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Matrices and Motion
MATHematics MAGnification™

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MATRICES and MOTION MAGNIFICATION

This is one of a series of very short books on math, statistics, and physics called "Math Magnifications." The "magnification" refers to focusing on a particular topic that is pivotal in or emblematic of mathematics.

OUTLINE

A matrix is a rectangular array of numbers. We will present the algebra of matrices. By representing motions of interest (projection, reflection, and rotation) with matrices, we will become able to extend geometry beyond what is visible to the naked eye, or even visualizable to the naked brain. In particular, we can perform a dizzying sequence of motions, by multiplying the corresponding matrices. Proofs are included for the curious, but are not needed for examples or homework.

This magnification will expose the reader to the interaction between algebra (calculation) and geometry (pictures).

For this magnification, students should be familiar with lines and line segments, angles between, and midpoints of, lines or line segments and the Cartesian plane (including distance between points in the Cartesian plane) and be able to do products and sums (including negative numbers, decimals, fractions, and irrational numbers) without calculators. Reference [4] is more than sufficient.

See [1] and [3] for a much more complete treatment of matrices.

Definition 1. For m, n positive integers, an $m \times n$ (reads "m by n") (real) **matrix** is a rectangular array of m rows (horizontal sequences of n real numbers) and n columns (vertical sequences of m real numbers)

$$A \equiv (a_{ij}) \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix};$$

a_{ij} denotes the ij^{th} entry, the number in the i^{th} row and j^{th} column.

Example 2. $A \equiv \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 2 & 1 \end{bmatrix}$ is a (3×2) matrix, with rows $\begin{bmatrix} 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & -1 \end{bmatrix}$, and $\begin{bmatrix} 2 & 1 \end{bmatrix}$, and columns $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$. The entry $a_{11} = 1, a_{12} = 2, a_{21} = 0$, etc.

Definition 3. The **transpose** of the $m \times n$ matrix A, denoted A^T ("A transpose") is the $n \times m$ matrix such that, for $1 \leq i \leq m$, the i^{th} column of A^T equals the i^{th} row of A.

Example 4. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

Definitions 5: Matrix algebra. Here we present multiplication of real numbers times matrices and addition of matrices. Multiplication of matrices times matrices will appear in Definitions 12.

In these definitions, c is a real number, A is as in Definition 1, and

$$B \equiv (b_{ij}) \equiv \begin{bmatrix} b_{11} & b_{12} & b_{13} & \dots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \dots & b_{2n} \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ b_{m1} & b_{m2} & b_{m3} & \dots & b_{mn} \end{bmatrix}.$$

Then $cA = (ca_{ij})$ and $(A + B) = (a_{ij} + b_{ij})$; that is, addition and multiplication of matrices is done entrywise.

The matrix $A - B$ is defined to be $A + (-1)B$.

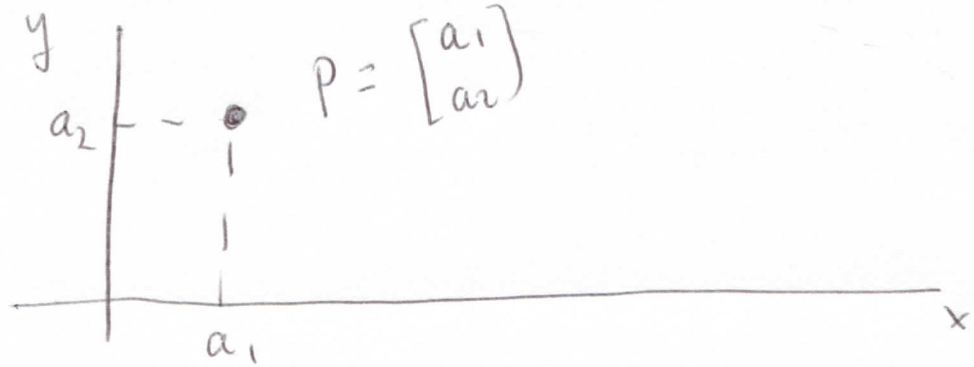
Examples 6. $3\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}; 2\begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 0 & 4 \\ 0 & 2 & 2 & 2 \end{bmatrix}$.

Definitions 7. For the remainder of this magnification, we will only consider m, n equal to 1 or 2 in Definition 1: (2×2) matrices $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, (2×1) matrices (also called **column 2-vectors**) $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, (1×2) matrices (also called **row 2-vectors**) $\begin{bmatrix} a_1 & a_2 \end{bmatrix}$, and (1×1) matrices $\begin{bmatrix} a \end{bmatrix}$, a number placed in a box.

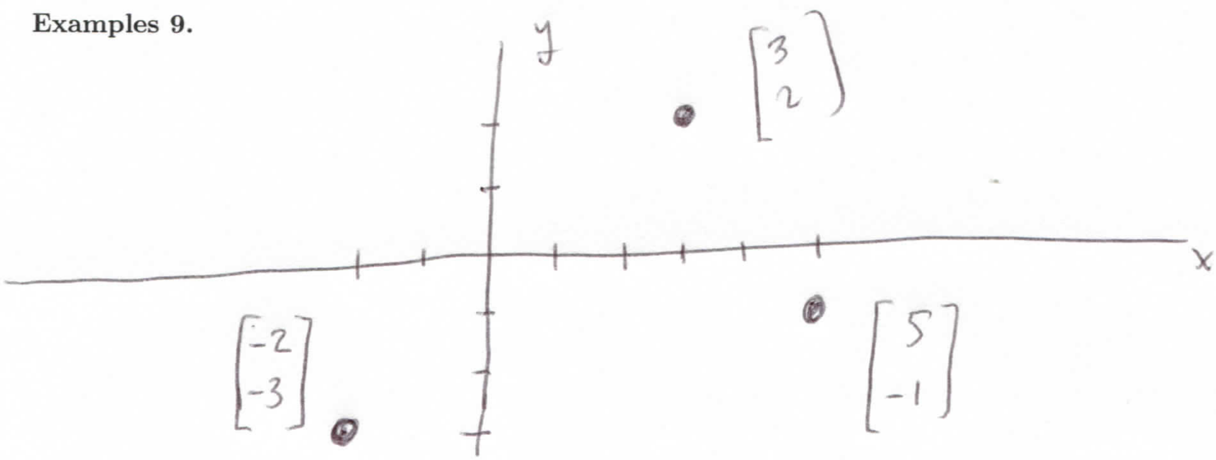
Notice that a row 2-vector is the transpose of a column 2-vector.

Definition 8: Pictorial convention. For any real a_1, a_2, we will associate the column 2-vector $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ with the point in the Cartesian plane (hereafter referred to as **the plane**) a_1 units to the right of the origin and a_2 units above the origin.

The statement "the point P $= \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$" refers to that correspondence.



Examples 9.



Here is a result relating geometry to matrix algebra.

Proposition 10. Suppose P, Q, and R are points in the plane.

(a) Q is on the line segment between P and R if and only if (Q − P) = c(R − Q) for some positive c if and only if Q = $(1-t)$P + tR, for some t with $0 \leq t \leq 1$.

(b) Q is the midpoint of the line segment between P and R if and only if (Q − P) = (R − Q) if and only if Q = $\frac{1}{2}$P + $\frac{1}{2}$R.

Examples 11. Suppose P equals $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and R equals $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$.

(a) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is the midpoint of the line segment from P to R, since

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

and

$$\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \left(\begin{bmatrix} -1 \\ 4 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right).$$

(b) $\frac{1}{2}\begin{bmatrix} -1 \\ 5 \end{bmatrix}$ is on the line segment between P and R but is not the midpoint, since

$$\frac{1}{2}\begin{bmatrix} -1 \\ 5 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \frac{3}{4}\begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

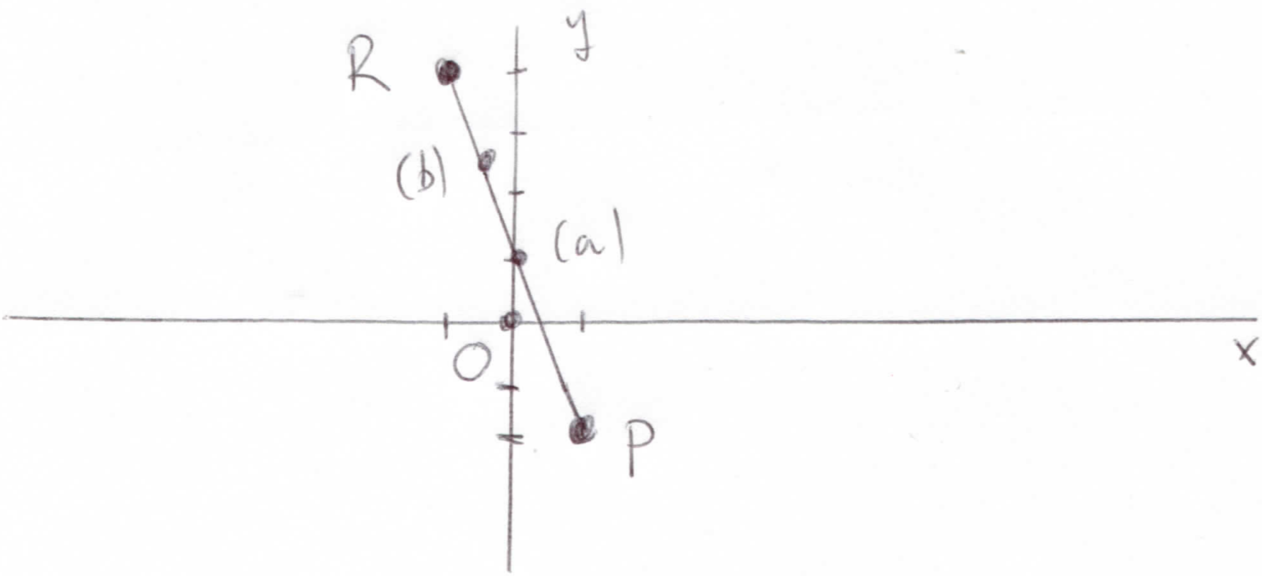
and

$$\left(\frac{1}{2}\begin{bmatrix} -1 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} -3 \\ 9 \end{bmatrix} = 3\left(\frac{1}{2}\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right) = 3\left(\begin{bmatrix} -1 \\ 4 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 5 \end{bmatrix}\right).$$

(c) The origin $O \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is not on the line segment between P and R, since

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix} = O - P = c(R - O) = c\begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

implies that $-1 = c(-1)$ and $2 = c(4)$, impossible.



Notice that, as t goes from 0 to 1 in Proposition 10(a), $Q = (1-t)P + tR$ moves along the line segment between P and R, from P to R.

Definitions 12: Matrix multiplication. Historically, matrices first appeared as a way of solving a set of linear equations, such as

$$\begin{array}{rcrcl} x & + & 2y & = & 3 \\ 4x & + & 5y & = & 6 \end{array}$$

by writing it as a single matrix equation

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

This rewriting leads to the following definitions, beginning with a row vector on the left times a column vector on the right.

$$\begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \left[(a_1 b_1 + a_2 b_2) \right];$$

that number $(a_1b_1 + a_2b_2)$ is the **dot product** or **inner product** of the vectors $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$; see [1] and [3].

