LINEAR ALGEBRA HOMEWORK

CHAPTER I HOMEWORK

HWI.1. Carry out the indicated matrix multiplication or assert that it's not defined:

(a) $\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. (b) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix}^2$. (c) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. (d) $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

(e) $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ 1 \\ 0 \end{bmatrix}$. (f) $\begin{bmatrix} 5 \\ -3 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$.

HWI.2. Let $A \equiv \frac{1}{2} \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$.

(a) Find all positive integral powers of A, denoted $A^k, k = 1, 2, 3, \ldots$.

(b) Solve the Difference Equation

$$\vec{x}_{k+1} = A\vec{x}_k, \quad k = 1, 2, 3, \ldots, \quad \vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

(c) Find the smallest positive n such that $A^n = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(d) Find the smallest positive n such that $A^n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

HWI.3. SAME as HWI.2, except $A = \frac{1}{2} \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$.

HWI.4. For rabbits and foxes as in Examples 1.25(4), (*) on page 58, suppose we begin with 60 foxes and 300 rabbits. How many rabbits and foxes will there be three years from now?

HWI.5. For Moonorgs as in Examples 1.25(5), if there are currently 100,000 Moonorgs on the moon, and 10,000 Moonorgs not on the moon, how many will be on the moon and how many will be off the moon, two years from now?

HWI.6. If $\vec{x} \equiv (0, 1, 2, -3)$ and $\vec{y} \equiv (-3, 2, 0, 1)$, get $(2\vec{x} - \vec{y})$.

HWI.7. Suppose $A \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, find

(a) A^2;

(b) A^3; and

(c) $A^k, k = 1, 2, 3 \ldots$.

Your answer to (c) should have a "k" in it.

HWI.8. Suppose

$$A \equiv \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B \equiv \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 0 \end{bmatrix}, \quad C \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Find AB, BC, ABC, and $(AB)^2$.

HWI.9. A manufacturer sells three products in two markets. The number of units of each product sold in each market in a particular year is given by

$$T \equiv \begin{bmatrix} 5,000 & 2,000 & 1500 \\ 2,000 & 3,000 & 1,000 \end{bmatrix},$$

where the row is the market and the column is the product.

The matrices

$$S_1 \equiv \begin{bmatrix} 2.00 \\ 4.00 \\ 3.00 \end{bmatrix} \quad S_2 \equiv \begin{bmatrix} 1.80 \\ 3.50 \\ 2.50 \end{bmatrix}$$

give, respectively, the unit sales price and the unit cost price for the products. What do TS_1, TS_2, and $(TS_1 - TS_2)$ mean?

CHAPTER II HOMEWORK

HWII.1. Denote by (*) the linear system

$$\begin{array}{rcrcrcrcrl} x_1 & & & + & x_3 & + & 3x_4 & + & 2x_5 & = 0 \\ & & x_2 & + & x_3 & - & 2x_4 & - & 3x_5 & = 1 \\ 2x_1 & & & + & 2x_3 & + & 6x_4 & + & 4x_5 & = 2 \\ x_1 & + & 2x_2 & + & 3x_3 & - & x_4 & - & 4x_5 & = 3 \end{array}$$

(a) Write (*) in matrix form (DON'T SOLVE).

(b) Write (*) in vector form (DON'T SOLVE).

HWII.2. Believe me again when I tell you that

$$\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

(i) Use this fact to find a solution of

$$\begin{array}{rcrcrl} x_1 & & & + & x_3 & = 0 \\ 2x_1 & - & x_2 & + & & = 0 \\ & & x_2 & + & 2x_3 & = 0 \\ x_1 & & & & & = 1 \end{array}$$

(ii) Use this fact to find $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ so that

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

HWII.3. In each of the following, solve (that is, find all solutions) or show inconsistency, with Gauss-Jordan elimination:

(a)

$$\begin{array}{rcrcrcr} x & + & y & & & = & 10 \\ 2x & + & y & - & z & = & 1 \\ -x & + & y & + & z & = & 2 \end{array}$$

(b)

$$\begin{array}{rcrcrcrcr} x & + & y & & & + & w & = & 0 \\ x & + & y & - & z & & & = & 1 \\ x & + & y & + & z & + & 2w & = & 2 \end{array}$$

(c)

$$\begin{array}{rcrcrcrcr} x & + & y & & & + & w & = & 0 \\ x & + & y & - & z & & & = & 1 \\ x & + & y & + & z & + & 2w & = & -1 \end{array}$$

HWII.4. In each of the following ((a), (b), and (c)), the last two rows of the augmented matrix of a linear system, after being placed in row echelon form with elementary operations, are given. In each of (a), (b), and (c), use this to determine if the linear system is consistent.

(a)

$$\begin{bmatrix} 0 & 1 & 0 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 0 & 1 & 0 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 0 & 1 & 0 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

HWII.5. In each of the following, solve or show there are no solutions, with Gauss-Jordan elimination.

$$(a)\ \begin{array}{rcrcr} x & + & y & = & 0 \\ x & - & y & = & -1 \\ x & + & 3y & = & 1 \end{array} \quad (b)\ \begin{array}{rcrcr} x & + & y & = & 1 \\ x & - & y & = & -1 \\ x & + & 3y & = & 1 \end{array} \quad (c)\ \begin{array}{rcrcr} x & + & y & = & 0 \\ x & - & y & = & -1 \\ x & + & 2y & = & 1 \end{array}$$

$$(d)\ \begin{array}{rcrcrcrcr} & & & & z & + & w & = & 1 \\ x & + & y & & & - & w & = & 0 \\ x & + & y & + & z & & & = & 1 \end{array} \quad (e)\ \begin{array}{rcrcrcrcr} & & & & z & + & w & = & 1 \\ x & + & y & & & - & w & = & 0 \\ & & y & + & z & & & = & 0 \end{array}$$

$$(f)\ \begin{array}{rcrcrcrcr} x & - & 2y & + & z & - & w & = & 1 \\ x & + & y & & & & & = & 0 \\ & & & & z & + & w & = & 1 \\ & & y & - & z & & & = & 0 \end{array} \quad (g)\ \begin{array}{rcrcrcrcr} x & + & 2y & & & + & w & = & 1 \\ x & + & y & & & & & = & 0 \\ & & & & z & + & w & = & 1 \\ & & y & - & z & & & = & 0 \end{array}$$

$$(h)\ \begin{array}{rcrcrcr} & & y & - & z & = & 1 \\ x & - & 2y & + & z & = & 0 \\ x & + & y & & & = & 0 \end{array} \quad (i)\ \begin{array}{rcrcrcr} & & y & - & z & = & 1 \\ x & + & 2y & - & z & = & 0 \\ x & + & y & & & = & 0 \end{array}$$

CHAPTER III HOMEWORK

HWIII.1. Believe me when I tell you that the rank of

$$\begin{bmatrix} 1 & 0 & 1 & 3 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 2 & 0 & 2 & 6 & 4 \\ 1 & 2 & 3 & -1 & -4 \end{bmatrix}$$

is two.

