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Probability and Counting
MATHematics MAGnification™

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PROBABILITY and COUNTING MAGNIFICATION

This is one of a series of very short books on math, statistics, and physics called "Math Magnifications." The "magnification" refers to focusing on a particular topic that is pivotal in or emblematic of mathematics.

OUTLINE

This magnification is one of four intuitive introductory magnifications about probability. See [2] for more topics and details, especially extensive examples and homework, or see [1, Chapter I] for an informal introduction to probability.

This magnification will deal with probabilities involving equally likely outcomes, where probability is **relative frequency**,

$$P(A) = \frac{\text{number of outcomes in } A}{\text{number of possible outcomes}},$$

where $P(A)$ denotes probability of the event A.

This means that, for each probability, we perform two acts of counting: one for the numerator of the fraction above, one for the denominator.

Examples will include dice rolling, coin flipping (and all the situations that coin flipping is a prototype for, such as number of daughters of a couple and pizza/tiger door choices), Pascal's triangle, permutations, combinations, poker hands, and the Three Stooges.

The only prerequisites for this magnification are arithmetic, including fractions and percents.

As in [1] and [2], a **sample space** is the set of all outcomes of an experiment and an **event** is a subset of a sample space.

Throughout this magnification, all outcomes are equally likely. We then have the following formula for probability, where we denote by $P(A)$ the probability of the event A.

$$P(A) = \frac{\text{number of outcomes in } A}{\text{number of possible outcomes}} \qquad (1).$$

For example, if there are 20 cookies in a cookie jar, 5 of which have chocolate chips, there is a $\frac{5}{20} = 25\%$ chance of getting a cookie with chocolate chips, if you choose a cookie at random from the cookie jar. The formula (1), equating probability with *relative frequency*, is intuitive.

Formula (1) requires that we *count*, first the number of outcomes in the event A for the numerator of our relative frequency fraction, then the number of possible outcomes for the denominator.

The word "count" unfortunately has an intellectually trivial sound; do we take off our shoes to count above 10? We shall see that the *counting arguments* needed for formula (1) can be quite subtle.

Examples 2. A **fair coin** or **honest coin** is one that, on any flip, is equally likely to come up heads as it is to come up tails. A **die** is a cube with the numbers one through six marked on each side with dots; a **fair die** or **honest die** is a die with each side equally likely to come up, when rolled.

By (1),

$$P(\text{heads}) = \frac{1}{2} = P(\text{tails})$$

when flipping a fair coin, and

$$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$$

when rolling a fair die.

(a) Suppose I roll an honest die twice. Find the probability that

(i) The total is 4;

(ii) One die is twice the other; and

(iii) The total is 1.

<u>SOLUTIONS</u>: Draw the sample space as the following matrix, where the horizontal numbers are the first roll, the vertical numbers the second.

	1	2	3	4	5	6
1		✓	x			
2	✓	x		✓		
3	x					✓
4		✓				
5						
6			✓			

Since there are 36 squares, the denominator in formula (i) is 36, for any probability requested. For numerator, we have marked with an "x" for (i) and a "✓" for (ii), in the matrix above.

(i) $\frac{3}{36} = \frac{1}{12}$

(ii) $\frac{6}{36} = \frac{1}{6}$

(iii) $\frac{0}{36} = 0.$

(b) If we flip a fair coin twice, find the probabilities of 0 heads, 1 head, and 2 heads.

SOLUTIONS: Make a tree diagram, writing H for heads, T for tails.

$$\left.\begin{array}{lll} & H & HH \\ H - & T & HT \\ T - & H & TH \\ & T & TT \end{array}\right\}$$ sample space, 4 possible outcomes