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} (a_{11}b_1 + a_{12}b_2) \\ (a_{21}b_1 + a_{22}b_2) \end{bmatrix}; \quad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} (a_{11}b_{11} + a_{12}b_{21}) & (a_{11}b_{12} + a_{12}b_{22}) \\ (a_{21}b_{11} + a_{22}b_{21}) & (a_{21}b_{12} + a_{22}b_{22}) \end{bmatrix}.$$

In general, one performs row-on-left times column-on-right wherever possible; here's a less popular example.

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \begin{bmatrix} a_1 & a_2 \end{bmatrix} = \begin{bmatrix} b_1 a_1 & b_1 a_2 \\ b_2 a_1 & b_2 a_2 \end{bmatrix};$$

notice how different $\left(\begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right)$ and $\left(\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \begin{bmatrix} a_1 & a_2 \end{bmatrix} \right)$ are.

Examples 13.

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \left[1 \times 3 + 2 \times 4 \right] = [11]; \quad \begin{bmatrix} 5 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \left[5 \times 3 + 6 \times 4 \right] = [39];$$

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = [2]; \quad \begin{bmatrix} 5 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = [6]; \quad \begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 11 \\ 39 \end{bmatrix}; \quad \begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 2 \\ 39 & 6 \end{bmatrix}.$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

With numbers, one gets the same product regardless of the order of operation: $ab = ba$, for any real numbers a, b; that is, numerical multiplication *commutes*. The last example of matrix multiplication shows that matrix multiplication does *not* commute, in general.

We will see numerous examples that might make this seem more natural, when we relate matrices to *motion* in 15 through 19. In general, we want to think of matrix multiplication as *doing things* to points or column vectors. For example, suppose you have a matrix A that represents "open the window" and a matrix B that represents "put your head through the window." Then you should believe that

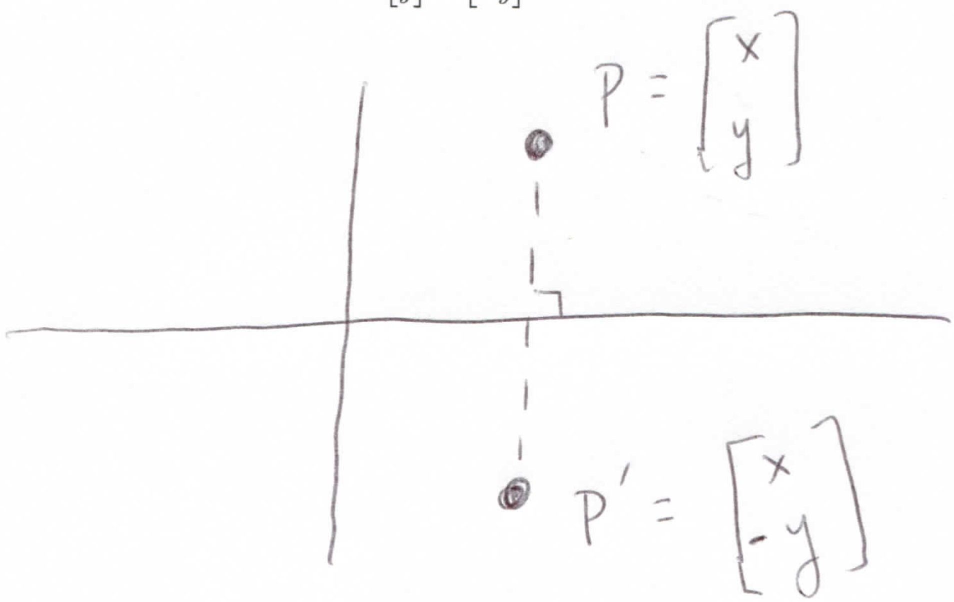
$$[A, \text{ followed by } B] \qquad \text{is not the same as} \qquad [B, \text{ followed by } A].$$

Definition 14. The ((2 × 2)) **identity** matrix is $I \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Notice that, for any real x, y, $I \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$. The identity matrix is the unique matrix that does *not* change any vectors, when you multiply by it.

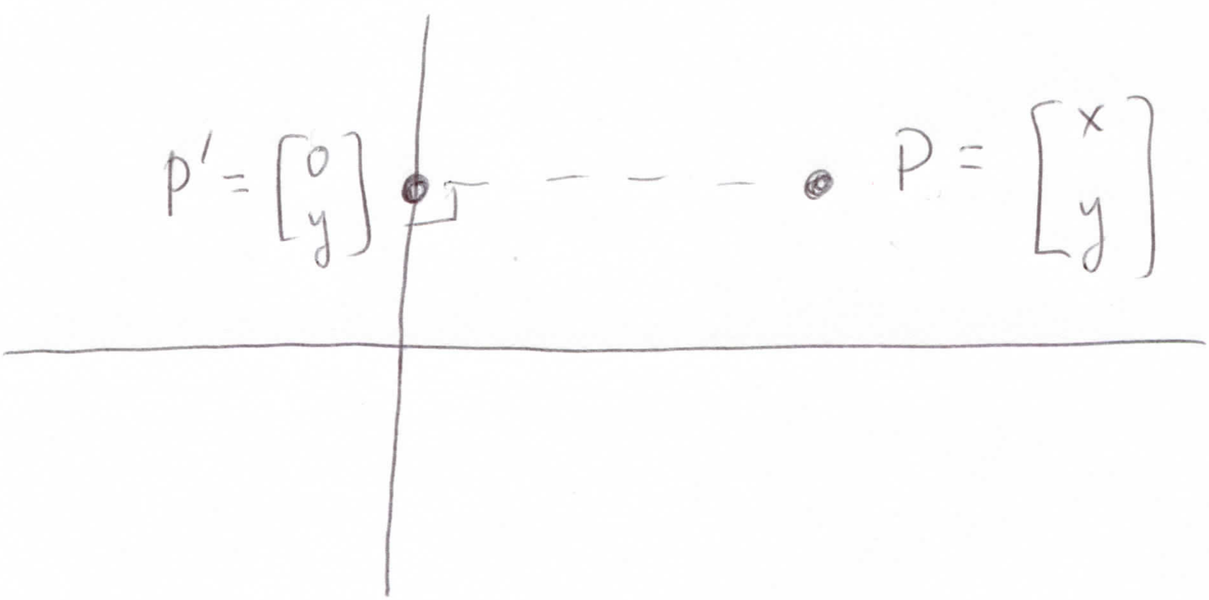
Examples 15. (a) If $A \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then, for any real x, y,

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}.$$



(b) If $A \equiv \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, then, for any real x, y,

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}.$$



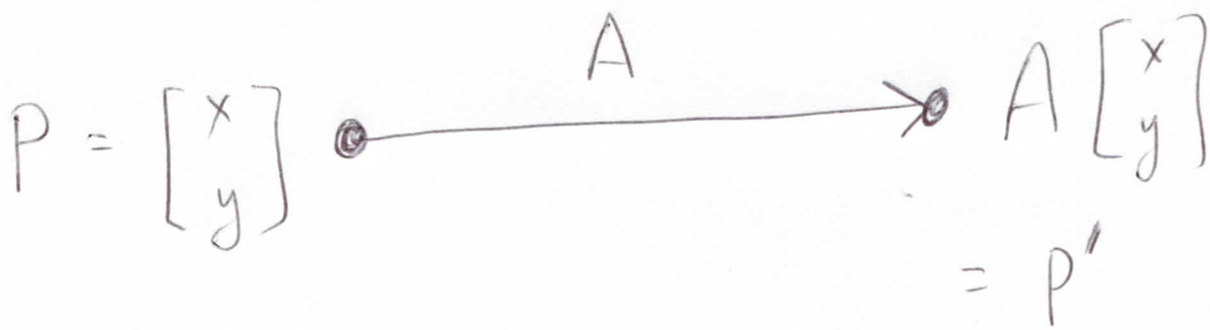
Notice the sense of motion, in passing from a point $\mathrm{P} = \begin{bmatrix} x \\ y \end{bmatrix}$ to the new point $\mathrm{P}' = A \begin{bmatrix} x \\ y \end{bmatrix}$. For $A \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, that motion is called a *reflection* (through the x axis), while for $A \equiv \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, the motion is called a *projection* (onto the y axis). See Definitions 17.

Matrix multiplication *does something* to points or vectors; in the examples just mentioned, but in a more dynamic verb form, one matrix reflected through the x axis, while the other projected onto the y axis.

Definitions 16. (See Definition 8.) Let A be an arbitrary (2×2) matrix. The **motion, with standard matrix** A, is the rule that assigns, to each point $\mathrm{P} = \begin{bmatrix} x \\ y \end{bmatrix}$ in the plane, the point $\mathrm{P}' = A \begin{bmatrix} x \\ y \end{bmatrix}$. This is denoted

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto A \begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{or} \quad \mathrm{P} \mapsto \mathrm{P}', \quad \text{or} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \mathrm{P} \mapsto \mathrm{P}' = A \begin{bmatrix} x \\ y \end{bmatrix}.$$

The point P' is the **image** of P, under the motion.



A more popular synonym for motion is a *linear transformation* from the plane to itself, a special type of *function* or *map*. See [1].

Thus, from Examples 15,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathrm{P} \mapsto \mathrm{P}' = \begin{bmatrix} x \\ -y \end{bmatrix}$$

has standard matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, while

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathrm{P} \mapsto \mathrm{P}' = \begin{bmatrix} 0 \\ y \end{bmatrix}$$

has standard matrix $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

The challenge of interest is to *start* with a motion

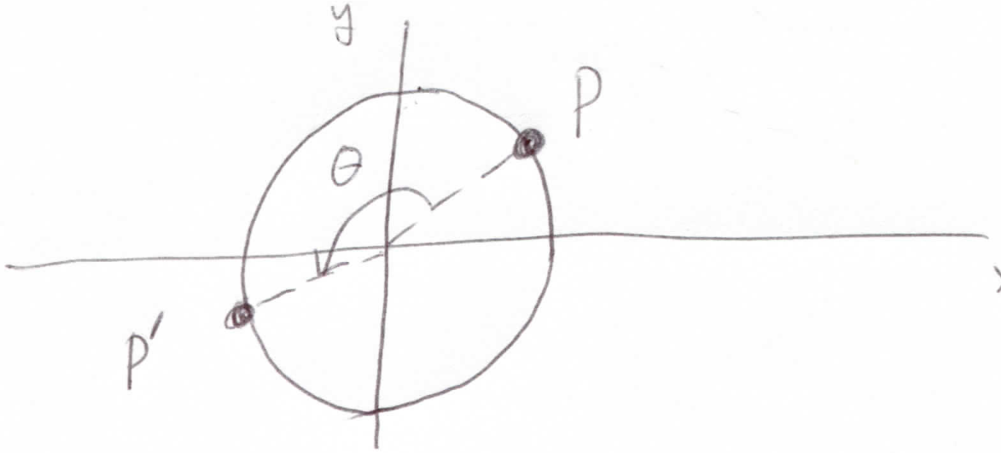
$$\mathrm{P} \mapsto \mathrm{P}'$$

and find the standard matrix for that motion.

