2 + 3x_3$$

x_2, x_3 arbitrary (free variables)

$$\left.\begin{array}{l} x_2 = 1 \\ x_3 = 0 \end{array}\right\} \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \equiv \vec{v}_1$$

$$\left.\begin{array}{l} x_2 = 0 \\ x_3 = 1 \end{array}\right\} \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \equiv \vec{v}_2$$

$$\boxed{\text{basis is } \{(2,1,0), (3,0,1)\}}$$

When you write the solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 + 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix},$$

basis

the basis jumps out at you, with an explicit linear combination of the basis vectors.

(2) $A = \begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -2 & 0 & 1 \\ 2 & -1 & 0 & 2 & -1 \end{bmatrix}$.

We must do some work, solving $A\vec{x} = \vec{0}$:

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 2 & -1 & 0 & 2 & -1 & 0 \end{array}\right] \longrightarrow (R_3 - 2R_1)$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & -1 & 0 \end{array}\right] \longrightarrow (R_3 + R_2)$$

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$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left(\text{already reduced}\right)$$

$$\begin{aligned} x_1 \quad - x_3 + x_4 \quad &= 0 \\ x_2 - 2x_3 \quad + x_5 &= 0 \end{aligned} \rightarrow$$

$$\begin{aligned} x_1 &= x_3 - x_4 \\ x_2 &= 2x_3 \quad - x_5 \end{aligned} \qquad \left(x_3, x_4, x_5 = \text{free variables}\right)$$

OR

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_3 - x_4 \\ 2x_3 - x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} =$$

$$x_3 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

basis

basis is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Now for our other favorite vector space (see 4.17).

4.35 CONSTRUCTION OF BASIS FOR a SPAN

For vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, let

$$V \equiv \operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$$

To get a basis for V:

1. Make $\vec{v}_1, \vec{v}_2, \ldots$ the <u>rows</u> of a matrix

$$B \equiv \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \end{bmatrix}$$

2. Perform elementary (row) operations on B, to put it into echelon form

3. basis = {nontrivial rows} left after Step 2.

Examples 4.36

(1) Find a basis for

span$\{(1,0,1,2), (2,1,0,3), (1,1,-1,1)\}$.

<u>SOLUTION</u> Make each vector the row of a matrix and perform the "Gauss" of Gauss Jordan!

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 3 \\ 1 & 1 & -1 & 1 \end{bmatrix} \rightarrow \begin{pmatrix} R_2 - 2R_1 \\ R_3 - R_1 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 1 & -2 & -1 \end{bmatrix} \rightarrow (R_3 - R_2)$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow$$

basis

$\{(1,0,1,2), (0,1,-2,-1)\}$

For the remaining examples, let

$$A = \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 2 & 0 \\ 1 & -1 & -1 & 1 & -1 & 1 \end{bmatrix}.$$

(2) Find a basis for the row space of A.

$$\begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 2 & 0 \\ 1 & -1 & -1 & 1 & -1 & 1 \end{bmatrix} \to (R_5 - R_1)$$

$$\begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 2 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 \end{bmatrix} \to \begin{pmatrix} R_3 - R_2 \\ R_4 - R_2 \\ R_5 + R_2 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow (R_4 + R_3)$$

$$\begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rightarrow$ basis $\{(1,0,-1,1,0,1),\ (0,1,0,0,1,0),\ (0,0,1,1,-1,0)\}$

(3) Find a basis for the column space of A.

Make each column a row, then Gauss away:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ -1 & 0 & 1 & -1 & -1 \\ 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 2 & -1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{pmatrix} R_3 + R_1 \\ R_4 - R_1 \\ R_6 - R_1 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to (R_5 - R_2),$$

then do

$$\begin{pmatrix} R_4 - R_3 \\ R_5 + R_3 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

basis $\left\{ \begin{array}{l} (1,0,0,0,1), \\ (0,1,1,1,-1), \\ (0,0,1,-1,0) \end{array} \right\}$

(4) Find a basis for the range space of A.