Use this fact to find the number of free variables in the set of solutions of

$$\begin{array}{ccccccccccc} x_1 & & & + & x_3 & + & 3x_4 & + & 2x_5 & = 0 \\ & & x_2 & + & x_3 & - & 2x_4 & - & 3x_5 & = 0 \\ 2x_1 & & & + & 2x_3 & + & 6x_4 & + & 4x_5 & = 0 \\ x_1 & + & 2x_2 & + & 3x_3 & - & x_4 & - & 4x_5 & = 0 \end{array}$$

DO NOT SOLVE.

HWIII.2. Suppose a, b, c, d are numbers such that the rank of

$$\begin{bmatrix} 1 & 0 & 1 & 3 & 2 & a \\ 0 & 1 & 1 & -2 & -3 & b \\ 2 & 0 & 2 & 6 & 4 & c \\ 1 & 2 & 3 & -1 & -4 & d \end{bmatrix}$$

equals two.

(a) Does

$$\begin{array}{ccccccccccc} x_1 & & & + & x_3 & + & 3x_4 & + & 2x_5 & = a \\ & & x_2 & + & x_3 & - & 2x_4 & - & 3x_5 & = b \\ 2x_1 & & & + & 2x_3 & + & 6x_4 & + & 4x_5 & = c \\ x_1 & + & 2x_2 & + & 3x_3 & - & x_4 & - & 4x_5 & = d \end{array}$$

have a solution (DON'T SOLVE)? Give a brief reason for your conclusion.

(b) Find the number of free variables in the set of solutions of the linear system in (a) (DON'T SOLVE).

HWIII.3. What can be said (number of free variables, number of solutions) about the set of solutions of

(i) a system of 15 linear equations, with 10 variables;

(ii) a system of 10 linear equations, with 15 variables.

HWIII.4. In each of the following linear systems, get the rank of the coefficient matrix and the rank of the augmented matrix, and use those to answer the following questions:

Is the linear system consistent?

and

If consistent, what is the number of free variables (DON'T SOLVE)?

(i)

$$\begin{array}{rcrcrl} x_1 & - & x_2 & + & 2x_3 & = 0 \\ -x_1 & + & x_2 & & & = 3 \\ x_1 & - & x_2 & + & 4x_3 & = 5 \end{array}$$

(ii)

$$\begin{array}{rcrcrl} x_1 & - & x_2 & + & 2x_3 & = 0 \\ -x_1 & + & x_2 & & & = 3 \\ x_1 & - & x_2 & + & 4x_3 & = 3 \end{array}$$

HWIII.5. Use elementary operations to get the inverse of $\begin{bmatrix} 3 & 5 & 7 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix}$.

HWIII.6. Believe me again when I tell you that

$$3(1,2,-1,1) - (2,1,1,2) - (1,5,-4,1) = (0,0,0,0).$$

Use this fact to answer the following.

(i) Find a nontrivial solution of

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & x_3 & = & 0 \\ 2x_1 & + & x_2 & + & 5x_3 & = & 0 \\ -x_1 & + & x_2 & - & 4x_3 & = & 0 \\ x_1 & + & 2x_2 & + & x_3 & = & 0 \end{array}$$

(ii) Find

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 5 \\ -1 & 1 & -4 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}.$$

(iii) Is $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 5 \\ -1 & 1 & -4 \\ 1 & 2 & 1 \end{bmatrix}$ singular?

HWIII.7. Believe me when I tell you that the rank of $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 0 & 5 \\ 1 & 4 & -7 \end{bmatrix}$ is 2. Use this fact to find the number of free variables in the set of solutions of

$$\begin{array}{rcrcrcl} x & + & 2y & - & z & = & 0 \\ 2x & + & 2y & + & 4z & = & 0 \\ x & + & & & 5z & = & 0 \\ x & + & 4y & - & 7z & = & 0 \end{array}$$

DO NOT SOLVE.

HWIII.8. Suppose a linear system has 10 variables and 7 equations. Assume it is consistent.

(a) What are the possibilities for the number of free variables in the set of solutions?

(b) Is it possible for this linear system to have a unique solution?

(c) Is it possible for this linear system to have infinitely many solutions?

HWIII.9. Suppose a linear system has 7 variables and 10 equations. Assume it is consistent.

(a) What are the possibilities for the number of free variables in the set of solutions?

(b) Is it possible for this linear system to have a unique solution?

(c) Is it possible for this linear system to have infinitely many solutions?

HWIII.10. Find the rank of each of the following matrices.

(a) $\begin{bmatrix} 1 & -1 & 1 \\ 2 & -3 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}$.

(b) $\begin{bmatrix} 1 & 3 & \sqrt{7} & 9 & 11 \end{bmatrix}$.

HWIII.11. In each of the following, either find the inverse or show that it doesn't exist.

(a) $\begin{bmatrix} 1 & 2 & 0 & 1 \\ -1 & -3 & 1 & 1 \\ 1 & 1 & 1 & 3 \\ 0 & -1 & 2 & 5 \end{bmatrix}$. (b) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}$.

HWIII.12. Believe me when I tell you that $\begin{bmatrix} 9 & -\frac{3}{2} & -5 \\ -5 & 1 & 3 \\ -2 & \frac{1}{2} & 1 \end{bmatrix}$ is the inverse of $A \equiv \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix}$.

Use this fact to find a solution of

$$\begin{array}{ccccccc} x_1 & + & 2x_2 & - & x_3 & = & 1 \\ 2x_1 & + & 2x_2 & + & 4x_3 & = & 0 \\ x_1 & + & 3x_2 & - & 3x_3 & = & 1 \end{array}$$

Will there be any other solutions? Why or why not?

HWIII.13. Let $C \equiv \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Find C^2.

HWIII.14. Let $B \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(a) Find B^2.

(b) Find B^3.

(c) For arbitrary $k = 1, 2, 3, 4, \ldots$, find B^k.

HWIII.15. Let $A \equiv \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$. Show that $(A^k + A^{k+1})$ is the same matrix, for $k = 1, 2, 3, \ldots$, and find out what that matrix is.

HWIII.16. Suppose $A \equiv \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 0 \\ 1 & 1 & 3 & -1 \end{bmatrix}$. Is $(1, 2, -1, 0)$ in the null space of A? Is $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ in the range of A?