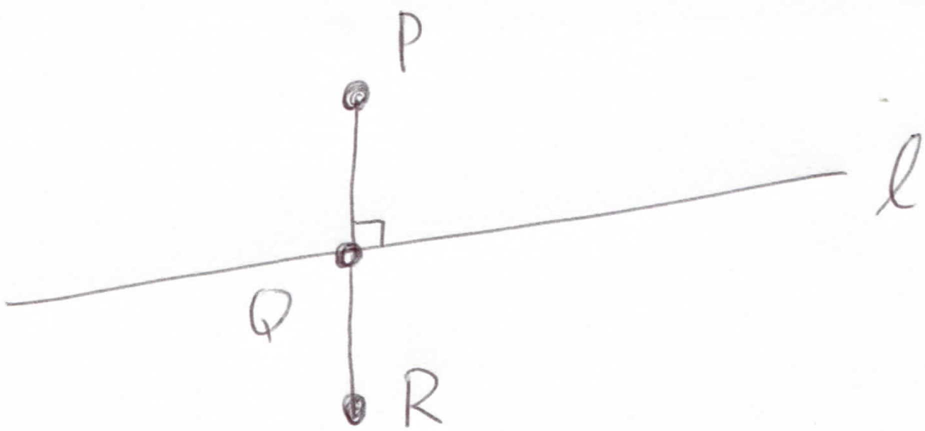
Definitions 17: Motions of interest. (a) Given an angle θ (spelled "theta", pronounced "thay-tuh"), the (counterclockwise) **rotation** of angle θ is the motion

$$\mathrm{P} \mapsto \mathrm{P}'$$

such that P′ is on the same circle centered at the origin as P and the counterclockwise angle, from the line segment between the origin and P, and the line segment between the origin and P′, is θ.



For (b) and (c), consider the following picture, where ℓ is a line through the origin and Q is a point on ℓ.



If the line segment from P to R is perpendicular to the line ℓ and Q is the midpoint of the line segment from P to R, then

(b) $\mathrm{P}' \equiv \mathrm{Q}$ is the **projection** of P onto ℓ and

(c) $\mathrm{P}' \equiv \mathrm{R}$ is the **reflection** of P through ℓ.

For reflection, think of ℓ as a mirror; $\mathrm{P}' \equiv \mathrm{R}$ is then literally the *image*, as in Definitions 16, of P.

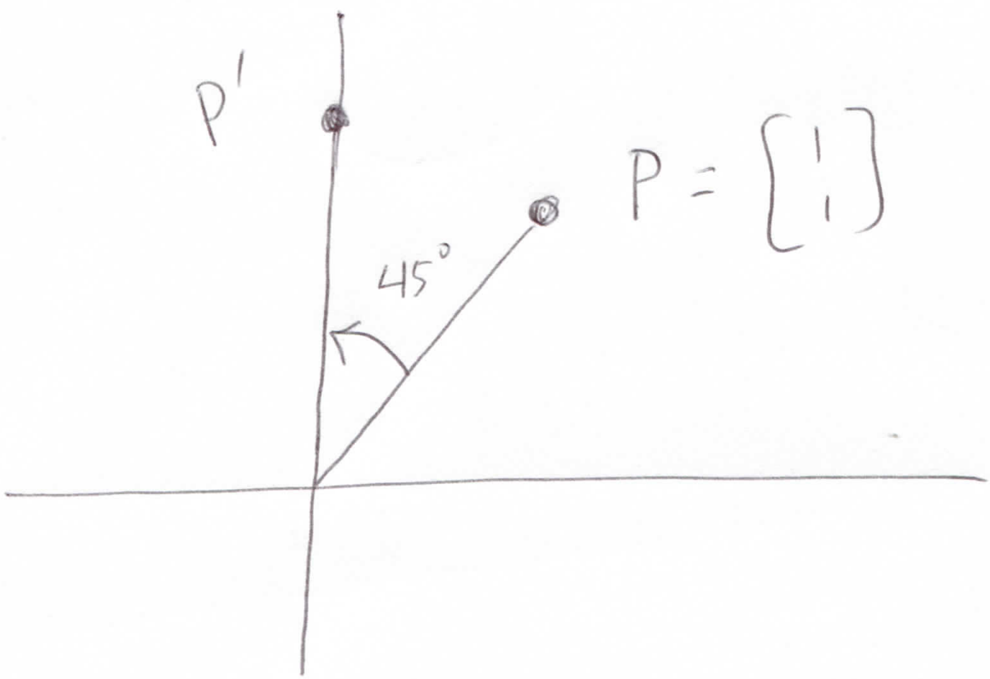
Theorem 18: Standard matrices for Definitions 17. (a) For those who have seen trigonometry (trig), as in [2], the standard matrix is $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

To avoid the necessity of trig, we will restrict ourselves to θ equal to some multiples of 45 degrees. See also Homework number 7.

With each of the following rotation matrices, we will illustrate its action by applying it to the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For a counterclockwise rotation of 45 degrees, the standard matrix is

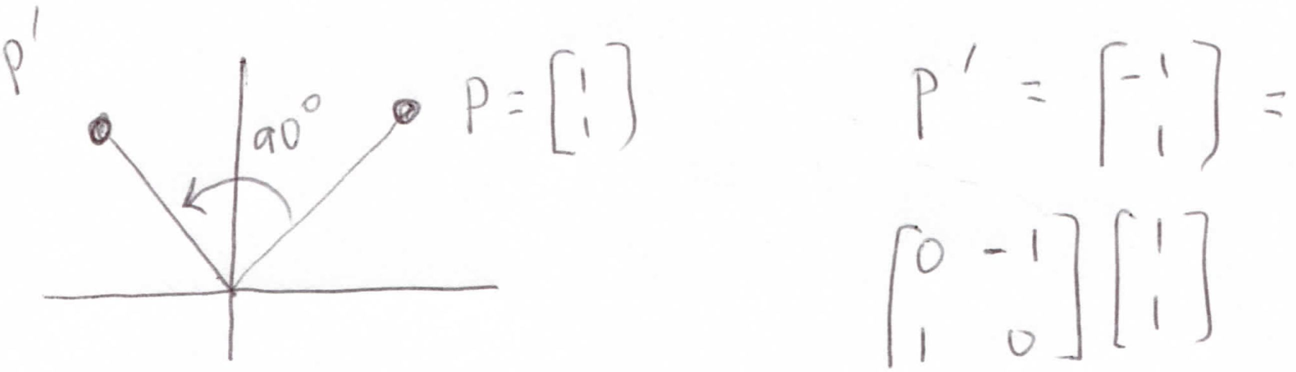
$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$



$$P' = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

It follows, by squaring the 45 degree rotation matrix, that the standard matrix for counterclockwise rotation of 90 degrees is

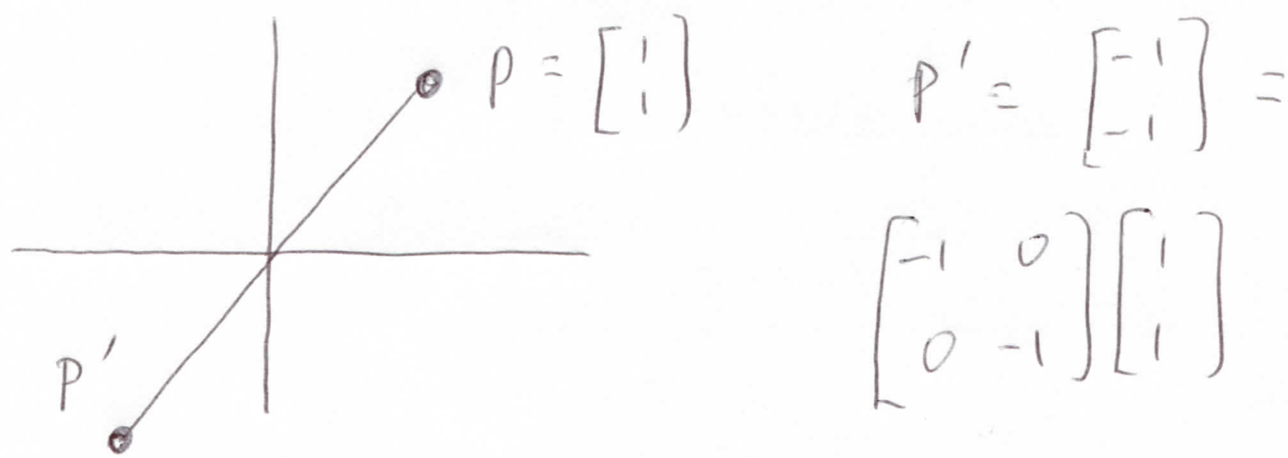
$$\left(\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\right)\left(\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\right) = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}.$$



We can next square the 90 degree matrix to get the standard matrix for rotation by 180 degrees:

$$\left(\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\right)\left(\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\right) = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix} = -I;$$

notice that this performs a reflection through the origin: $\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}-x\\ -y\end{bmatrix}$, for real x, y.



For (b) and (c), let $\begin{bmatrix}a\\ b\end{bmatrix}$ be a point other than the origin on the line ℓ; that is, ℓ is the line through the origin $\begin{bmatrix}0\\ 0\end{bmatrix}$ and $\begin{bmatrix}a\\ b\end{bmatrix}$.

(b) The standard matrix for projection onto ℓ is

$$\frac{1}{a^2+b^2}\begin{bmatrix}a^2 & ab\\ ab & b^2\end{bmatrix}.$$

(c) The standard matrix for reflection through ℓ is

$$\frac{1}{a^2+b^2}\begin{bmatrix}(a^2-b^2) & 2ab\\ 2ab & (b^2-a^2)\end{bmatrix}.$$

Examples 19. All rotations are counterclockwise.

(a) Find the standard matrix for the following motion: rotate 45 degrees, then reflect through $y = 2x$.

(b) Find the standard matrix for the following motion: reflect through $y = 2x$, then rotate 45 degrees.

(c) Find the standard matrix for the following motion: rotate 90 degrees, then reflect through $y = -x$.

(d) Find the standard matrix for the following motion: reflect through $y = -x$, then rotate 90 degrees.

(e) Find the standard matrix for the following motion: rotate 180 degrees, then reflect through $y = 3x$.

(f) Find the standard matrix for the following motion: reflect through $y = 3x$, then rotate 180 degrees.

(g) Find the standard matrix for the following motion: project onto $y = x$, then rotate 90 degrees.

(h) Find the standard matrix for the following motion: rotate 90 degrees, then project onto $y = x$.

(i) Find the standard matrix for the following motion: project onto $y = x$, then rotate 90 degrees, then project onto $y = x$.

(j) Find the standard matrix for the following motion: project onto $2y = -x$, then reflect through the y axis.

(k) Find the standard matrix for the following motion: project onto $2y = -x$, then reflect through the y axis, then rotate 90 degrees.

(ℓ) Find the image of the point $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ under the motion in (a).