$$P(0\ H) = P(\{TT\}) = \frac{1}{4}$$

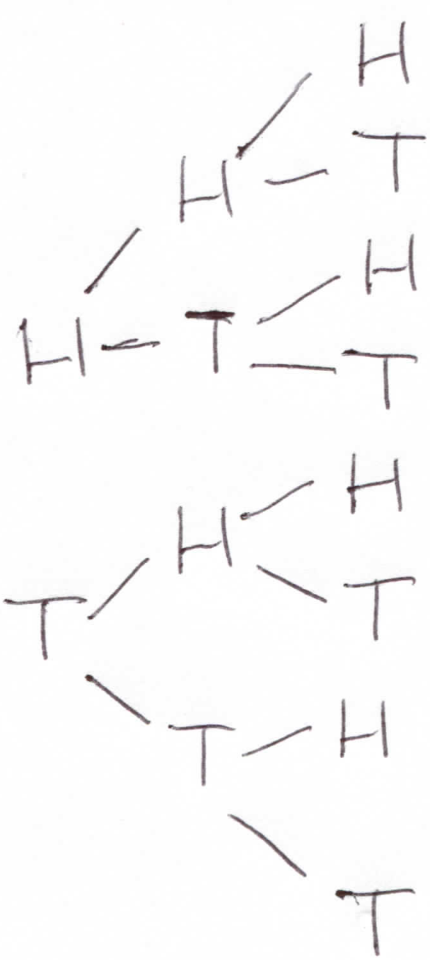
$$P(1\ H) = P(\{HT, TH\}) = \frac{2}{4}$$

$$P(2\ H) = P(\{HH\}) = \frac{1}{4}$$

(c) If we flip a fair coin three times, find the probabilities of 0 heads, 1 head, 2 heads, and 3 heads.

SOLUTIONS:

Add another coin flip to our tree in (b):



HHH
HHT
HTH
HTT
THH
THT
TTH
TTT

sample space, 8 possible outcomes

Counting outcomes, as in (b), we read off

$$P(0\,H) = \frac{1}{8}, \quad P(1\,H) = \frac{3}{8},$$

$$P(2\,H) = \frac{3}{8}, \quad P(3\,H) = \frac{1}{8}.$$

A tree diagram for four flips is already looking too messy for our collective taste.

Here is a different picture.

Pascal's Triangle 3

This has infinitely long sides of 1s, with each interior number the sum of the two numbers directly above it; for example, $\begin{matrix}6 & & 4\\ & 10 & \end{matrix}$.

We have embellished Pascal's triangle with vertical columns on each side.

PASCAL'S △

flips		sum
0	1	1
1	1 1	2
2	1 2 1	4
3	1 3 3 1	8
4	1 4 6 4 1	16
5	1 5 10 10 5 1	32
6	1 6 15 20 15 6 1	64
7	1 7 21 35 35 21 7 1	128
8	1 8 28 56 70 56 28 8 1	256
9	1 9 36 84 126 126 84 36 9 1	512
⋮	⋮	⋮

Discussion 4

Let's organize our results about fair coin flipping.

1 flip

number of heads	number of ways	probability
0	1	1/2
1	1	1/2

2 flips

number of heads	number of ways	probability
0	1	1/4
1	2	2/4
2	1	1/4

3 flips

number of heads	number of ways	probability
0	1	1/8
1	3	3/8
2	3	3/8
3	1	1/8

Notice that the

number of flips gives us the row of Pascal's triangle; the numerators of the probabilities appear in the interior and the denominators in the right-most column.

For example, without drawing trees, if we wanted probabilities for 4 flips of a fair coin, we'd look, in Pascal's triangle, at the row

flips		sum
⋮		⋮
4	1 4 6 4 1	16

to conclude

$P(0\ H) = \frac{1}{16}$, $P(1\ H) = \frac{4}{16}$,

$P(2\ H) = \frac{6}{16}$, $P(3\ H) = \frac{4}{16}$,

$P(4\ H) = \frac{1}{16}$

Examples 5

Get each of the following probabilities from Pascal's triangle.

(a) $P(4 \text{ heads in } 6 \text{ flips})$

(b) $P(1 \text{ head in } 5 \text{ flips})$

(c) $P(5 \text{ heads in } 8 \text{ flips})$

(d) $P(0 \text{ heads in } 5 \text{ flips})$

(e) $P(3 \text{ heads in } 9 \text{ flips})$

(f) $P(6 \text{ heads in } 9 \text{ flips})$

All coins are fair.