HWIII.17. Find the rank of each of the following.

$$(a)\quad \begin{bmatrix} 1 & -1 & 1 & 0 \end{bmatrix} \quad (b)\quad \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (c)\quad \begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$(d)\quad \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix} \quad (e)\quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad (f)\quad \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$(g)\quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 \end{bmatrix} \quad (h)\quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 2 & 0 & 0 \end{bmatrix} \quad (i)\quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

HWIII.18. Suppose $d_1 \neq d_2$. Find all (2×2) matrices that commute with $A \equiv \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$. That is, find all matrices B such that $AB = BA$.

HWIII.19. For what numbers $d_1, d_2, \ldots, d_n$ will the diagonal matrix with diagonal entries $d_1, d_2, \ldots, d_n$ be invertible?

HWIII.20. Find the range space and the null space of each of the matrices (a)–(f) in HWIII.17.

HWIII.21. Suppose A is an invertible $(n \times n)$ matrix. What can be said about the range space of A and the null space of A?

HWIII.22. Find all real numbers a for which the matrix $\begin{bmatrix} 1 & 1 \\ a & 1 \end{bmatrix}$ is invertible. For such a, find the inverse.

HWIII.23. Find the rank, range space and null space of $A \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

CHAPTER IV HOMEWORK

HWIV.1.

(a) Use rank to determine the number of free variables in the set of solutions of

$$\begin{array}{rcrcrcl} x_1 & - & x_2 & + & x_3 & = & 0 \\ 2x_1 & + & x_2 & + & 5x_3 & = & 0 \\ -x_1 & + & 3x_2 & + & x_3 & = & 0. \end{array}$$

DON'T SOLVE THE LINEAR SYSTEM.

(b) Use part (a) to determine if

$$\begin{array}{rcrcrcl} x_1 & - & x_2 & + & x_3 & = & 0 \\ 2x_1 & + & x_2 & + & 5x_3 & = & 0 \\ -x_1 & + & 3x_2 & + & x_3 & = & 0. \end{array}$$

has a nontrivial solution. DON'T SOLVE THE LINEAR SYSTEM.

(c) Use rank to determine if

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & x_3 & = & 2 \\ 2x_1 & + & x_2 & + & 5x_3 & = & 13 \\ -x_1 & + & 3x_2 & + & x_3 & = & 4 \end{array}$$

is consistent. DON'T SOLVE THE LINEAR SYSTEM.

(d) Use rank or part (c) to determine if $(2, 13, 4)$ is a linear combination of $\{(1, 2, -1), (-1, 1, 3), (1, 5, 1)\}$.

HWIV.2. Write $(0, 12, 8, 16)$ as a linear combination of $\{(-1, 1, 2, 3), (1, 2, 0, 1), (2, 7, 2, 6)\}$.

HWIV.3. In each of the following, write the vector $\vec{b}$ as a linear combination of the given set of vectors S, or determine that $\vec{b}$ is not a linear combination of S.

(a) $\vec{b} \equiv (0, 0, 3, 0); S \equiv \{(1, 0, 1, -1), (0, 1, 1, 0), (1, 2, 0, -1)\}$.

(b) $\vec{b} \equiv (0, 0, 3, 0); S \equiv \{(1, 0, 1, -1), (0, 1, 1, 0)\}$.

HWIV.4. (a) Find a basis for the null space of $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 1 \\ 1 & 1 & 2 & 3 \end{bmatrix}$.

(b) Find the nullity of $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 1 \\ 1 & 1 & 2 & 3 \end{bmatrix}$.

HWIV.5. Find a basis for the span of $\{(1, 0, 0, 1, 2), (0, 2, 1, 0, 0), (1, 2, 1, 1, 2), (1, -2, -1, 1, 2)\}$.

HWIV.6. Suppose the null space of a matrix A may be described, via Gauss-Jordan elimination, as

$$\{x_1(1, 0, 1, 2) + x_2(1, 2, 3, 4) + x_3(5, 6, 7, 8) \,|\, x_1, x_2, x_3 \text{ real } \}.$$

Get a basis for the null space of A.

HWIV.7. Let $A \equiv \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 2 & 1 & 3 \end{bmatrix}$.

(a) Get the dimension of the range of A.

(b) Get the dimension of the null space of A.

HWIV.8. Believe me when I tell you that

$$\operatorname{rank}\begin{bmatrix}1 & 0 & 1 & -1\\ 0 & 1 & 1 & 0\\ 1 & 1 & 2 & -1\\ 1 & -1 & 0 & 0\end{bmatrix} = 3 \quad\text{and}\quad \operatorname{rank}\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 1 & 1 & 2\\ 1 & -1 & 0\end{bmatrix} = 2.$$

Use those facts to answer the following questions.

(a) Is $(-1, 0, -1, 0)$ a linear combination of $\{(1, 0, 1, 1), (0, 1, 1, -1), (1, 1, 2, 0)\}$?

(b) Is

$$\begin{array}{ccccccc} x_1 & + & & + & x_3 & = & -1\\ & & x_2 & + & x_3 & = & 0\\ x_1 & + & x_2 & + & 2x_3 & = & -1\\ x_1 & - & x_2 & & & = & 0\end{array}$$

consistent? DON'T SOLVE THE LINEAR SYSTEM.

HWIV.9. Use rank to answer each of the following.

(a) Is $\{(1, 2, -1), (-1, 1, 3), (1, 5, 1)\}$ linearly independent?

(b) Is $\begin{bmatrix}1 & -1 & 1\\ 2 & 1 & 5\\ -1 & 3 & 1\end{bmatrix}$ singular?

(c) Does

$$\begin{array}{ccccccc} x_1 & - & x_2 & + & x_3 & = & 0\\ 2x_1 & + & x_2 & + & 5x_3 & = & 0\\ -x_1 & + & 3x_2 & + & x_3 & = & 0\end{array}$$

have a nontrivial solution?

(d) Are there numbers b_1, b_2, b_3 so that

$$\begin{array}{ccccccc} x_1 & - & x_2 & + & x_3 & = & b_1\\ 2x_1 & + & x_2 & + & 5x_3 & = & b_2\\ -x_1 & + & 3x_2 & + & x_3 & = & b_3\end{array}$$

is inconsistent?

(e) Is $\begin{bmatrix}1 & -1 & 1\\ 2 & 1 & 5\\ -1 & 3 & 1\end{bmatrix}$ invertible?

HWIV.10. Which of the following are *possible*, just from the information given? For each part, give a brief argument/relevant calculation.

(a) A set of 4 3-vectors linearly independent.