(m) Find the image of the point $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ under the motion in (b).

(n) Find the image of the point $\begin{bmatrix} -3 \\ 5 \end{bmatrix}$ under the motion in (c).

(o) Find the image of the point $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ under the motion in (g).

(p) Find the image of the point $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ under the motion in (h).

(q) Find the image of the point $\begin{bmatrix} \sqrt{3} \\ 77 \end{bmatrix}$ under the motion in (i).

(r) Denote by $A_1 \equiv \frac{1}{a^2+b^2} \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$ the standard matrix for projection and by $A_2 \equiv \frac{1}{a^2+b^2} \begin{bmatrix} (a^2-b^2) & 2ab \\ 2ab & (b^2-a^2) \end{bmatrix}$ the standard matrix for reflection.

Argue both geometrically and algebraically that $A_1^2 \equiv A_1 A_1 = A_1$ and $A_2^2 = I$ (see Definition 14).

Solutions. For a line ℓ with equation $y = mx$, we choose $(a, b) \equiv (1, m)$ in (b) and (c) of Theorem 18. For ℓ equal to the y axis, we choose $(a, b) \equiv (0, 1)$.

Also see Theorem 18 for rotation matrices.

In general, for a motion with standard matrix A_1, followed by a motion with standard matrix A_2, follow the motion, starting with a point P $= \begin{bmatrix} x \\ y \end{bmatrix}$:

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto A_1 \begin{bmatrix} x \\ y \end{bmatrix} \mapsto A_2 \left(A_1 \begin{bmatrix} x \\ y \end{bmatrix} \right) = (A_2 A_1) \begin{bmatrix} x \\ y \end{bmatrix},$$

thus the standard matrix for that sequence of motions is $(A_2 A_1)$.

Because we multiply matrices on the left, but read normal words from left to right, we get a counterintuitive result for sequences of motions.

(a)

$$\frac{1}{5}\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{5\sqrt{2}} \begin{bmatrix} 1 & 7 \\ 7 & -1 \end{bmatrix}.$$

(b)

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix} = \frac{1}{5\sqrt{2}} \begin{bmatrix} -7 & 1 \\ 1 & 7 \end{bmatrix}.$$

(c)

$$\frac{1}{2}\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(d)

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

(e)

$$\frac{1}{10}\begin{bmatrix} -8 & 6 \\ 6 & 8 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 8 & -6 \\ -6 & -8 \end{bmatrix}.$$

(f)

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{10}\begin{bmatrix} -8 & 6 \\ 6 & 8 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 8 & -6 \\ -6 & -8 \end{bmatrix}.$$

(g)

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}.$$

(h)

$$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}.$$

(i)

$$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(j)

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{5}\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -4 & 2 \\ -2 & 1 \end{bmatrix}.$$

We could have used $(a, b) = (2, -1)$ for this projection.

(k)

$$\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\frac{1}{5}\begin{bmatrix}4 & -2\\-2 & 1\end{bmatrix}=\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\frac{1}{5}\begin{bmatrix}-4 & 2\\-2 & 1\end{bmatrix}=\frac{1}{5}\begin{bmatrix}2 & -1\\-4 & 2\end{bmatrix}.$$

(ℓ)

$$\frac{1}{5\sqrt{2}}\begin{bmatrix}1 & 7\\7 & -1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}3\\1\end{bmatrix}.$$

(m)

$$\frac{1}{5\sqrt{2}}\begin{bmatrix}-7 & 1\\1 & 7\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}.$$

(n)

$$\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}-3\\5\end{bmatrix}=\begin{bmatrix}3\\5\end{bmatrix}.$$

(o)

$$\frac{1}{2}\begin{bmatrix}-1 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}.$$

(p)

$$\frac{1}{2}\begin{bmatrix}1 & -1\\1 & -1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}.$$

(q)

$$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}\sqrt{3}\\77\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}.$$

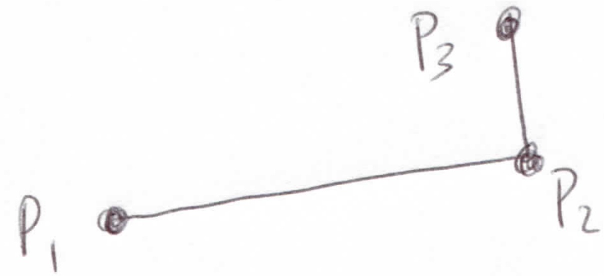
(r) For both A_1 and A_2, "algebraically" means matrix multiplication; check by multiplying that $A_1^2 = A_1$ and $A_2^2 = I$.

The geometry of multiplying by A_1 is that you've dropped to the line ℓ. Applying A_1 again does not change anything, since you're already on the line; for any $\begin{bmatrix}x\\y\end{bmatrix}$ on the line, $A_1\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\y\end{bmatrix}$, like dropping to the earth when you're already on the earth.

The geometry of multiplying by A_2 is that you reflect back to where you started, after applying A_2 again; see the second drawing in Definitions 17.

Partial proofs 20, of Theorem 18. Begin with

Lemma 21. Let P_1, P_2, and P_3 be points in the plane. The line segment from P_3 to P_2 is perpendicular to the line segment from P_1 to P_2 if and only if $(P_3 - P_2)^T(P_1 - P_2) = \begin{bmatrix}0\end{bmatrix}$.



Lemma 21 is a sort of opposite of Proposition 10, in the sense that being parallel is the opposite of being perpendicular.

Partial Proof of Theorem 18(a), for θ equal to 90 degrees. For arbitrary real x, y, we need to show that

$$\begin{bmatrix}-y\\x\end{bmatrix} = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

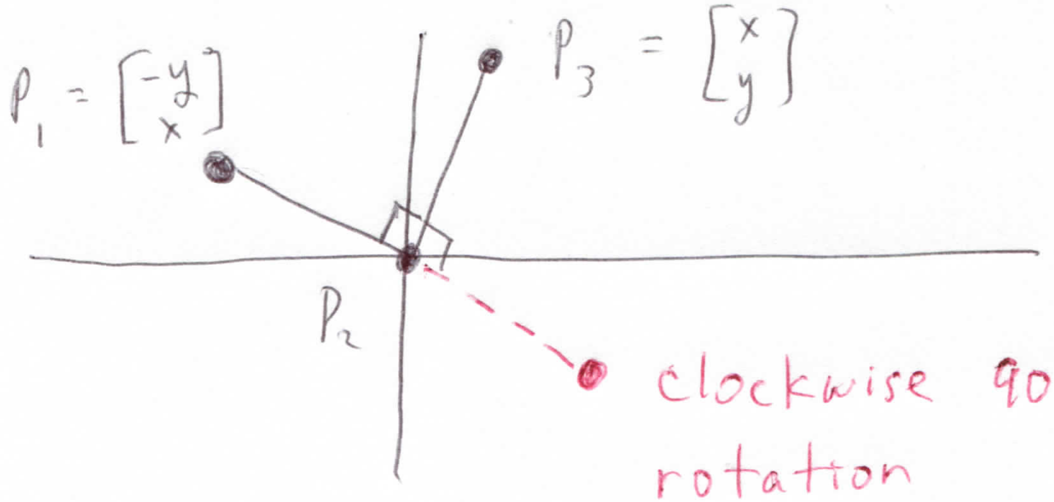
is the 90 degree counterclockwise rotation of $\begin{bmatrix}x\\y\end{bmatrix}$, as in Definitions 17(a).

Note first that both $\begin{bmatrix}x\\y\end{bmatrix}$ and $\begin{bmatrix}-y\\x\end{bmatrix}$ are on the same circle centered at the origin, since their distances to the origin are the same.

For the 90 degree angle as in Definitions 17(a), apply Lemma 21, with P_2 equal to the origin, P_3 equal to $\begin{bmatrix}x\\y\end{bmatrix}$, P_1 equal to $\begin{bmatrix}-y\\x\end{bmatrix}$ (see drawing below):

$$(P_3 - P_2)^T(P_1 - P_2) = \begin{bmatrix}x\\y\end{bmatrix}^T\begin{bmatrix}-y\\x\end{bmatrix} = \begin{bmatrix}(-yx + xy)\end{bmatrix} = \begin{bmatrix}0\end{bmatrix}.$$

This shows that the line segment between the origin and $\begin{bmatrix}x\\y\end{bmatrix}$ is perpendicular to the line segment between the origin and $\begin{bmatrix}-y\\x\end{bmatrix}$, so that $\begin{bmatrix}-y\\x\end{bmatrix}$ is the counterclockwise or clockwise rotation of 90 degrees of $\begin{bmatrix}x\\y\end{bmatrix}$; we are leaving unproved the fact that it is the *counterclockwise* rotation. □



Proof of Theorem 18(b). For arbitrary real x, y, we need to show that

$$P_2 = \frac{1}{a^2+b^2}\begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

is the projection of $P_3 = \begin{bmatrix} x \\ y \end{bmatrix}$ onto ℓ.

To complete our identification with Lemma 21, let P_1 equal the origin.

Note first that

$$P_2 = \frac{1}{a^2+b^2}\begin{bmatrix} a^2x + aby \\ abx + b^2y \end{bmatrix} = \frac{ax+by}{a^2+b^2}\begin{bmatrix} a \\ b \end{bmatrix}. \quad (*)$$

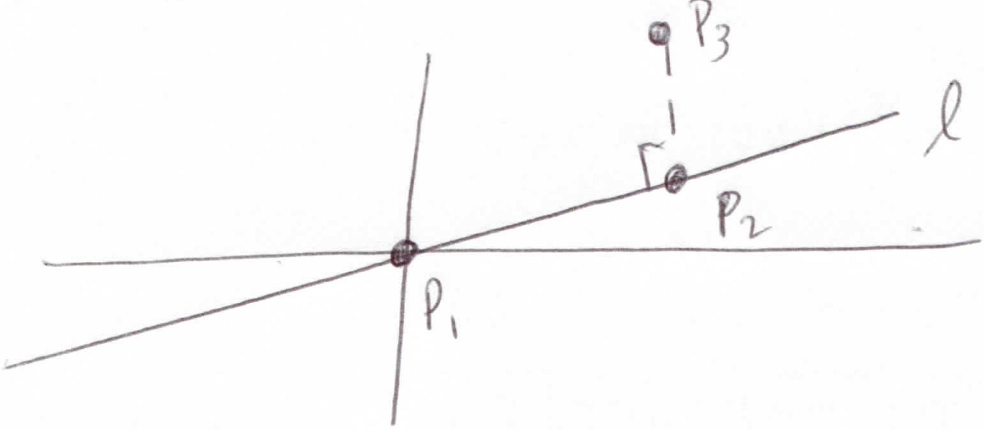
Since P_2 is a real multiple of $\begin{bmatrix} a \\ b \end{bmatrix}$, P_2 is on the line ℓ.