SOLUTIONS:

(a) Here's the row for 6 flips

1	6	15	20	15	6	1	sum 64
0H	1H	2H	3H	4H	5H	6H	

so our fraction is $15/64$

(b) For 5 flips, look at

1	5	10	...	sum 32;
	1H			

our fraction is $5/32$

(c) $56/256$

(d) $1/32$

(e) $84/512$

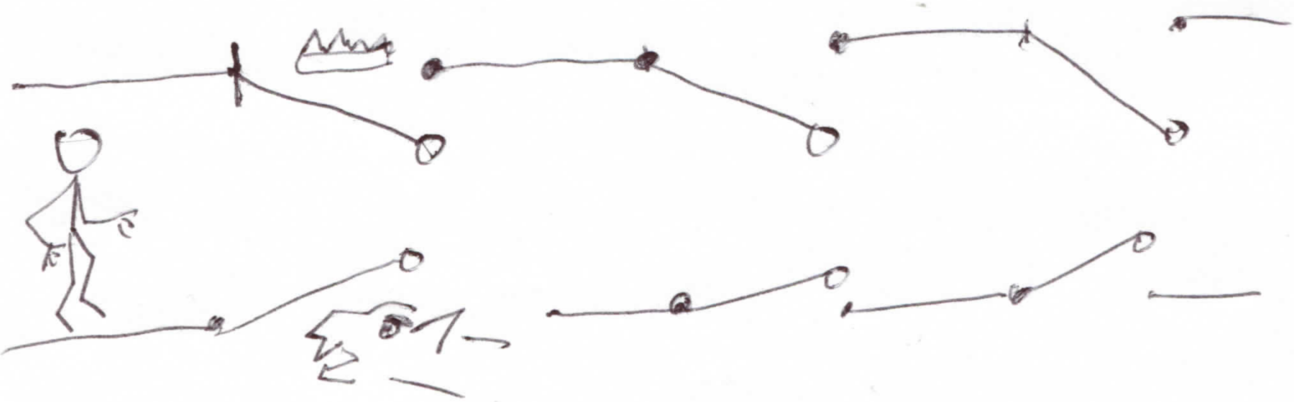
(f) $84/512$

NOTE that $3+6=9$

Examples 6

There are many situations that are the same as our fair coin flipping in Examples 5, after a small translation. Here are two examples.

(a) You walk down a corridor with three pairs of doors. Each pair consists of a door with a pizza behind it and a door with an irritated tiger behind it, who will hurt you if you open his door.



At each pair of doors, you open one at random.

What is the probability you will be hurt exactly once?

If we translate

"being hurt at a pair of doors" as

"get heads when flipping a fair coin"

our question is equivalent to getting

P(1 head in 3 flips)

= 3/8, from Pascal's triangle.

(6) A quiz consists of five true/false questions. If you guess randomly on each question, what is the probability you will get 60% or less on the quiz?

Translate "guess correctly on question" as "get heads on fair coin flip", then we want

P(3 H in 5 flips) + P(2 H in 5 flips)

$+ \cdots = \frac{10}{32} + \frac{10}{32} + \frac{5}{32} + \frac{1}{32}$

$= 26/32$

Discussion 7

Counting the number of "successes" (success defined as whatever you want) in an independently repeated experiment, with a fixed probability p of success on each repetition is equivalent to counting the number of heads when flipping a coin, if said coin has a probability p of coming up heads when flipped.

This counting is called a binomial random variable; see [2, Chapter VII]. When $p = 0.5$, the counting is equivalent to counting the number of heads when flipping a fair coin.

Counting the number of daughters among ten children produced by a particular couple is an example of a binomial random variable with p, the probability a child is a daughter, probably not equal to 0.5

Discussion 8

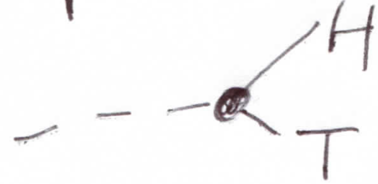
Let's look at the denominator in the answers to Examples 5, corresponding to the right-most column in Pascal's triangle.

What is being counted in that right-most column is the number of arrangements of heads and tails when we flip a coin.