(b) A set of 4 3-vectors linearly dependent.

(c) A set of 3 4-vectors linearly independent.

(d) A set of 3 4-vectors linearly dependent.

HWIV.11. Let $A \equiv \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 1 & 1 & 3 & -1 & 1 \end{bmatrix}$. For each of the following, find a basis.

(a) the null space of A.

(b) the row space of A.

(c) the column space of A.

(d) the range of A.

(e) the span of $\{(1, 0, 1), (0, 1, 1), (1, 2, 3), (-1, 0, -1), (0, 1, 1)\}$.

(f) the span of $\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

(g) the set of all $\vec{x} \equiv (x_1, x_2, x_3, x_4, x_5)$ such that

$$\begin{array}{ccccccccccc} x_1 & & & + & x_3 & - & x_4 & & & = 0 \\ & & x_2 & + & 2x_3 & & & + & x_5 & = 0 \\ x_1 & + & x_2 & + & 3x_3 & - & x_4 & + & x_5 & = 0 \end{array}$$

(h) the set of all $\vec{y} \equiv (y_1, y_2, y_3)$ such that, for some $\vec{x} \equiv (x_1, x_2, x_3, x_4, x_5)$,

$$\begin{array}{ccccccccccc} x_1 & & & + & x_3 & - & x_4 & & & = y_1 \\ & & x_2 & + & 2x_3 & & & + & x_5 & = y_2 \\ x_1 & + & x_2 & + & 3x_3 & - & x_4 & + & x_5 & = y_3 \end{array}$$

HWIV.12. Find a basis for the null space of A, and a basis for the range of A, if $A \equiv \begin{bmatrix} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$.

HWIV.13. Suppose A is a matrix and Gauss-Jordan elimination produces the following set of solutions of the homogeneous linear system $A\vec{x} = \vec{0}$:

$$x_1 = 2x_3 - 7x_4, \;\; x_2 = x_3 + 9x_4, \;\; x_3, x_4 \text{ arbitrary.}$$

Find a basis for the null space of A.

HWIV.14. Let A be the matrix

$$\begin{bmatrix} 1 & -2 & 3 & -1 & 5 & 0 & -5 \\ 2 & 1 & 1 & 3 & 0 & 5 & 5 \\ 3 & 0 & 3 & 3 & 3 & 6 & 3 \end{bmatrix}$$

Believe me when I say that the rank of A is 2. Use this fact to find

(a) the dimension of the row space of A;

(b) the dimension of the column space of A (also known as the range of A);

(c) the dimension of the null space of A. (DON'T FIND null space of A.)

(d) the dimension of the span of $\{(1, -2, 3, -1, 5, 0, -5), (2, 1, 1, 3, 0, 5, 5), (3, 0, 3, 3, 3, 6, 3)\}$.

HWIV.15. In each of the following, state the possible values for the dimension of the null space of A and the dimension of the range of A.

(a) A is 5×8.

(b) A is 8×5.

HWIV.16. In each of the following, what can be said about the dimension of the subspace V?

(a) V contains seven linearly independent vectors.

(b) V is spanned by seven vectors.

(c) V is spanned by seven linearly independent vectors.

(d) V is spanned by eight vectors and contains six linearly independent vectors.

HWIV.17. If the dimension of the subspace V is 20, what can be said about

(a) A set of ten vectors in V;

(b) A set of thirty vectors in V.

HWIV.18. In each of the following, determine if the set is a basis for $\mathbf{R}^4$.

(a) $\{(1, 2, 0, 1), (0, 1, 2, 3), (1, 0, 1, 1), (1, 3, -3, -2)\}$;

(b) $\{(1, 2, 3, 4), (0, 5, 6, 7), (0, 0, 8, 9), (0, 0, 0, 10)\}$.

HWIV.19. Believe me when I tell you that

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 5 \\ -1 \end{bmatrix}.$$

Use this fact to

(a) Write $(-2, 4, 5, -1)$ as a linear combination of $\{(1, 2, 0, 2), (0, 1, 1, 0), (-1, 0, 1, -1)\}$;

(b) Find a solution of

$$\begin{array}{rcrcrcr} x_1 & & & - & x_3 & = & -2 \\ 2x_1 & + & x_2 & & & = & 4 \\ & & x_2 & + & x_3 & = & 5 \\ 2x_1 & & & - & x_3 & = & -1 \end{array}$$

(c) Determine if $\{(-2, 4, 5, -1), (1, 2, 0, 2), (0, 1, 1, 0), (-1, 0, 1, -1)\}$ is linearly independent.

HWIV.20. Use HWIII.6 to determine if $\{(1, 2, -1, 1), (2, 1, 1, 2), (1, 5, -4, 1)\}$ is linearly independent.

HWIV.21. Use rank to answer each of the following.

(a) Is $\{(1, 2, -1), (-1, 1, 3), (1, 2, 1)\}$ linearly independent?

(b) Is $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 2 \\ -1 & 3 & 1 \end{bmatrix}$ singular?

(c) Does

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & x_3 & = & 0 \\ 2x_1 & + & x_2 & + & 2x_3 & = & 0 \\ -x_1 & + & 3x_2 & + & x_3 & = & 0 \end{array}$$

have a nontrivial solution?

(d) Is

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & x_3 & = & 2 \\ 2x_1 & + & x_2 & + & 2x_3 & = & 13 \\ -x_1 & + & 3x_2 & + & x_3 & = & 4 \end{array}$$

consistent (don't solve)?

(e) Are there numbers b_1, b_2, b_3 so that

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & x_3 & = & b_1 \\ 2x_1 & + & x_2 & + & 2x_3 & = & b_2 \\ -x_1 & + & 3x_2 & + & x_3 & = & b_3 \end{array}$$

is inconsistent?

(f) Is $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 2 \\ -1 & 3 & 1 \end{bmatrix}$ invertible?

HWIV.22. Which of the following are *possible*, just from the information given?

(a) A set of 6 7-vectors linearly independent.

(b) A set of 6 7-vectors linearly dependent.

(c) A set of 7 6-vectors linearly independent.

(d) A set of 7 6-vectors linearly dependent.

HWIV.23. Solve an appropriate linear system to determine if the set of vectors

$$\{(1,2,0,1),(-1,-3,1,1),(1,1,1,3),(0,0,0,1)\}$$

is linearly independent.

HWIV.24. Is $(1,2,3)$ a linear combination of $(1,1,-1)$ and $(1,0,-5)$? Answer this question in 2 ways:

(a) with a rank calculation; and

(b) by considering an appropriate linear system.