SAME as (3) (see 4.17)

SECTION

$\boxed{\text{IV}}$ E:

DIMENSION

The two defining qualities of a basis act in opposition to each other. To span a vector space we like more vectors, while for linear independence we like fewer vectors; more precisely, adding vectors cannot decrease span, while removing them might, and a linearly independent set is still linearly independent if vectors are removed, but might lose its independence if vectors are added

Given these conflicting demands, it is surprising that bases exist at all; if, for a given vector space, they do exist, we expect them to have something in common,

THEOREM 4.37

If V is a vector space, then any two bases for V have the same number of vectors.

Example 4.38

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \text{ and } \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

are both bases for $\mathbb{R}^2$; each of them consists of two vectors.

Theorem 4.37 makes the following definition unambiguous.

DEFINITION 4.39

The **dimension** of a vector space V, denoted

$$\dim(V),$$

is the number of vectors in a basis for V.

Example 4.40

For any natural number n, $\dim(\mathbb{R}^n) = n$, since its standard basis

$$\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$$

has n vectors.

We want dimensions of our vector spaces in 4.17. We will begin with

$$V \equiv \operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\},$$

as in 4.35.

The first two steps in 4.35 for constructing a basis for V are the first two steps in calculating rank; the third step for calculating rank is to count the number of nontrivial rows. By the third step of 4.35, that's counting the number of vectors in the basis for V, that is, dimension. In Examples 4.36 (1), the rank of the relevant matrix is two, the number of basis vectors.

We are hinting at the following (see Rank Properties 3.30).

THEOREM 4.41

$$\dim\left(\operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}\right) =$$

$$\operatorname{rank}\begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_n \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}.$$

Example 4.42

Get the dimension of the span of $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix} \right\}$.

SOLUTION: We need a rank calculation:

$$\operatorname{rank} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & -3 & 2 & -3 \end{bmatrix} = \begin{pmatrix} R_2 - R_1 \\ R_3 - 2R_1 \end{pmatrix}$$

$$\operatorname{rank} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & -5 & 0 & -5 \end{bmatrix} = (R_3 + 5R_2)$$

$$\operatorname{rank} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 2$$

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OR we could have calculated

$$\operatorname{rank}\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & -3 \\ 1 & 1 & 2 \\ 1 & 2 & -3 \end{bmatrix} = \begin{pmatrix} R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{pmatrix}$$

$$\operatorname{rank}\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \\ 0 & 1 & -5 \end{bmatrix} = (R_4 - R_2)$$

$$\operatorname{rank}\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 2.$$

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COROLLARY 4.43

For any matrix A,

$$\dim(R(A)) = \text{rank}(A) = \dim(R(A^T)).$$

For our other favorite vector space $\mathcal{N}(A) \equiv$ null space of the matrix A, we have special terminology for dimension.

DEFINITION 4.44

If A is a matrix, the

nullity of $A \equiv \dim(\mathcal{N}(A))$.

As with our construction of a basis for $\mathcal{N}(A)$ in 4.33, our focus in on the free variables in $\mathcal{N}(A) \equiv \{\vec{x} \mid A\vec{x} = \vec{0}\}$. The following is consistent with $\dim(\mathbb{R}^n) = n$ (see 4.40) and, along with 4.33 and 4.32, expresses our intuition about a vector space with n

free variables being like $\mathbb{R}^n$.

THEOREM 4.45

For any matrix A, nullity of $A \equiv \dim(\mathcal{N}(A))$ equals the number of free variables in $\{\text{solutions of } A\vec{x}=\vec{0}\} \equiv \mathcal{N}(A)$.

Example 4.46 Let

$$A = \begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -2 & 0 & 1 \\ 2 & -1 & 0 & 2 & -1 \end{bmatrix}.$$

In Example 4.34(2) we calculated $\mathcal{N}(A)$ to be

$$\left\{ x_3 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; \begin{matrix} x_3, x_4, \\ x_5 \\ \text{real} \end{matrix} \right\}$$

with the three free variables x_3, x_4, x_5, thus $\dim(\mathcal{N}(A)) = 3$.