It remains to show that the line segment from P_3 to P_2 is perpendicular to the line segment from P_2 to P_1. This will follow from Lemma 21, after using (*) in the following calculation.

$$(P_3 - P_2)^T(P_1 - P_2) = (P_2 - P_3)^T(P_2 - P_1) = (P_2 - P_3)^T P_2 = P_2^T P_2 - P_3^T P_2$$

$$= \left[\frac{(ax+by)}{(a^2+b^2)}\right]^2 \begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} - \frac{(ax+by)}{(a^2+b^2)}\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \left[\frac{(ax+by)^2}{(a^2+b^2)} - \frac{(ax+by)^2}{(a^2+b^2)}\right] = [0],$$

so that Lemma 21 implies that the line segment from P_3 to P_2 is perpendicular to the line segment from P_2 to the origin, as desired. □



Proof of Theorem 18(c). By Proposition 10(b), for P, Q, and R as in Definitions 17(b) and (c),

$$(\mathrm{Q} - \mathrm{P}) = (\mathrm{R} - \mathrm{Q});$$

this implies that

$$\mathrm{R} = (2\mathrm{Q} - \mathrm{P}),$$

so that, by (b), since the identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the standard matrix for $P \mapsto P$, the standard matrix for reflection $P \mapsto R$ is

$$2\left(\frac{1}{a^2+b^2}\begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}\right) - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{a^2+b^2}\begin{bmatrix} (2a^2 - (a^2+b^2)) & 2ab \\ 2ab & (2b^2 - (a^2+b^2)) \end{bmatrix} = \frac{1}{a^2+b^2}\begin{bmatrix} (a^2-b^2) & 2ab \\ 2ab & (b^2-a^2) \end{bmatrix}.$$

□

Definition 22. If A is the standard matrix for a motion and Ω (Greek letter, spelled "Omega," pronounced "Oh-may-guh") is a subset of the plane, then the **image** of Ω under the motion is

$$\Omega' \equiv A(\Omega) \equiv \{A \begin{bmatrix} x \\ y \end{bmatrix} \mid \begin{bmatrix} x \\ y \end{bmatrix} \text{ is in } \Omega\}.$$

Theorem 23. (a) The image of a line segment under a projection is either a single point or a line segment.

(b) The image of a line segment under a rotation or reflection is a line segment.

Proof. Let A be the standard matix of a motion. By Proposition 10(a), a line segment Ω between two (different) points P and R has the form $\{(1-t)\mathrm{P} + t\mathrm{R} \mid 0 \leq t \leq 1\}$. If $\mathrm{P} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\mathrm{R} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$, it follows that

$$A(\Omega) = \{(1-t)A \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + tA \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \mid 0 \leq t \leq 1\}.$$

If $\mathrm{P}' \equiv A \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \neq \mathrm{R}' \equiv A \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$, $A(\Omega)$ is a line segment, while if $\mathrm{P}' = \mathrm{R}'$, $A(\Omega)$ is a single point.

To finish the proof of (b), it is sufficient to show that, in the preceding paragraph, P$'$ cannot equal R$'$ if P does not equal R, when the motion is a rotation or reflection.

To this end, suppose P does not equal R.

If A is the standard matrix for a rotation of angle θ degrees, with $0 \leq \theta \leq 360$, let B be the standard matrix for a rotation of angle $(360 - \theta)$ degrees. Then $P = BA \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = B\mathrm{P}'$ and $R = BA \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = B\mathrm{R}'$, so that, since P does not equal R, P$'$ cannot equal R$'$.

If A is the standard matrix for a reflection, then A^2 equals the identity matrix, thus $P = A^2 \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = A\mathrm{P}'$ and $R = A^2 \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = A\mathrm{R}'$, so that, again, since P does not equal R, P$'$ cannot equal R$'$. □

Theorem 23 implies that, to get the image of a polygon under one of the motions of Definitions 17, we need only get the images of the vertices of the polygon, then draw lines between those images that correspond to the lines in the original polygon.

Examples 24. (a) Let Ω be the triangle with vertices $\mathrm{P}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathrm{P}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\mathrm{P}_3 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

Find the image of Ω under reflections through and projections onto the axes, and under counter-clockwise rotations of 90 degrees, 180 degrees, and 270 degrees.

(b) Let Ω be the quadrilateral with vertices $\mathrm{P}_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\mathrm{P}_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\mathrm{P}_3 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, and $\mathrm{P}_4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Find the image of Ω under the motion in Examples 19(a).

(c) Let Ω be the rectangle with vertices $\mathrm{P}_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, $\mathrm{P}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\mathrm{P}_3 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ and $\mathrm{P}_4 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$.

Find the image of Ω under reflection through the x axis.

Solutions. (a) For $j = 1, 2, 3$, let P'_j be the image of P_j, under a given motion.

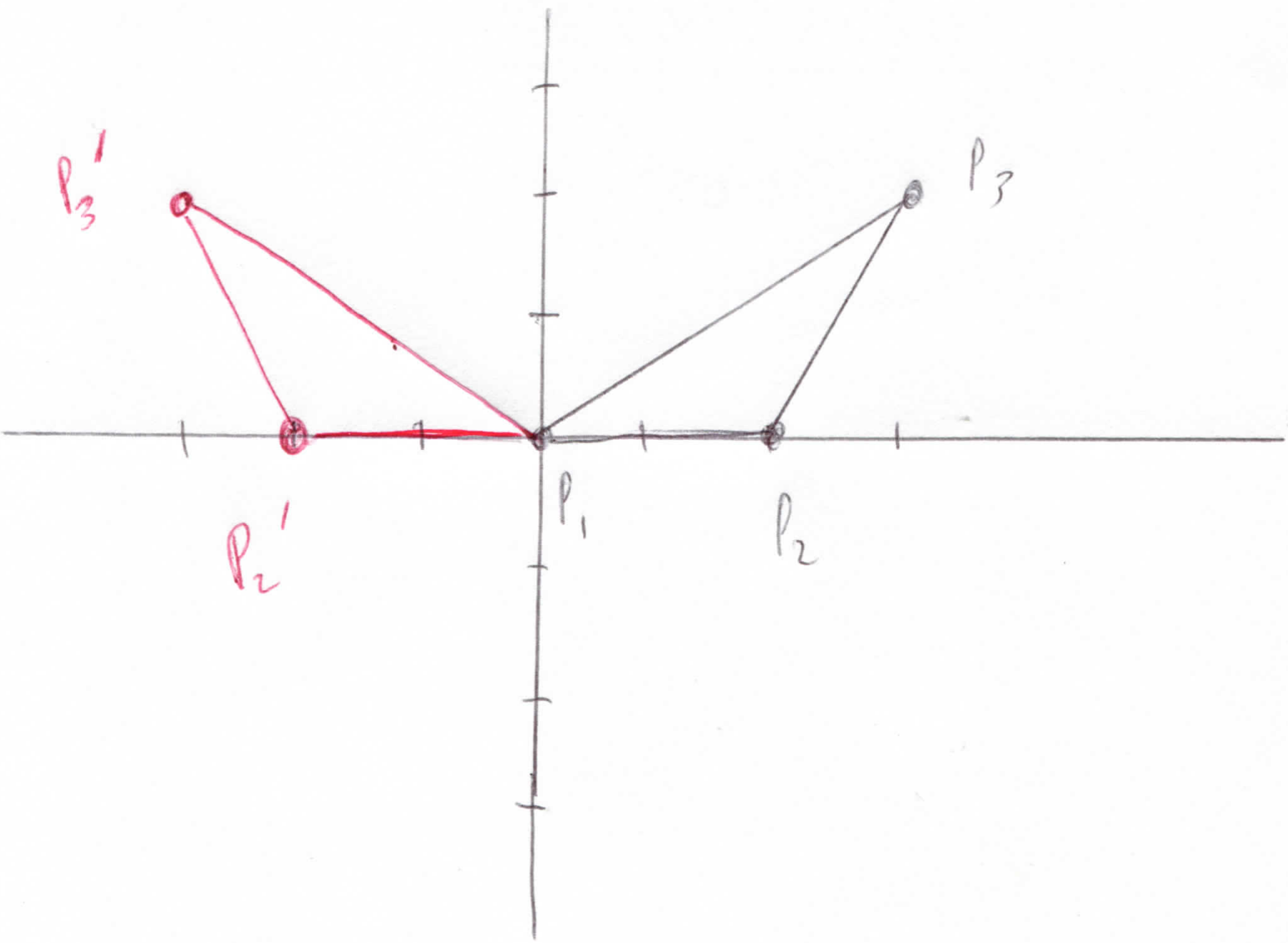
By Theorem 23, for each motion, we only need to calculate P'_1, P'_2, and P'_3, then draw the line segments from P'_1 to P'_2, from P'_2 to P'_3, and from P'_3 to P'_1. In other words, Ω', the image of Ω, is the triangle with vertices P'_1, P'_2, and P'_3.

In each picture, Ω is drawn in black and Ω', the image of Ω, is drawn in red.

All rotations are counterclockwise.

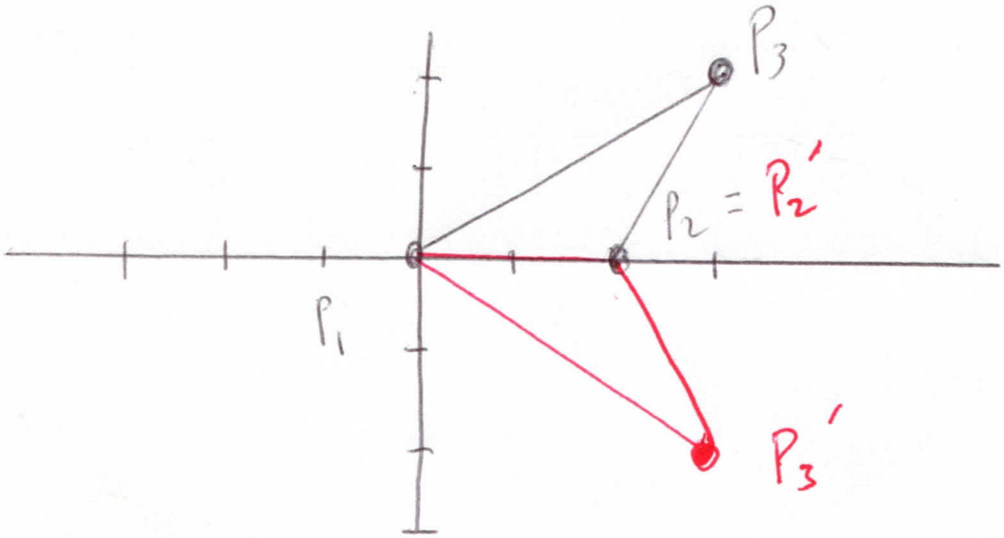
Reflection through y axis. $A \equiv \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ implies that

$$\mathrm{P}'_1 = A\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathrm{P}'_2 = A\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \mathrm{P}'_3 = A\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}.$$



Reflection through x axis. $A \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ implies that

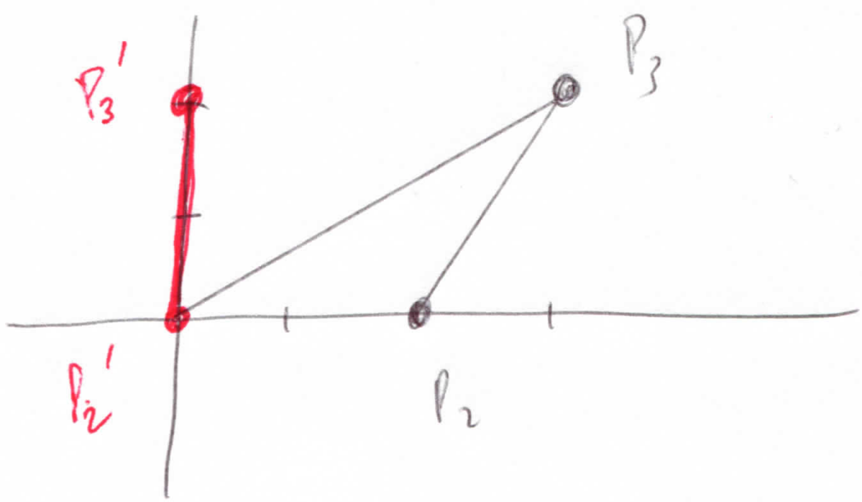
$$P_1' = A\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}, P_2' = A\begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}, P_3' = A\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}3\\-2\end{bmatrix}.$$



With reflection through a line ℓ, think of ℓ as being a mirror, with Ω' being the *mirror image* of Ω, that is, what you see in the mirror.