HWIV.25. Without getting a basis, find the dimension of

(a) the span of $\{(1,0,1,2,0),(2,1,0,1,0),(0,-1,2,3,0)\}$;

(b) the null space of $\begin{bmatrix} 1 & 0 & 1 & 2 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 3 & 0 \end{bmatrix}$;

(c) the null space of $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \\ 2 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$.

HWIV.26. Write the linear system

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & 3x_3 & = & 4 \\ 5x_1 & + & 6x_2 & & & = & 7 \end{array}$$

as

(a) A matrix equation $A\vec{x} = \vec{b}$; and

(b) A linear combination $x_1\vec{A}_1 + x_2\vec{A}_2 + x_3\vec{A}_3 = \vec{b}$.

HWIV.27. Write $(3,4,3,6)$ as a linear combination of $\{(1,0,1,2),(-1,1,1,1),(2,3,0,1)\}$.

HWIV.28. Use the facts that

$$\operatorname{rank}\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & 1 \\ 1 & 1 & 0 & 3 \end{bmatrix} = 2 \text{ and } \operatorname{rank}\begin{bmatrix} 1 & 2 & -1 & 4 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 3 & 0 \end{bmatrix} = 3$$

to answer the following questions.

(a) Is the linear system

$$\begin{array}{rcrcrcrl} x_1 & + & 2x_2 & - & x_3 & + & 4x_4 & = 1 \\ & & x_2 & - & x_3 & + & x_4 & = 0 \\ x_1 & + & x_2 & & & + & 3x_4 & = 0 \end{array}$$

consistent?

(b) Is $(1,0,0)$ a linear combination of $\{(1,0,1),(2,1,1),(-1,-1,0),(4,1,3)\}$?

HWIV.29. Is $\{(1,0,0,-1),(0,2,1,1),(1,0,1,-1)\}$ linearly independent? Answer this question in 2 ways:

(a) with a rank calculation; and

(b) by considering an appropriate linear system.

HWIV.30. Use the facts that

$$\operatorname{rank}\begin{bmatrix}1&2&0&0&-1\\0&1&1&1&0\\1&1&-1&-1&-1\end{bmatrix}=2=\operatorname{rank}\begin{bmatrix}1&2&0&0&-1&3\\0&1&1&1&0&2\\1&1&-1&-1&-1&1\end{bmatrix}\text{ and }\operatorname{rank}\begin{bmatrix}1&2&0&0&-1\\0&1&1&1&0\\1&1&-1&-1&-1\\1&2&3&4&0\end{bmatrix}=3$$

to answer the following questions.

(a) Is $(3,2,1)$ in the column space of $\begin{bmatrix}1&2&0&0&-1\\0&1&1&1&0\\1&1&-1&-1&-1\end{bmatrix}$?

(b) Is $(1,2,3,4,0)$ in the row space of $\begin{bmatrix}1&2&0&0&-1\\0&1&1&1&0\\1&1&-1&-1&-1\end{bmatrix}$?

(c) Is the linear system

$$\begin{array}{ccccccccccc} x&+&2y&&&&&-&v&=3\\ &&y&+&z&+&w&&&=2\\ x&+&y&-&z&-&w&-&v&=1\end{array}$$

consistent?

(d) Find the nullity, without getting a basis, of $\begin{bmatrix}1&2&0&0&-1\\0&1&1&1&0\\1&1&-1&-1&-1\end{bmatrix}$.

(e) Find the dimension of the span of $\{(1,2,0,0,-1),(0,1,1,1,0),(1,1,-1,-1,-1)\}$, without getting a basis.

(f) Find the dimension of the span of $\{(1,0,1),(2,1,1),(0,1,-1),(-1,0,-1)\}$, without getting a basis.

(g) Is $(1,2,3,4,0)$ a linear combination of $(1,2,0,0,-1),(0,1,1,1,0)$, and $(1,1,-1,-1,-1)$?

(h) Is $(3,2,1)$ a linear combination of $(1,0,1),(2,1,1),(0,1,-1),(-1,0,-1)$?

HWIV.31. Suppose A is a 10×14 matrix.

(a) What are the possible values of the dimension of the range of A?

(b) What are the possible values of the dimension of the row space of A?

(c) What are the possble values of the dimension of the column space of A?

(d) What are the possible values of the dimension of the null space of A?

HWIV.32. Same question as no. 31, for a 14×10 matrix.

HWIV.33. Suppose V is a subspace of dimension 32, and S is contained in V.

(a) If S contains 20 vectors, could S be linearly independent? Could S span V?

(b) If S contains 40 vectors, could S be linearly independent? Could S span V?

(c) If S is linearly independent and nonempty, what is the maximum number of vectors it could contain? What is the minimum?

(d) If S spans V, what is the maximum number of vectors it could contain? What is the minimum?

HWIV.34. Suppose V is a subspace and S is contained in V. What can be said about $\dim(V)$ if

(a) S consists of 13 vectors and S spans V;

(b) S is linearly independent and consists of 17 vectors;

(c) S is a linearly independent set of 20 vectors that spans V.

HWIV.35. Suppose A is a 15×97 matrix of rank 10.

(a) What is the dimension of the row space of A?

(b) What is the dimension of the column space of A?

(c) What is the dimension of the null space of A?

HWIV.36. Suppose the dimension of the range of a matrix A is 13 and the dimension of the null space of A is 10.

(a) What can be said about the number of rows of A?

(b) What can be said about the number of columns of A?

HWIV.37. Let $V \equiv \{(x_1, x_2, x_3, x_4) \,|\, x_1 + x_2 + x_3 + x_4 = 0\}$. For each of the following sets of vectors in V, determine if (i) it is linearly independent; ii) it spans V; and (iii) it is a basis for V.

(a) $\{(1, 1, 0, -2), (0, 1, -3, 2), (2, 0, -1, -1)\}$

(b) $\{(1, 1, 0, -2), (0, 1, -3, 2)\}$

(c) $\{(1, 1, 0, -2), (0, 1, -3, 2), (2, 0, -1, -1), (0, 0, 1, -1)\}$

(d) $\{(1, -1, 2, 0), (1, -1, 0, 2), (0, 0, 1, -1)\}$

HWIV.38. Is $\{(1, 2, 3), (-1, 0, 1), (1, 4, 7)\}$ a basis for R^3?

HWIV.39. Suppose a subspace V has $\{(1, 0, 1, 2), (0, 1, -2, 1), (0, 0, 7, 9)\}$ as a basis. What is the dimension of V?

HWIV.40. Find a basis for $R(A)$, if $A \equiv \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & -2 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 2 & 0 & 2 & 2 \end{bmatrix}$.

CHAPTER V HOMEWORK.

HWV.1. Find the determinants of each of the following matrices.