Note that our basis for $\mathcal{N}(A)$ is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\},$$

containing three vectors.

Combining Theorems 4.45 and 3.28 gives us the following relatively simple way of calculating nullity.

THEOREM 4.47

For any matrix A,

$$\dim(\mathcal{N}(A)) = \begin{pmatrix} \text{number} \\ \text{of columns} \\ \text{of } A \end{pmatrix} - \text{rank}(A)$$

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Example 4.48

Get the nullity of

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 & 3 & 2 \end{bmatrix}.$$

SOLUTION: $\text{rank}(A) = \begin{pmatrix} R_2 - R_1 \\ R_3 - R_1 \end{pmatrix}$

$$\text{rank} \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 2 & 2 \end{bmatrix} = (R_3 - 2R_2)$$

$$\text{rank} \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = 2$$

$\rightarrow$ $\boxed{\dim(\mathcal{N}(A)) = 6 - 2 = 4}$

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REMARK 4.49

For getting a basis for $\mathcal{N}(A)$, you must solve

$$(*) \quad A\vec{x} = \vec{0}.$$

If you only want the dimension of $\mathcal{N}(A)$, you do not need to solve the linear system $(*)$; we recommend you then use Theorem 4.47.

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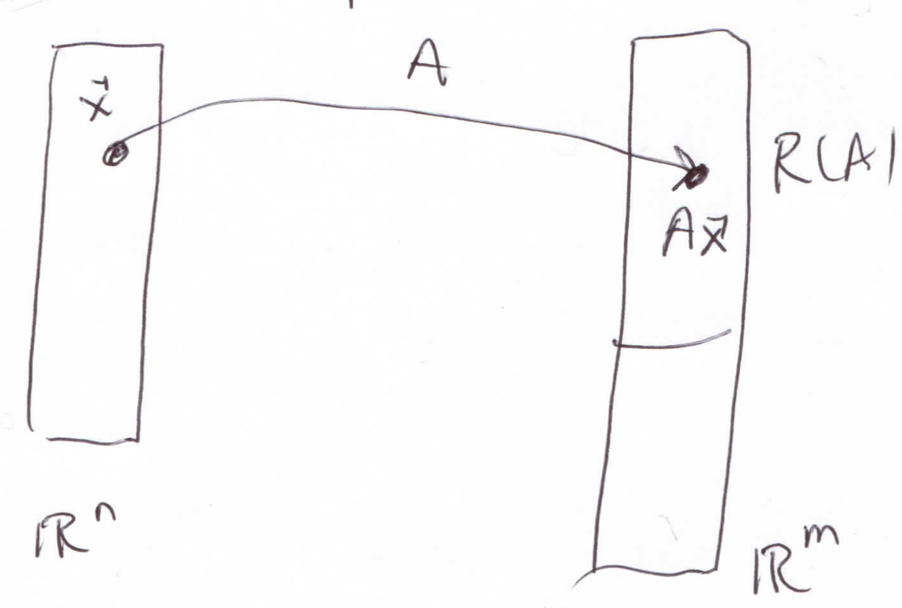
By way of putting 3.28 and Theorem 4.47 in perspective and making them more believable, let's state Theorem 4.47 entirely in terms of dimensions of vector spaces

COROLLARY 4.50

If A is an $(m \times n)$ matrix, then

$$\underset{\overset{\uparrow}{n}}{\dim(\mathbb{R}^n)} = \underset{\overset{\uparrow}{(n-r)}}{\dim(\mathcal{N}(A))} + \underset{\overset{\uparrow}{r \equiv \operatorname{rank}(A)}}{\dim(\mathcal{R}(A))}.$$

Return now to the discussion and pictures after the definition of the range space $\mathcal{R}(A)$ (3.43); in particular, we think of multiplication by A as changing a vector $\vec{x}$ in $\mathbb{R}^n$ to a vector $A\vec{x}$ in $\mathbb{R}^m$; more precisely, $A\vec{x}$ is in the subspace $\mathcal{R}(A)$ of $\mathbb{R}^m$.