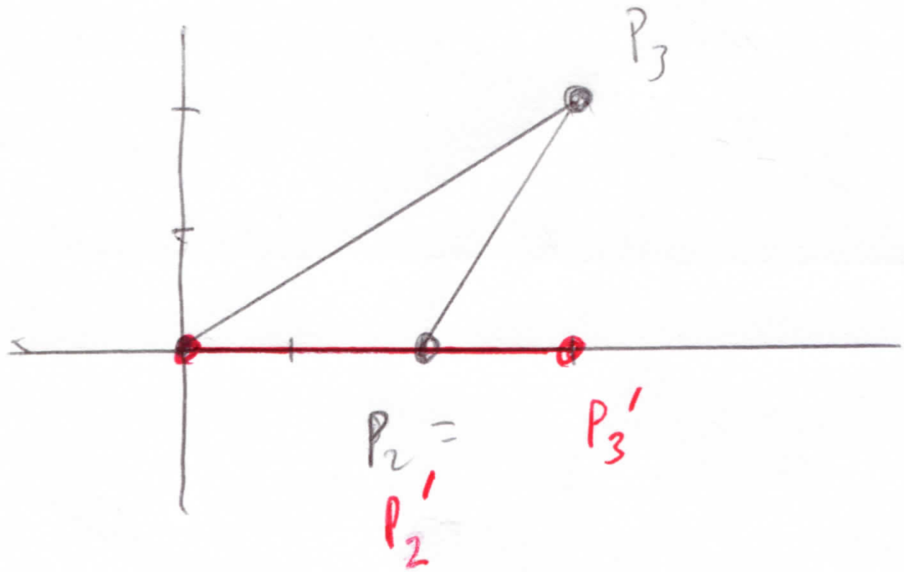
Projection onto y axis. $A \equiv \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ implies that

$$P_1' = A\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}, P_2' = A\begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}, P_3' = A\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix}.$$



Projection onto x axis. $A \equiv \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ implies that

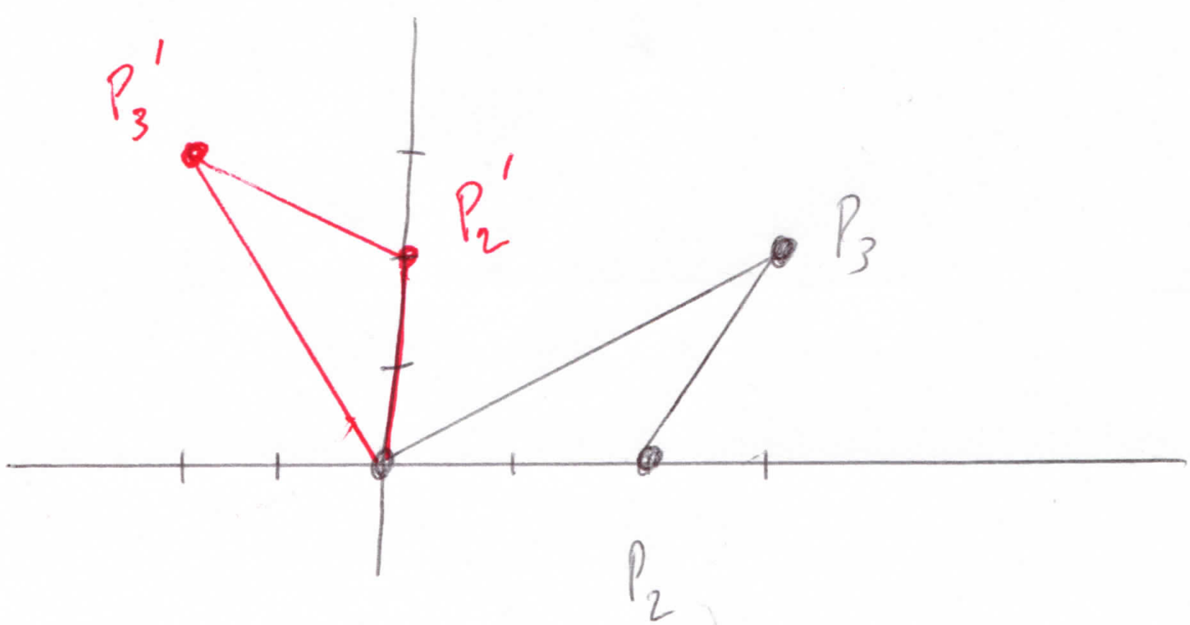
$$P_1' = A\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, P_2' = A\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, P_3' = A\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$$



Notice that the projections above crushed a triangle into a line segment; this is in contrast to reflections and rotations, where the image of a triangle is guaranteed to be a triangle (see Theorem 23).

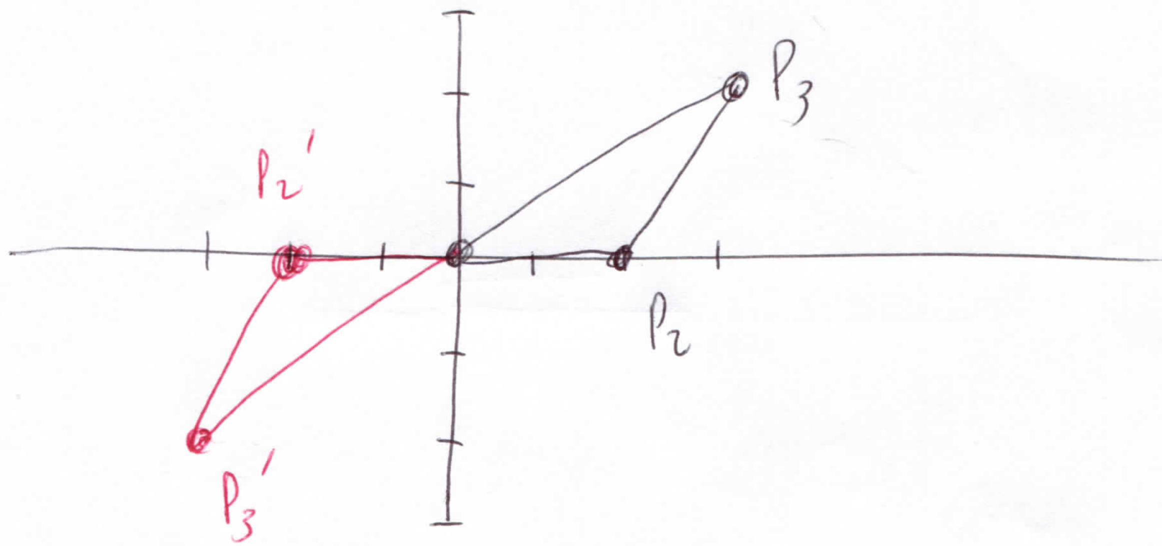
90 degree rotation. $A \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ implies that

$$P_1' = A\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, P_2' = A\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, P_3' = A\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$



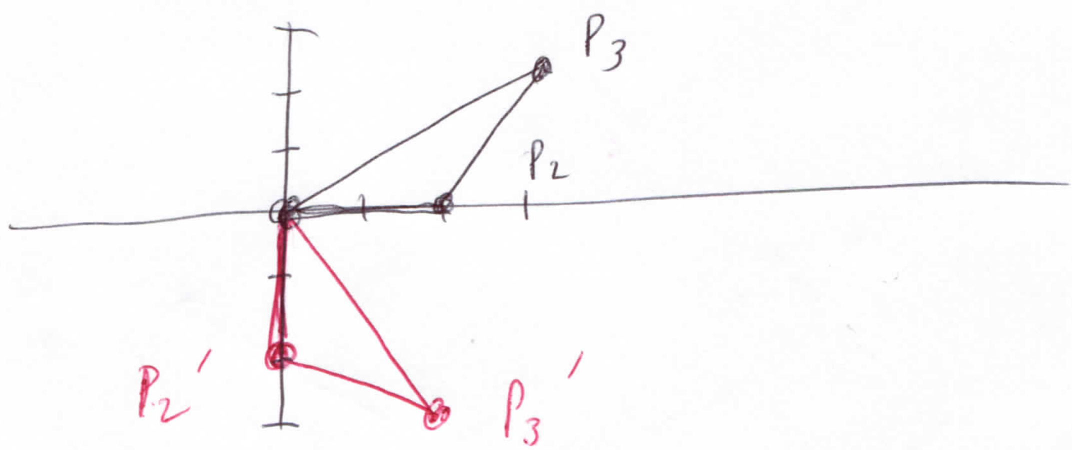
180 degree rotation. $A \equiv \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ implies that

$$P'_1 = A\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}, P'_2 = A\begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}-2\\0\end{bmatrix}, P'_3 = A\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}-3\\-2\end{bmatrix}.$$



270 degree rotation. $A \equiv \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (90 degree rotation times 180 degree rotation) implies that

$$P'_1 = A\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}, P'_2 = A\begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}0\\-2\end{bmatrix}, P'_3 = A\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}2\\-3\end{bmatrix}.$$



Notice, in the motions of Examples 24(a), that the points P_1, P_2, P_3 (counterclockwise) are changed to clockwise motion by reflection, but remain counterclockwise under rotation.