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

(b) $\begin{bmatrix} 17 & \frac{1}{9} & 5 \\ 1 & 2 & 0 \\ 3 & 4 & 0 \end{bmatrix}$.

(c) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 5 & 1 \\ -1 & 0 & 2 \end{bmatrix}$.

(d) $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 3 & 1 \end{bmatrix}$.

HWV.2. Find the area of the parallelogram with vertices $(1, 1), (2, 3), (2, 5), (3, 7)$.

HWV.3. Use determinant to determine if

$$\begin{bmatrix} 1 & 17 & \sqrt{2} & 9 \\ 0 & 3 & 19 & \frac{1}{3} \\ 0 & 0 & 2 & 3 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

is invertible.

HWV.4. Suppose A is a (10×10) matrix and $\det A = 3$. What is $\det(2A)$? What is $\det(A^7)$?

HWV.5. Suppose A and B are square matrices with $AB = I$.
Show that both A and B are invertible, with $A = B^{-1}$ and $B = A^{-1}$.

CHAPTER VI HOMEWORK

HWVI.1. Find the orthogonal projection of $(1,2,3,4,5)$ onto the span of

$$\{(1,1,0,0,-2),(1,1,1,1,1),(0,0,2,-2,0)\}.$$

HWVI.2. Find the orthogonal projection of $(1,0,0,-2)$ onto

$$W \equiv \text{ the span of } \{(0,1,0,1),(1,2,0,0),(0,2,1,0)\}.$$

(NOTE: W also equals the span of $\{(0,1,0,1),(1,1,0,-1),(-2,1,3,-1)\}$).

HWVI.3. Use the orthogonality of $S \equiv \{\vec{w}_1 \equiv (1,0,0,1), \vec{w}_2 \equiv (0,1,1,0), \vec{w}_3 \equiv (-1,1,-1,1), \vec{w}_4 \equiv (1,1,-1,-1)\}$ to write $(5,0,-3,7)$ as a linear combination of S.

You may leave arithmetic undone. All dot products and norms should be expanded.

DO NOT DO Gauss-Jordan on this problem.

HWVI.4. Use normal equations to find the least-squares approximating line thru

$$(-5,-1),(0,1),(5,2),(10,4),$$

and compute the corresponding least-squares error.

HWVI.5. Use normal equations to find the least-squares approximating parabola for the points $(1,1),(2,-2),(3,3),(4,4)$.

HWVI.6. Find all least-squares solutions of

$$\begin{array}{ccccccc} x & + & y & & & = & 0 \\ & & y & + & z & = & 0 \\ x & & & - & z & = & 1 \end{array}$$

HWVI.7. Set up, but DO NOT SOLVE, normal equations whose solution produces the least-squares approximating cubic $y = a + bx + cx^2 + dx^3$ (that is, produces a, b, c, and d) for the points $\{(-3,1),(-1,5),(0,2),(2,3),(4,-7)\}$.

HWVI.8. For the points

$$(2,0),(2,6),$$

find

(a) all least-squares approximating horizontal lines;

(b) all least squares approximating lines that are not vertical;

(c) all least squares approximating lines thru the origin; and

(d) all least squares approximating parabolas.

What do the answers of (a)–(d) have in common?

HWVI.9. Find the least squares errors for nos. 8(a)—(d).

HWVI.10. Find a unit vector that points in the same direction as $(1,2,0,-2,0)$.

HWVI.11. Get the orthogonal projection of $(1,0,0,-1)$ onto the subspace spanned by the orthogonal vectors

$$\{(1,1,0,0),(1,-1,3,6),(0,0,2,-1)\}.$$

HWVI.12. Write $(1,0,-1)$ as a linear combination of the orthogonal vectors

$$(1,1,-1),(1,1,2),(1,-1,0).$$

DO NOT do Gauss-Jordan. Use the orthogonality of $\{(1,1,-1),(1,1,2),(1,-1,0)\}$.

HWVI.13. Is the following orthogonal set orthonormal? Why or why not?

$$\{\frac{1}{3}(1,2,2),(0,-1,1),(-4,1,1)\}.$$

HWVI.14. Find a vector orthogonal to the plane containing $(1,2,3),(1,3,2)$, and $(2,2,0)$.

HWVI.15. Find the area of the triangle with vertices $(1,2,3),(1,5,3)$, and $(2,2,4)$.

HWVI.16. Find the area of the parallelogram with vertices $(1,2,3),(1,3,5),(1,5,5)$, and $(1,4,3)$.

HWVI.17. Find the area of the parallelogram formed by the vectors $(1,0,5)$ and $(2,1,0)$.

HWVI.18. Suppose $(1,2,3,4)=\alpha_1(1,0,1,2)+\alpha_2\vec{x}+\alpha_3\vec{y}$, for scalars $\alpha_1,\alpha_2,\alpha_3$, where $\{(1,0,1,2),\vec{x},\vec{y}\}$ is a set of orthogonal vectors in $\mathbf{R}^4$.

Find α_1.

HWVI.19. Let $\vec{a}\equiv(1,2,3),\vec{b}\equiv(4,5,6)$. Find orthogonal $\vec{v}_1,\vec{v}_2$ so that $\vec{a}=\vec{v}_1+\vec{v}_2$ and $\vec{v}_1$ is parallel to $\vec{b}$.

HWVI.20. Apply Gram-Schmidt orthogonalization to $\{(0,1,1,0),(-1,1,0,1),(2,1,0,0)\}$.

HWVI.21. Use normal equations to find the least-squares approximating line through the origin for the points

$$\{(-1,0),(0,2),(1,5),(3,0)\}.$$

HWVI.22. Set up (all matrices must be multiplied out), but DO NOT SOLVE, the normal equations whose solution is the coefficients a,b,c,d for the least-squares approximating cubic $a+bx+cx^2+dx^3$ for the points

$$\{(-2,0),(-1,1),(0,5),(1,2),(2,1)\}.$$

HWVI.23. Find all least-squares solutions, and the least-squares error, of

$$\begin{array}{rcrcl} x & + & 2y & = & 1 \\ -x & & & = & 0 \\ x & - & y & = & 0 \\ -x & + & y & = & 1 \end{array}$$

HWVI.24. A matrix is **orthogonal** if it is square and its columns form an orthonormal set.

Suppose A is an orthogonal $(n\times n)$ matrix and $\vec{x}$ and $\vec{y}$ are vectors in $\mathbf{R}^n$.

(a) Show that A^T is the inverse of A.

(b) Show that $A\vec{x}\cdot A\vec{y}=\vec{x}\cdot\vec{y}$.