As in the discussion after (3.43), first let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We leave it to the reader to show that

$$\mathcal{N}(A) = \left\{ \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} \;\middle|\; x_3 \text{ is real} \right\},$$

$$\mathcal{R}(A) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{bmatrix} \;\middle|\; x_1, x_2 \text{ are real} \right\},$$

$$\mathcal{R}(A^T) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} \;\middle|\; x_1, x_2 \text{ are real} \right\}$$

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Since A is (4×3),
A changes vectors $\vec{x}$ in $\mathbb{R}^3$
to vectors $A\vec{x}$ in $\mathbb{R}^4$. Let's
focus on bases for $\mathcal{N}(A)$,
$\mathcal{R}(A^T)$, and $\mathcal{R}(A)$:

span $= \mathcal{R}(A^T)$; dim $= 2$ $\equiv r$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

span $= \mathcal{R}(A)$; dim $= 2$ $\equiv r$

span $= \mathcal{N}(A)$; dim $= 1$ $= (3-2)$ $\equiv (n-r)$

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dashrightarrow \right.$$

$$\mathbb{R}^3 \xrightarrow{\quad A \quad} \mathbb{R}^4$$

$(n=3)$ $\qquad$ $(m=4)$

The basis

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

for $\mathbb{R}^3$ is the sum of two pieces:

$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ (For $\mathcal{N}(A)$) dies

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ (for $\mathcal{R}(A^T)$) mutates into $\mathcal{R}(A)$

$$3 = \dim(\mathbb{R}^3) = \underset{\text{dies}}{1} + \underset{\text{mutates}}{2}$$