Here's a gotted version of our picture of Pascal's triangle, showing only the left-most and right-most columns.

number of flips	number of arrangements of heads and tails
0	$1 = 2^0$
1	$2 = 2^1$
2	$4 = 2^2$
3	$8 = 2^3$
4	$16 = 2^4$
⋮	

You can see the doubling
in the right column:
in a tree diagram an
H and T get pasted onto
each prior branch



If we wanted the number of
arrangements of heads and tails in
150 flips, we could exploit the
doubling pattern:

$2^{150} \sim 1.4 \times 10^{45}$.

We don't want to write down all those arrangements that we counted.

We would like now for you to think of coin flipping in the following way. Think of each flip as an experiment, count the number of outcomes in each experiment, then multiply those countings, to get the number of outcomes in the combined experiment, meaning experiment 1 followed by experiment 2, etc.

Here's the picture for flipping a coin 3 times!

2 2 2

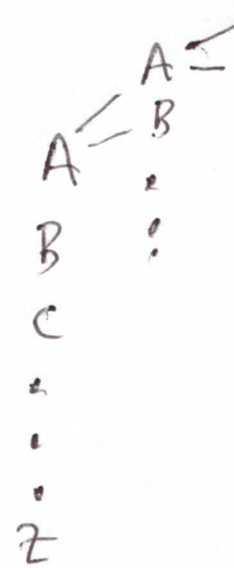
experiment	1st flip	2nd flip	3rd flip
number of outcomes	2	2	2

→ $2 \cdot 2 \cdot 2 = 8$ outcomes for flipping 3 times.

Examples 9

(a) How many 3-letter words are there? ("word" means any sequence of 3 letters, from A - Z)

<u>SOLUTION</u>: We could make a tree diagram



[no room]

Think instead of

three experiments

Choose 1^{st} letter;

Choose 2^{nd} letter;

Choose 3^{rd} letter:

experiment	1^{st} letter	2^{nd} letter	3^{rd} letter
number of outcomes	26	26	26

For the combined experiment,

producing 3-letter words, multiply

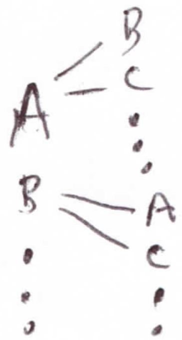
outcomes: $26 \cdot 26 \cdot 26 = 26^3 =$

17,576 3-letter words

(b) How many 3-letter words with no letter repeated are there?

<u>SOLUTION</u>:

Again, we could try to draw a tree diagram:



• • • ; BETTER:

experiment	1st letter	2nd letter	3rd letter
number of outcomes	26	25	24

can't use 1st letter

can't use 1st or 2nd letter

MULTIPLY:

$26 \cdot 25 \cdot 24 = 15,600.$

(c) If a 3-letter word is chosen at random, what is the probability there will be no repeated letters?

SOLUTION: Use the relative frequency formula (1), with numbers from (a) and (b):

$$\frac{26 \cdot 25 \cdot 24}{26 \cdot 26 \cdot 26} \sim 89\%$$

These examples are meant to lead up to the following.

Basic Counting Principle 10

Suppose there are

n_1 outcomes for the 1st experiment;

n_2 outcomes for the 2nd experiment, regardless of the outcome of the 1st;

n_3 outcomes for the 3rd experiment, regardless of the outcomes of the 1st & 2nd,

⋮

Then there are $(n_1 n_2 n_3 \cdots)$ outcomes for the combined sequence of experiments.

Definition 11

A *permutation* of n things taken r at a time is an ordered set of r things chosen from n available.

$_nP_r$ is the number of such permutations

Example 12

A 3-letter word with no letter repeated is a permutation of 26 things taken 3 at a time:

(think of letter blocks)

choose 3, put in order

We have already calculated, in Examples 9(b),

$$_{26}P_3 = 26 \cdot 25 \cdot 24$$

Definition 13

The following is convenient for many formulas.