(c) Show that A preserves orthogonality and norm. That is, show that

$$\|A\vec{x}\|=\|\vec{x}\| \quad \text{and, if } \vec{x}\perp\vec{y}, \text{ then } A\vec{x} \text{ is orthogonal to } A\vec{y}.$$

HWVI.25. For arbitrary real numbers $x_1,x_2,x_3,\ldots,x_n,y_1,y_2,y_3,\ldots y_n$, find the least-squares approximating line to the data $\{(x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)\}$ in terms of

$$\overline{x}\equiv\frac{1}{n}\sum_{k=1}^n x_k, \qquad \overline{y}\equiv\frac{1}{n}\sum_{k=1}^n y_k, \qquad \overline{(x^2)}\equiv\frac{1}{n}\sum_{k=1}^n x_k^2, \qquad \overline{(xy)}\equiv\frac{1}{n}\sum_{k=1}^n x_k y_k.$$

CHAPTER VII HOMEWORK

HWVII.1. For each of the following linear transformations T, find the standard matrix for T; that is, find a matrix A so that

$$T(\vec{x}) = A\vec{x},$$

for all $\vec{x}$ in the domain of T.

Believe (Appendix One) that the standard matrix for rotation by 60 degrees counterclockwise is $\frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$.

(a) $T(x, y, z) \equiv (3x + 5z, 0, y - 19z, x + y + z, 3z)$.

(b) $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined to be the projection onto the line $y = 3x$.

(c) $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined to be the reflection thru the line $y = 3x$.

(d) $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined to be rotation counterclockwise by sixty degrees, followed by projection onto the line $y = 3x$.

(e) $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined to be the projection onto the line $y = 3x$, followed by counterclockwise rotation by sixty degrees.

HWVII.2. For each of the following linear transformations T, find the standard matrix for T. The answer in each part should be a single matrix. Believe (Appendix One) that the standard matrix for 30 degree counterclockwise rotation is $\frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$.

(a) $T(x, y, z) \equiv (x - 5z, 0, y - 19z, x + 2y + 3z, z, 2y)$.

(b) T defined to be the projection onto the line $y = 2x$.

(c) T defined to be rotation counterclockwise by thirty degrees, followed by projection onto the line $y = 2x$.

(d) T defined to be the projection onto the line $y = 2x$, followed by counterclockwise rotation by thirty degrees.

(e) T defined to be reflection thru the line $y = 2x$.

HWVII.3. Get the first ten Fibonacci numbers $F_1, F_2, \ldots, F_{10}$.

HWVII.4. Suppose I have three jars of water. Consider the following operations.

(a) Pour half the contents of jar one into jar two.

(b) Pour one third of the contents of jar two into jar three.

(c) Pour half the contents of jar three into jar one.

Find a matrix A so that, if $\vec{x} \equiv (x_1, x_2, x_3)$ and x_i equals the amount of water in jar i ($i = 1, 2, 3$), then, for $i = 1, 2, 3$, the i^{th} component of $A\vec{x}$ equals the amount of water in jar i, after performing operations (a), (b), and (c), in that order.

HWVII.5. Same as HWVII.5, except the operations are performed in the order (b), (c), and (a).

HWVII.6. Suppose I have three buckets of slime. Every day I do the following. I pour $\frac{1}{3}$ the contents of bucket one into bucket three. Then I take all the contents of bucket three, put half of them into bucket one and half in bucket two.

Find a matrix A such that, if $\vec{x} \equiv (x_1, x_2, x_3)$ and x_i equals the amount of slime in bucket i ($i = 1, 2, 3$) today, then, for $i = 1, 2, 3$, the i^{th} component of $A\vec{x}$ equals the amount of slime in bucket i the next day.

HWVII.7. Suppose I have three canisters of mutagen. Every day I do the following. I pour $\frac{3}{4}$ of the contents of canister one into canister two. Then I pour $\frac{1}{7}$ the contents of canister two into canister three. Then I consume all the contents of canister three.

Find a matrix A such that, if $\vec{x} \equiv (x_1, x_2, x_3)$ and x_i equals the amount of mutagen in canister i $(i = 1, 2, 3)$ today, then, for $i = 1, 2, 3$, the i^{th} component of $A\vec{x}$ equals the amount of mutagen in canister i the next day.

CHAPTER VIII HOMEWORK

A **simple** eigenvalue is an eigenvalue of algebraic multiplicity one. Its geometric multiplicity is then automatically one.

HWVIII.1. Suppose
$$A \equiv \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$
Find the eigenspace of the eigenvalue -1 (for A).

HWVIII.2 Same question as no. 1, for
$$A \equiv \begin{bmatrix} 3 & -1 & 2 \\ 2 & 0 & 4 \\ -1 & 1 & 0 \end{bmatrix}$$
and the eigenvalue 2.

HWVIII.3. Suppose
$$A \equiv \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \end{bmatrix}.$$
Find the eigenvalue that the eigenvector $(-2, 1, 1)$ corresponds to.

HWVIII.4. Suppose A is a 2×2 matrix with eigenvector $(1, 0)$ corresponding to the eigenvalue 2 and eigenvector $(0, 1)$ corresponding to the eigenvalue 3. What must A equal?

HWVIII.5. Suppose $A \equiv \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ has eigenvectors $(1, 1, 1), (1, 1, -2), (-1, 1, 0)$. Diagonalize A; that is, find diagonal D and invertible P so that $A = PDP^{-1}$.

HWVIII.6. Suppose $P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$, where $P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$. Get A^k, for arbitrary k.

HWVIII.7. Suppose $A \equiv \begin{bmatrix} 1 & 1 & 0 \\ 2 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ has eigenvalues $1, 2$, and -3. Diagonalize A; that is, find diagonal D and invertible P so that $A = PDP^{-1}$.

HWVIII.8. In each of the following, diagonalize A.

(a) $A \equiv \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$.

(b) $A \equiv \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 0 & 2 \end{bmatrix}$.

(c) A has eigenspaces $E_0 = \{(s, 2s, 3t, 4t, 5t) \mid s, t \text{ arbitrary}\}$, $E_1 = \{(0, s, s+r, s-t, r+2t) \mid r, s, t \text{ arbitrary }\}$.

HWVIII.9. Find all eigenvalues of $\begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}$.

HWVIII.10. For A as in no. 8(c), find $(A^k - A)$, for all k. You do not need to calculate A; all that's relevant is that A is diagonalizable and has only 0 and 1 as eigenvalues.

HWVIII.11. In each part (a)–(f) below, choose one of the following

(1) A is guaranteed, just from the information given, to be diagonalizable.

(2) A is guaranteed, just from the information given, to be not diagonalizable.