$$= \dim(\mathcal{N}(A)) + \dim(\mathcal{R}(A))$$

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IN GENERAL,

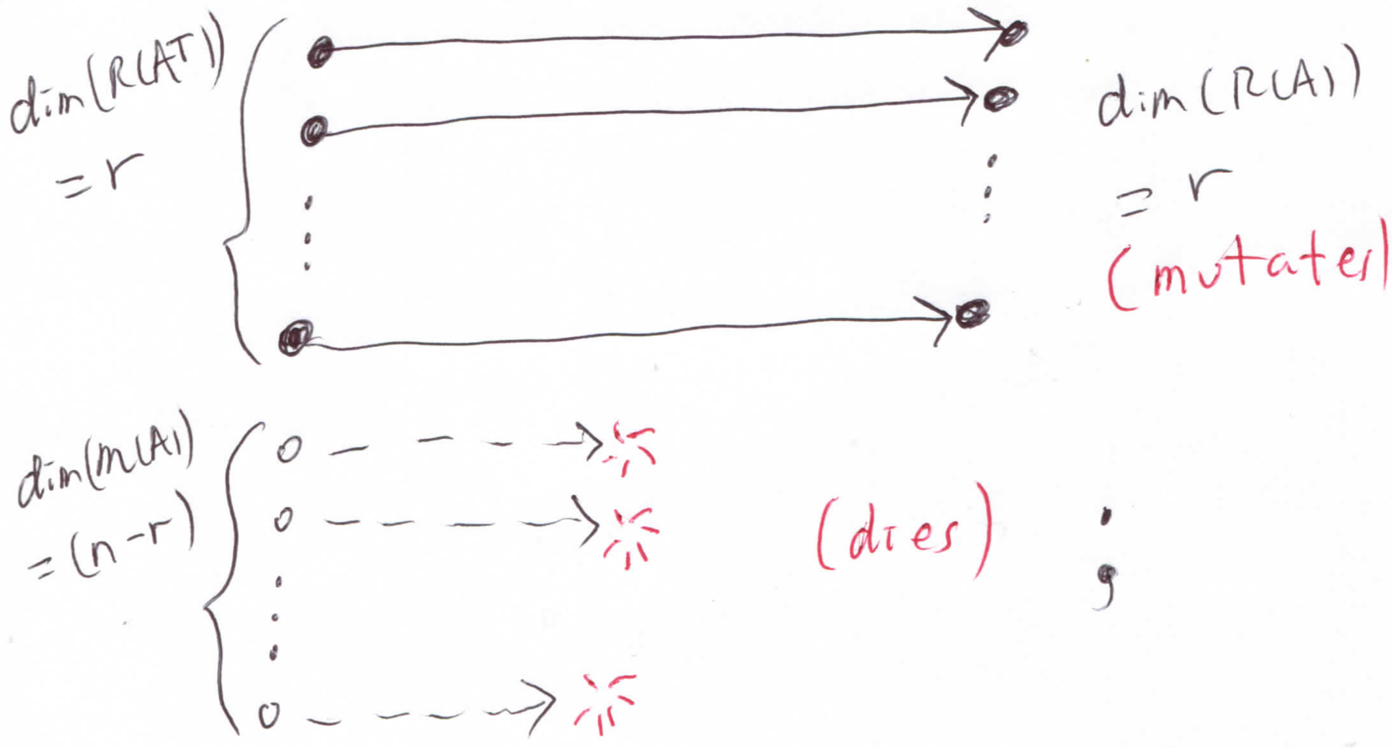
if A is $(m \times n)$, let

$r \equiv \text{rank}(A)$

$$\mathbb{R}^n \xrightarrow{A} R(A) \left(\begin{array}{c} \text{contained} \\ \text{in } \mathbb{R}^m \end{array} \right)$$

breaks down into bases for

$N(A)$, $R(A^T)$, $R(A)$ as follows



;

$$n = \dim(\mathbb{R}^n) = \underbrace{(n-r)}_{\text{dies}} + \underbrace{r}_{\text{mutates}}$$

$$= \underbrace{\dim(\mathcal{N}(A))}_{\text{dies}} + \underbrace{\dim(\mathcal{R}(A))}_{\text{mutates}}$$

Everything dies or mutates.

The relationship between $\mathcal{R}(A^T)$ and $\mathcal{N}(A)$ will be made clear in Theorem 6.51(h).

BASIS, SPAN, and LINEAR INDEPENDENCE 4.51

A basis is both

a minimal {spanning vectors}

and

a maximal {linearly independent vectors}

Example 4.52

A basis for $\mathbb{R}^2$ may be controcted by beginning with a spanning set, then removing vectors until linear independence is achieved!

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \rightarrow \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

too many – linearly dependent

OR, begin with a linearly independent set and add on vectors that aren't in the span of the prior vectors, until spanning is achieved:

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \rightarrow \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

too few –
doesn't span $\mathbb{R}^2$

Since dimension is merely the number of vectors in a basis, 4.51 leads to the following characterization of dimension.

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DIMENSION, SPAN and LINEAR INDEPENDENCE

4.53

If V is a vector space, then $\dim(V)$ is both

(1) The maximum number of vectors in linearly independent subsets of V; and

(2) The minimum number of vectors required to span V.

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More informally:

(4.54)

$$\begin{pmatrix}\text{number of} \\ \text{vectors in} \\ \text{linearly} \\ \text{independent} \\ \text{subset}\end{pmatrix} \le \dim \le \begin{pmatrix}\text{number of} \\ \text{vectors in} \\ \text{spanning} \\ \text{set}\end{pmatrix}$$

Examples 4.55

1. What can be said about $\dim(V)$, if

(a) V contains {5 linearly ind. vectors}

(b) V equals span {9 vectors} ?

SOLUTIONS:

(a) $\dim(V) \geq 5$

(b) $\dim(V) \leq 9$

2. If $\dim(V) = 15$, what can be said about

(a) {20 vectors in V}

(b) {12 vectors in V}.

SOLUTIONS:

(a) must be linearly dependent, since $20 > 15$

(b) can't span V, since $12 < 15$.

If we know that we have the correct <u>number</u> of vectors for a basis, it is easier to determine if we have a basis.

THEOREM 4.56

Suppose W is a vector space and $p \equiv \dim(W)$. If $S \equiv \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_p\}$ is contained in W, then the following are equivalent.

(1) S is a basis for W.

(2) S is linearly independent

(3) S spans W

(4) $\text{rank}\left[\vec{w}_1\ \vec{w}_2 \cdots \vec{w}_n\right] = \underline{p}$.