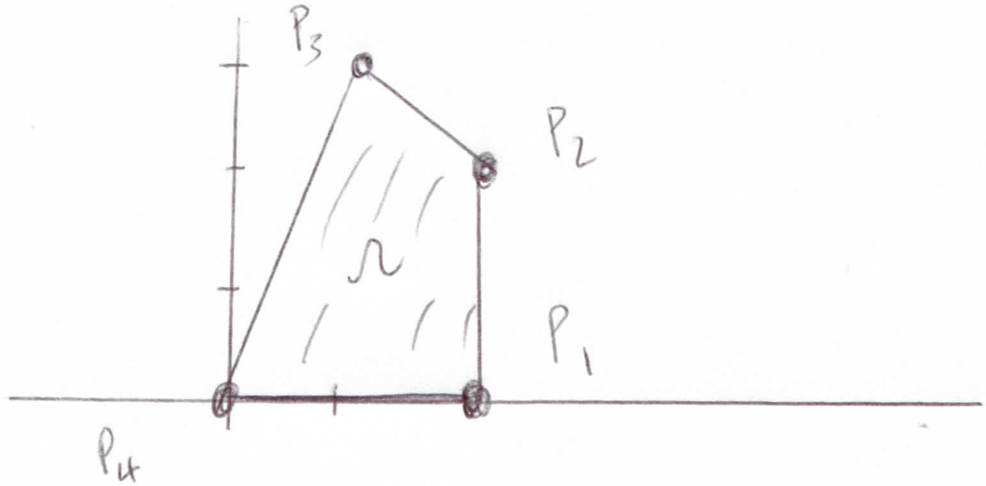
Examples 24(b) and (c) follow the same strategy as (a), with images of vertices $P'_j, j = 1, 2, 3, 4$. Notice that the motions in both (b) and (c) are changing counterclockwise to clockwise, when one follows the vertices in order. Notice also that the right angle at the vertex P_1 is moved to a right angle at P'_1. More generally, lengths and angles are preserved under rotation and reflection. See [3, Chapters VIII and IX].

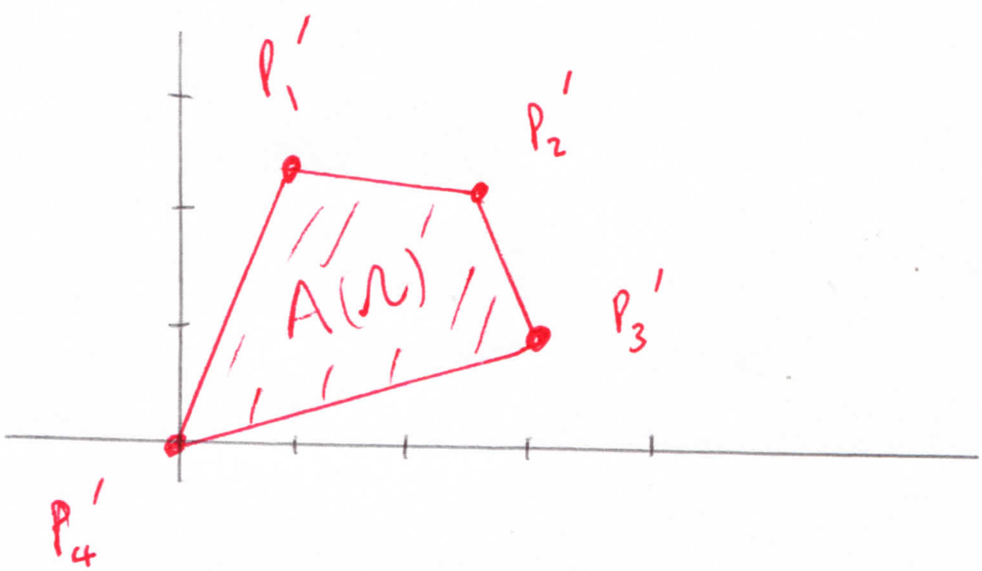
Because of this preservation, reflection and rotation are called *rigid motions.*

(b) Here $A = \frac{1}{5\sqrt{2}} \begin{bmatrix} 1 & 7 \\ 7 & -1 \end{bmatrix}$, so that the image of Ω is the quadrilateral with vertices

$$P'_1 = A \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{1}{5\sqrt{2}} \begin{bmatrix} 2 \\ 14 \end{bmatrix}, P'_2 = A \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \frac{1}{5\sqrt{2}} \begin{bmatrix} 16 \\ 12 \end{bmatrix}, P'_3 = A \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{1}{5\sqrt{2}} \begin{bmatrix} 22 \\ 4 \end{bmatrix}, \text{ and } P'_4 = A \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

In the two drawings below, Ω is drawn in black, Ω' in red.





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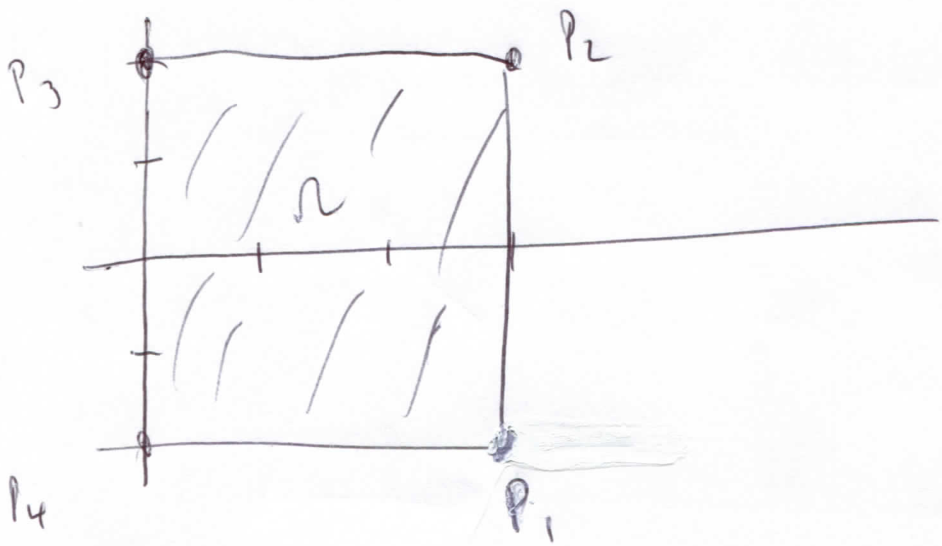
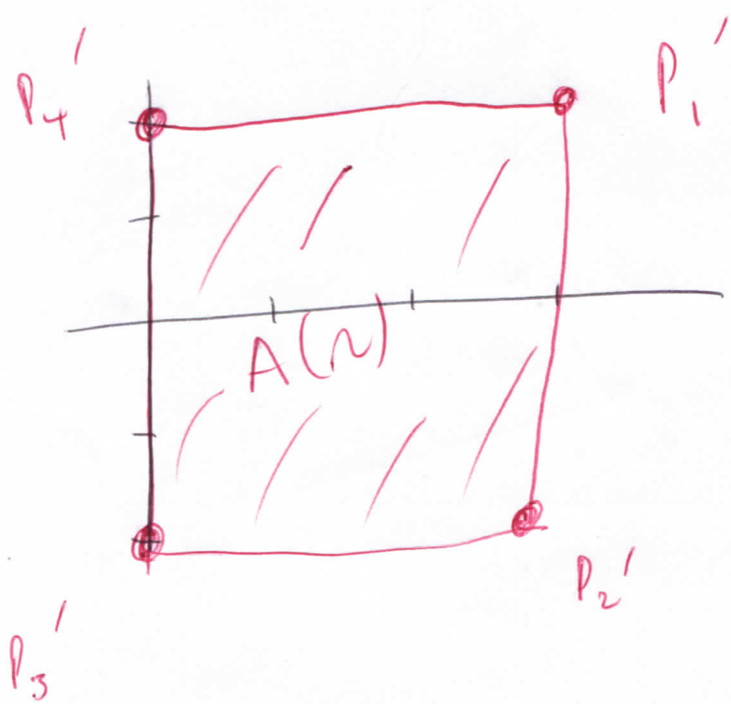
(c) Here $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, so Ω' is the rectangle with vertices

$$\mathrm{P}_1' = A\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \mathrm{P}_2' = A\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \mathrm{P}_3' = A\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}, \text{ and } \mathrm{P}_4' = A\begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Notice that Ω (drawn in black below) is the same set of points as $\Omega' \equiv A(\Omega)$ (drawn in red below). Yet there is movement of individual points; any point not on the x axis is getting moved, specifically, reflected through the x axis. It is the *totality* of points that is unchanged under this movement.

This is called *symmetry* (see Definition 25).





Definition 25. Suppose a motion has standard matrix A and Ω is a subset of the plane. We say that Ω is **symmetric** with respect to the motion if $A(\Omega) = \Omega$.

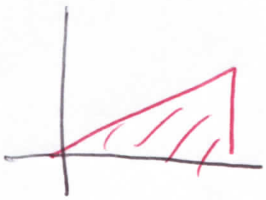
Examples 26. (a) Which of the figures below (i, ii, iii, iv, v, and/or vi) are symmetric with respect to 90 degree counterclockwise rotation?

(b) Which of the figures below (i, ii, iii, iv, v, and/or vi) are symmetric with respect to 180 degree counterclockwise rotation?

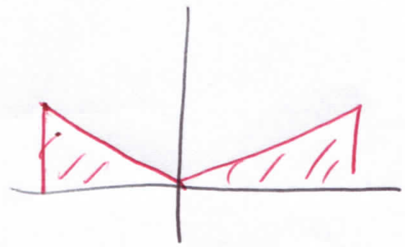
(c) Which of the figures below (i, ii, iii, iv, v, and/or vi) are symmetric with respect to reflection through the x axis?

(d) Which of the figures below (i, ii, iii, iv, v, and/or vi) are symmetric with respect to reflection through the y axis?

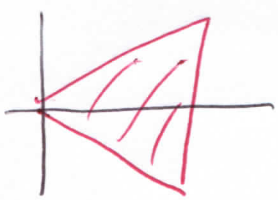
(i)



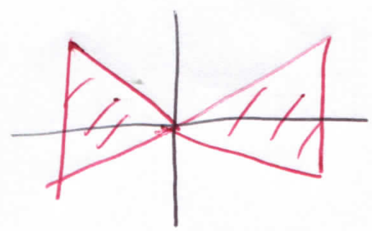
(ii)



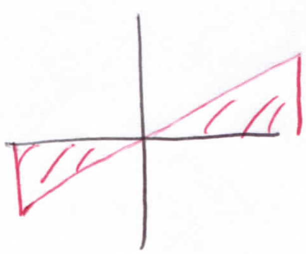
(iii)



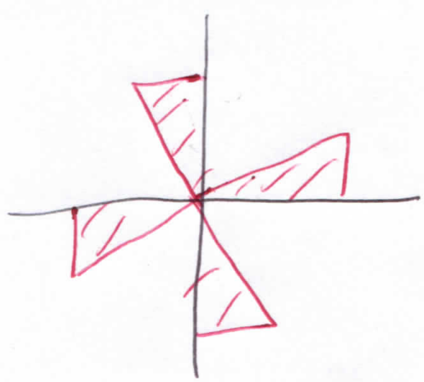
(iv)



(v)



(vi)



Solutions. These can be done visually; that is, visualize applying the specified rotations and reflections to the figure, and see if we end up with the same figure, that is, the same subset of the plane covered by the figure.

Figure (i) has none of the desired symmetries: rotating 90 degrees moves the triangle from the first quadrant to the second quadrant, as in (vi); rotating 180 degrees moves the triangle into the third quadrant, as in (v); reflection through the y axis moves Figure (i) into the second quadrant, as in (ii); reflection through the x axis moves Figure (i) into the fourth quadrant, as in (iii).