(3) There is insufficient information to determine if A is diagonalizable.

DO NOT DIAGONALIZE.

(a) A is 7×7 and has eigenvalues $0, 1, 5, 9$.

(b) A is 6×6 and has eigenvalues $0, 1, 5, 9, 10, 11$.

(c) There's a diagonalizable matrix B such that $B^2 = A$.

(d) A has exactly one eigenvalue.

(e) $A \equiv \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 7 \\ 3 & 6 & 8 & 9 \\ 4 & 7 & 9 & 10 \end{bmatrix}$. (f) $A \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$.

HWVIII.12. Suppose $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & -2 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $A = PDP^{-1}$. Find A^k, for $k = 1, 2, ...$ Your answer should be a single matrix, with ks and numbers in each entry.

HWVIII.13. (a) Find the solution of

$$\vec{x}_{n+1} = \begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix} \vec{x}_n \quad (n = 0, 1, 2, ...).$$

such that $\vec{x}_0 = (1, 0)$. Your answer should be a single vector, with numbers and ns in each component.

(b) Find $\vec{x}_{10}$ when $\vec{x}_0 = (1, 0)$.

HWVIII.14. Suppose $A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \vec{0}$. Find A^k, for $k = 1, 2, 3, \ldots$

HWVIII.15. In each of the following, either diagonalize A or state (with reasons) that it can't be diagonalized.

(a) $c_A(t) = t^2(1-t)^3(4-t)$, $E_0 = \{(s, 2t, s+t, 0, 0, 0) \mid s, t \text{ arbitrary }\}$, $E_4 = \{(s, 0, 2s, 3s, 0, 4s) \mid s \text{ arbitrary }\}$, $E_1 = \{(0, 0, s, t, r, s - r) \mid r, s, t \text{ arbitrary }\}$.

(b) $c_A(t) = t^2(1-t)^3(4-t)$, $E_0 = \{(s, 2t, s+t, 0, 0, 0) \mid s, t \text{ arbitrary }\}$, $E_4 = \{(s, 0, 2s, 3s, 0, 4s) \mid s \text{ arbitrary }\}$, $E_1 = \{(0, 0, s, 0, r, s - r) \mid r, s \text{ arbitrary }\}$.

(c) A is 5×5, with eigenspaces $E_{-1} = \{(t, 0, t, t, s) \mid s, t \text{ arbitrary }\}$, $E_2 = \{(0, t, 2t, 3t, 0) \mid t \text{ arbitrary }\}$, $E_5 = \{(s, s, t, -t, 0) \mid s, t \text{ arbitrary }\}$.

(d) A is 5×5, with eigenspaces $E_{-1} = \{(t, 0, t, t, t) \mid t \text{ arbitrary }\}$, $E_2 = \{(0, t, 2t, 3t, 0) \mid t \text{ arbitrary }\}$, $E_5 = \{(s, s, t, -t, 0) \mid s, t \text{ arbitrary }\}$.

HWVIII.16. Find all eigenvalues of $\begin{bmatrix} 1 & -4 \\ -3 & 2 \end{bmatrix}$.

HWVIII.17. Suppose
$$A \equiv \begin{bmatrix} 3 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 3 & 0 \end{bmatrix}.$$
Find the eigenspace of the eigenvalue 3 (for A).

HWVIII.18. Suppose
$$A \equiv \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix}.$$
Find the eigenvalue that the eigenvector $(2, -1, 1)$ corresponds to.

HWVIII.19. Diagonalize $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$.

HWVIII.20. Suppose $A \equiv \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ has eigenvectors $(1,1,1), (1,1,-2), (-1,1,0)$. Diagonalize A.

HWVIII.21. Suppose $A \equiv \begin{bmatrix} 4 & 1 & 0 \\ 2 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$ has eigenvalues $4, 5$, and 0. Diagonalize A.

HWVIII.22. Diagonalize A, if A has the following eigenspaces.
$E_0 = \{(t, 2t, 3t, 0, 0) \mid t \text{ arbitrary }\}, E_1 = \{(s+t, 0, 0, s-t, 0) \mid s, t \text{ arbitrary }\}, E_4 = \{(0, 0, s, t, 2s+t) \mid s, t \text{ arbitrary }\}.$

HWVIII.23. Suppose $A \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, A \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, A \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 2 \end{bmatrix}, A \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \end{bmatrix}$. Diagonalize A.

HWVIII.24. Find the eigenvalues, including multiplicity, of each of the following matrices.
$$(a) \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \quad (b) \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \quad (c) \begin{bmatrix} 5 & 1 & 2 \\ 0 & 3 & 0 \\ 2 & 1 & 5 \end{bmatrix} \quad (d) \begin{bmatrix} 2 & 3 & 0 \\ -1 & -2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$
$$(e) \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \quad (f) \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

HWVIII.25. Find the nontrivial eigenspaces of the matrices in HWVIII.24.

HWVIII.26. Believe us when we tell you that $(1, 2, 2), (2, 1, -2)$, and $(2, -2, 1)$ are eigenvectors of
$$\begin{bmatrix} 2 & -4 & 2 \\ -4 & 2 & -2 \\ 2 & -2 & -1 \end{bmatrix}.$$
Find the corresponding eigenvalues.

HWVIII.27. For each of the matrices in HWVIII.24, either diagonalize, or show that it cannot be diagonalized.

HWVIII.28. Suppose A has eigenvalues $1, 1, 0$, with corresponding eigenvectors $(1, 2, -1), (0, 1, 1), (1, 2, 1)$. Find A.

HWVIII.29. Find S so that $S^{-1}AS$ is diagonal, if $A \equiv \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}$.

HWVIII.30. Find a diagonal matrix that $A \equiv \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is similar to.

HWVIII.31. Find A^k, for arbitrary k, for each of the following matrices A.

$$(a) \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (b) \quad \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \quad (c) \;\; (c) \text{ from HWVIII.24} \quad (d) \;\; (e) \text{ from HWVIII.24.}$$

HWVIII.32. Suppose A has eigenvalues $1, \frac{1}{2}, \frac{1}{3}$, with corresponding eigenvectors $(1, 2, -1), (0, 1, 1), (1, 2, 1)$. Find A^k, for arbitrary k.

HWVIII.33. Suppose A has eigenvalues $2, 3$ and corresponding eigenspaces

$$E_2 = \{(s+t, s, t) \mid s, t \text{ are arbitrary}\}, E_3 = \{(0, t, t) \mid t \text{ is arbitrary}\}.$$

Find A^k, for arbitrary k.

HWVIII.34. Find the Fibonacci numbers F_{42} and F_{29}.