Examples 4.57

(1) Is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\}$

a basis for $\mathbb{R}^4$?

SOLUTION: Since we have 4 vectors and $\dim(\mathbb{R}^4) = 4$,

we look at

$$\text{rank}\begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 3 \\ 1 & 0 & 1 & 4 \end{bmatrix} = \begin{pmatrix} R_3 - R_1 \\ R_4 - R_1 \end{pmatrix}$$

$$\text{rank}\begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \end{bmatrix} = \begin{pmatrix} R_3 - R_2 \\ R_4 - R_2 \end{pmatrix}$$

$$\text{rank}\begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = (R_3 \leftrightarrow R_4)$$

$$\text{rank}\begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 3 \neq 4, \text{ so}$$

no, not a basis

(dim(span) = 3, so set of vectors fails to span $\mathbb{R}^4$; also, is 4 vectors, with dim(span) $\neq 4 \rightarrow$ lin. dep.)

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(2) Is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

a basis for $\mathbb{R}^3$?

SOLUTION: 3 vectors & $\dim(\mathbb{R}^3) = 3$,

so calculate

$$\text{rank} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = (R_3 - R_1)$$

$$\text{rank} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & -2 & 0 \end{bmatrix} = \left(\begin{array}{c} R_3 + 2R_2, \\ \text{then } \frac{1}{2}R_3 \end{array} \right)$$

$$\text{rank} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = 3 \rightarrow \boxed{\text{yes, a basis}}$$

(3) Is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

a basis for $\mathbb{R}^3$?

SOLUTION: no, since $\dim(\mathbb{R}^3) = 3$, so any basis for $\mathbb{R}^3$ has 3 vectors. (Our set won't span $\mathbb{R}^3$; not enough vectors)

(4) Is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

a basis for $\mathbb{R}^3$?

SOLUTION: no, as in (3), since there are more than 3 vectors. (Our set is linearly dependent; too many vectors)

SECTION IVF: VALUE-BASED EQUIVALENCES

Chapters II - IV presented a somewhat confusing array of binary definitions: consistent vs. inconsistent, unique vs. nonunique, trivial vs. nontrivial, singular vs. nonsingular, (linearly) dependent vs. independent, invertible vs. not invertible; these are all of the form right vs. left or heads vs. tails, and risk the random ambience of a coin flip. Also important but confusing is the fact that some of these definitions involve vectors, some matrices, and some linear systems. Many of our equivalences jump nimbly between these three entities. Coherence demands that we distinguish

between these entities,
even as we pursue equivalences.

Memory experts agree that definitions are easier to remember if they are associated with ethics and emotions, that is, things that can be called "good" or "bad" (insert synonym appropriate to culture) without irony.

In this short section,
we will show how to remember equivalences between the binary definitions listed at the beginning of the section, by identifying "bad" and "good" in our binary choices; then bad things are equivalent to other bad things while good things are equivalent to other good things.

In the following,

A is an $(m \times n)$ matrix

with columns the m-vectors

$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$; that is,

$$A = [\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_n],$$

and, for $1 \le j \le n$, $\vec{v}_j$ is the

j^{th} column of A.

"Badness" in the following

could be replaced by "Goodness"

by replacing each assertion

with its negation.

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EQUIVALENCES of BADNESS 4.58

Assertion (a)-(e) are equivalent; that is, if one is true, then so are all the others.

(a) $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly dependent.

(b) A is singular.

(c) The homogeneous linear system $A\vec{x} = \vec{0}$ has nontrivial solutions.

(d) The linear system $A\vec{x} = \vec{b}$ has non-unique solutions whenever it is consistent.

(e) $\text{rank}(A) < n$.

If $n = m$ (that is, A is a square matrix) then (a) - (e) are also equivalent to (f) and (g) below.

(f) A is not invertible.

(g) There's an m-vector $\vec{b}$ so that the linear system $A\vec{x} = \vec{b}$ is inconsistent; that is, has no solution.