Figure (ii) is symmetric with respect to reflection through the y axis. Figure (iii) is symmetric with respect to reflection through the x axis. Figure (iv) is symmetric with respect to both reflections and (this follows automatically) 180 degree counterclockwise rotation. Figure (v) is symmetric with respect to 180 degree counterclockwise rotation (this is the same as reflecting through the origin). Figure (vi) is symmetric with respect to 90 degree counterclockwise rotation, hence also with respect to 180 degree counterclockwise rotation.

Now let's put that information into literal answers to (a)–(d):

(a) (vi).

(b) (iv), (v), and (vi).

(c) (iii) and (iv).

(d) (ii) and (iv).

These can also be done algebraically, using the standard matrices for the desired rotations and reflections. Estimate the coordinates of vertices and check (using the standard matrix for a motion) that the motion takes each vertex to another vertex in the figure.

For example, in (ii), relevant vertices look like

$$\mathrm{P}_1 \equiv \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \mathrm{P}_2 \equiv \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathrm{P}_3 \equiv \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \mathrm{P}_4 \equiv \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \text{ and } \mathrm{P}_5 \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

For reflection through the y axis, the standard matrix is $A \equiv \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$; check that

$$A\mathrm{P}_1 = \mathrm{P}_2, A\mathrm{P}_2 = \mathrm{P}_1, A\mathrm{P}_3 = \mathrm{P}_4, A\mathrm{P}_4 = \mathrm{P}_3, \text{ and } A\mathrm{P}_5 = P_5,$$

implying symmetry for Figure (ii) with respect to reflection through the y axis.

For reflection through the x axis, the standard matrix is $A \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$; since $A\mathrm{P}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, which is *not* in the figure, Figure (ii) does not have symmetry with respect to reflection through the x axis.

The same failure occurs with rotation symmetries for Figure (ii).

HOMEWORK

1. Suppose $P = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $R = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$.

(a) Find the midpoint of the line segment from P to R.

(b) Which of the following points are on the line segment from P to R?

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \frac{1}{2}\begin{bmatrix} 5 \\ 0 \end{bmatrix}, \ \frac{1}{2}\begin{bmatrix} 5 \\ -4 \end{bmatrix}.$$

2. Find each of the following sums and/or products. On each part, your answer should be a single matrix.

(a) $\begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

(b) $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

(c) $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$.

(d) $\begin{bmatrix} 1 \\ -2 \end{bmatrix} - 3 \begin{bmatrix} 4 \\ 0 \end{bmatrix}$.

3. Which of the following pairs of line segments are perpendicular?

(a) line segment from $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and line segment from $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ to $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(b) line segment from $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and line segment from $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

4. (a) Find the standard matrix for the following motion: rotate 45 degrees, then project onto $y = 2x$.

(b) Find the standard matrix for the following motion: project onto $y = 2x$, then rotate 45 degrees.

(c) Find the standard matrix for the following motion: reflect through $y = x$, then rotate 90 degrees, then reflect through $y = -x$.

(d) Find the standard matrix for the following motion: project onto $y = -x$, then rotate 180 degrees, then reflect through $y = 3x$.

(e) For an arbitrary line through the origin ℓ, find the standard matrix for the following motion: project onto ℓ, then rotate 90 degrees, then project onto ℓ.

(f) Find the standard matrix for the following motion: rotate 90 degrees, then project onto $y = x$, then reflect through $y = x$.

(g) Find the standard matrix for the following motion: reflect through $y = x$, then rotate 90 degrees.

(h) Find the standard matrix for the following motion: reflect through $y = x$, then rotate 90 degrees, then project onto the y axis.

(i) Find the standard matrix for the following motion: project onto $y = 2x$, then rotate 90 degrees, then project onto $y = -\frac{1}{2}x$.

(j) Find the image of the point $\begin{bmatrix} \frac{1}{3} \\ \sqrt{2} \end{bmatrix}$ under the motion in (e).

(k) Find the image of the point $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ under the motion in (b).

(ℓ) Find the image of the point $\begin{bmatrix} -3 \\ 5 \end{bmatrix}$ under the motion in (g).

(m) Find the image of the point $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ under the motion in (f).

(n) Find the image of the point $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ under the motion in (c).

5. (a) Let Ω be the triangle with vertices $P_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $P_2 = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$ and $P_3 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

Find the image of Ω under the motion in number 4(g).

(b) Let Ω be the quadrilateral with vertices $P_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $P_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $P_3 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and $P_4 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

Find the image of Ω under the motion in number 4(c).

6. A **fixed point** of a motion with standard matrix A is a point $\begin{bmatrix} x \\ y \end{bmatrix}$ such that $A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$. In general, a motion moves points; a fixed point is one that *doesn't* get moved.

(a) What are the fixed points of a rotation?

(b) What are the fixed points of a projection onto a line ℓ through the origin?

(c) What are the fixed points of a reflection through a line ℓ through the origin?

7. For any 2×2 matrix B, define *powers* of B exactly as you do with numbers:

$$B^1 \equiv B,\ B^2 \equiv BB,\ B^3 \equiv B(B^2) = BBB,\ \ldots B^{n+1} \equiv B(B^n),\quad n = 1, 2, 3, \ldots$$

(a) Let A_1 and A_2 be as in Examples 19(r). For arbitrary $n = 1, 2, 3, \ldots$, characterize A_1^n in terms of A_1 and A_2^n in terms of A_2 and I (see Definition 14).

(b) Let $A_3 \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, the standard matrix for counterclockwise rotation of 45 degrees. Find A_3^4, A_3^8, and A_3^9. For what n does $A_3^n = I$? For what n does $A_3^n = A_3$?

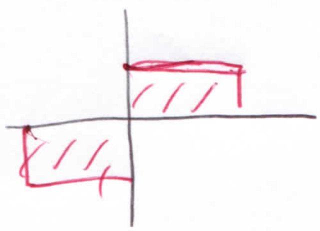
8. (a) Which of the figures below (i, ii, iii, iv, v, vi, vii, and/or viii) are symmetric with respect to 90 degree counterclockwise rotation?

(b) Which of the figures below (i, ii, iii, iv, v, vi, vii, and/or viii) are symmetric with respect to 180 degree counterclockwise rotation?

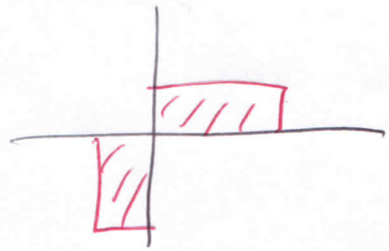
(c) Which of the figures below (i, ii, iii, iv, v, vi, vii, and/or viii) are symmetric with respect to reflection through the x axis?

(d) Which of the figures below (i, ii, iii, iv, v, vi, vii, and/or viii) are symmetric with respect to reflection through the y axis?

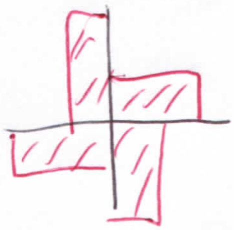
(i)



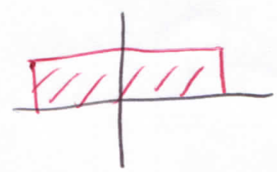
(ii)



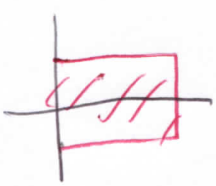
(iii)



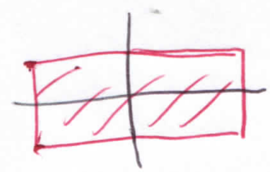
(iv)



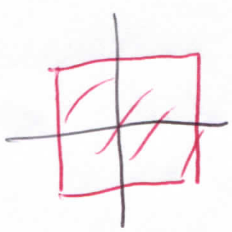
(v)



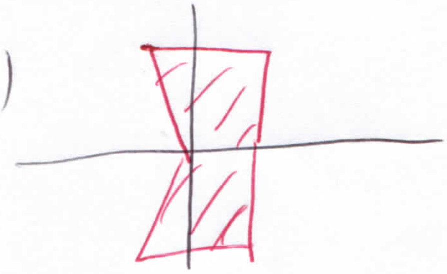
(vi)



(vii)



(viii)



HOMEWORK HINTS

1. Proposition 10 and Examples 11.

2. Definitions 5, Examples 6, Definition 12, and Examples 13.

3. Lemma 21.

4. Theorem 18 and Examples 19.

5. Theorem 23 and Examples 24.

6. Geometry.

7. Geometry; see especially Examples 19(r). For powers of A_2, look separately at even powers and odd powers. See Theorem 18 for rotation matrices. Also possibly useful is the fact that

$$IB = B = BI,$$

for any 2×2 matrix B (see Definition 14).

8. Definition 25 and Examples 26.

HOMEWORK ANSWERS

1. (a) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(b) $\frac{1}{2}\begin{bmatrix} 5 \\ -4 \end{bmatrix}$ only.

2. (a) $[2]$.

(b) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

(c) $\begin{bmatrix} 2 & 2 \\ 3 & 2 \end{bmatrix}$.

(d) $\begin{bmatrix} -11 \\ -2 \end{bmatrix}$.

3. (a) only.

4. (a) $\frac{1}{5\sqrt{2}}\begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$.

(b) $\frac{1}{5\sqrt{2}}\begin{bmatrix} -1 & -2 \\ 3 & 6 \end{bmatrix}$.

(c) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

(d) $\frac{1}{10}\begin{bmatrix} 7 & -7 \\ 1 & -1 \end{bmatrix}$.

(e) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (this can be deduced geometrically or algebraically).

(f) $\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$.

(g) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

(h) $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

(i) $\frac{1}{5}\begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix}$.

(j) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(k) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(ℓ) $\begin{bmatrix}3\\5\end{bmatrix}$.

(m) $\frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}$.

(n) $\begin{bmatrix}-2\\1\end{bmatrix}$.

5. (a) triangle with vertices $\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}3\\5\end{bmatrix}, \begin{bmatrix}-3\\0\end{bmatrix}$.

(b) quadrilateral with vertices $\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}-2\\1\end{bmatrix}, \begin{bmatrix}-1\\3\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}$.

6. (a) $\begin{bmatrix}0\\0\end{bmatrix}$ is the only fixed point of a rotation, *unless* it's a rotation of an integral multiple of 360 degrees; then every point is a fixed point.

(b) and (c) ℓ.

7. (a) For $n = 1, 2, 3, \ldots, A_1^n = A_1,\ A_2^n = I$ if n is even, $A_2^n = A_2$ if n is odd.

(b) $A_3^4 = -I$, rotation of 180 degrees (note that $4 \times 45 = 180$). $A_3^8 = I, A_3^9 = A_3$; more generally,

$$A_3^n = I \iff n = 8k,\ \ k = 1, 2, 3, \ldots;\ A_3^n = A_3 \iff n = (8k+1),\ \ k = 0, 1, 2, 3, \ldots.$$

8. (a) (iii), (vii).

(b) (i), (iii), (vi), (vii).

(c) (v), (vi), (vii), (viii).

(d) (iv), (vi), (vii).

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