4.59 WHY THE EQUIVALENCES in 4.58 ARE BAD.

(a) A vector in the set is redundant: it could be removed without changing the span.

(b) A nontrivial vector $\vec{x}$ is being killed by A; that is, $A\vec{x} = \vec{0}$.

(c) SAME as (b); also uniqueness of the trivial solution $\vec{x} \equiv \vec{0}$ is lost.

(d) Nonunique solutions mean nonunique realities or parallel universes, disturbing or at least confusing.

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(e) rank we think of as information; at least one column of A is not contributing information.

(f) Invertibility means the power to undo whatever we've done: if $\vec{x}$ is changed by A, to $A\vec{x}$, and we regret that change, we may change back to $\vec{x}$ by multiplying by A^{-1}, $\vec{x} = A^{-1}(A\vec{x})$

$$\begin{tikzcd} \vec{x} \arrow[r, bend left, "A"] & A\vec{x} \arrow[l, bend left, "A^{-1}\ (\text{undo } A)"] \end{tikzcd}$$

(g) We want to solve linear systems; it is worrisome when there are choices of $\vec{b}$, in $A\vec{x} = \vec{b}$, that we must avoid.

Examples 4.60

1. (a) $\{\vec{v}_1 = (1, 0, 1, -1), \vec{v}_2 = (1, 1, 0, 1), (1, -1, 2, -3)\}$ is linearly dependent since

$$(-2)\begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ -1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(b) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & -3 \end{bmatrix}$ is singular, since

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & -3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

(c) The homogeneous linear system

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{(matrix form)}$$

AKA

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{(vector form)}$$

OR

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_2 - x_3 &= 0 \\ x_1 \qquad + 2x_3 &= 0 \\ -x_1 + x_2 - 3x_3 &= 0 \end{aligned}$$

has the nontrivial solution

$(x_1, x_2, x_3) = (-2, 1, 1)$.

(d) The linear system

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ 7 \\ -8 \end{bmatrix}$$

has nonunique solution,

$(x_1, x_2, x_3) = (1, 2, 3)$ or $(-1, 3, 4)$;

notice that $(-1, 3, 4) = (1, 2, 3) + (-2, 1, 1)$,

where $(-2, 1, 1)$ is a solution of

$A\vec{x} = \vec{0}$.

(e) The rank of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & -3 \end{bmatrix} = 2 < 3 = \text{number of columns of } A.$$

2. (a) The vectors $\vec{v}_1 = (1,2)$ and $\vec{v}_2 = (3,6)$ are linearly dependent, since $3\vec{v}_1 - \vec{v}_2 = \vec{0}$.

(b) The matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ is singular, since $A\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \vec{0}$.

(c) The linear system

$$\begin{aligned} x_1 + 3x_2 &= 0 \\ 2x_1 + 6x_2 &= 0 \end{aligned}$$

has the nontrivial solution $\vec{x} = (3,-1)$.

(d) The linear system

$$x_1 + 3x_2 = 7$$
$$2x_1 + 6x_2 = 14$$

has nonunique solution,

$(x_1, x_2) = (1, 2)$ or $(4, 1)$;

$(4, 1) - (1, 2) = (3, -1)$, a solution of $A\vec{x} = \vec{0}$.

(e) $\text{rank}\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} = 1 < 2 =$ number of column of A

(f) $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ is not invertible.

(g) The linear system

$$x_1 + 3x_2 = 0$$
$$2x_1 + 6x_2 \quad 1$$

is inconsistent;

$$A\vec{x} = \vec{b} \rightarrow \left[\begin{array}{cc|c} 1 & 3 & b_1 \\ 2 & 6 & b_2 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 3 & b_1 \\ 0 & 0 & b_2 - 2b_1 \end{array}\right] \rightarrow$$

$b_2 = 2b_1$, if $A\vec{x} = \vec{b}$ is consistent.

In the language of Chapter III,

$$R(A) = \left\{ \begin{bmatrix} b_1 \\ 2b_1 \end{bmatrix} \middle| \ b_1 \text{ is arbitrary} \right\}$$

$$\neq \mathbb{R}^2$$