<u>0 factorial</u>, denoted $0!$, is 1;

<u>1 factorial</u>, denoted $1!$, is 1;

<u>2 factorial</u>, denoted $2!$, is 2;

$3! = 3 \cdot 2 \cdot 1 = 6$;

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$;

in general, for $n = 1, 2, 3, 4, 5, \dots$

<u>n factorial</u> is

$n! = n(n-1)(n-2) \cdots 2 \cdot 1$.

NOTE that $_{26}P_3 = 26 \cdot 25 \cdot 24$

$= \dfrac{26!}{23!}$, by applying massive

cancellation to the ratio

of factorials.

The Basic Counting Principle gives the following formulas for counting permutations.

Theorem 14

$$
{}_nP_r = \underset{\text{1st choice}}{n}\ \underset{\text{2nd choice}}{(n-1)}\ \underset{\text{3rd choice}}{(n-2)} \cdots \underset{\text{rth choice}}{(n-r+1)}
$$

$$
= \frac{n!}{(n-r)!}
$$

Examples 15

$$_{52}P_5 = \frac{52!}{47!} = 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48$$
$$= 311{,}875{,}200$$

$$_7P_3 = \frac{7!}{4!} = 7 \cdot 6 \cdot 5 = 210$$

$$_{100}P_2 = \frac{100!}{98!} = 100 \cdot 99 = 9{,}900$$

Definition 16

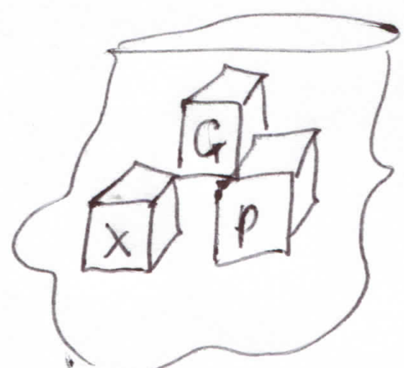
A combination of n things taken r at a time is an (unordered) set of r things chosen from n available.

$\binom{n}{r}$ (pronounced "n choose r")

denotes the number of such combinations.

Example 17

A sack of 3 letter blocks is a combination of 26 things taken r at a time.



HOW MANY such sacks?

${}_{26}P_3 = 15,600$ is too large a number, because each set of 3 letter blocks is being counted 6 times:

For example, $\{A, B, C\}$ is counted as

$$\left.\begin{array}{ccc} A & B & C \\ A & C & B \\ B & A & C \\ B & C & A \\ C & A & B \\ C & B & A \end{array}\right\} \text{6 permutations of } \{A, B, C\}$$

To correct this overcounting, divide by 6:

$$\frac{{}_{26}P_3}{6} = \binom{26}{3} = \text{number of such sacks}$$

$$\left(= \frac{15{,}600}{6} = 2{,}600 \right)$$

More generally, ${}_nP_r$ counts each combination of n things taken r at a time $r!$ times, by permuting the r things, so we correct as follows.

Theorem 18

$$\binom{n}{r} = \frac{{}_nP_r}{r!} = \frac{n!}{r!\,(n-r)!}$$

Examples 19

$${}_{30}P_2 = 30 \cdot 29 = 870;$$

$$\binom{30}{2} = \frac{870}{2} = 435$$

$$\binom{9}{5} = \frac{9!}{5!4!} = \frac{9\cdot 8\cdot 7\cdot 6\cdot 5}{5!}$$

$= 126.$

$$\binom{9}{4} = \frac{9\cdot 8\cdot 7\cdot 6}{4!} = 126$$

(NOTE: $5+4=9$)

$$\binom{52}{5} = \frac{52!}{5!\,47!} = \frac{52\cdot 51\cdot 50\cdot 49\cdot 48}{5!}$$

$= 2{,}598{,}960$ = number of poker hands (5 cards chosen from deck of 52 cards)

Examples 20

We have just described a poker hand; also relevant is that there are 4 aces in a deck of 52 cards.

What is the probability a poker hand contains

(a) no aces

(b) all 4 aces

(c) precisely 2 aces

SOLUTIONS:

(a) A poker hand has the same power regardless of how the cards are ordered, so combinations seem natural!

Formula (1) gives the probability as

$$\frac{\binom{48}{5}}{\binom{52}{5}} = \frac{48\cdot 47\cdot 46\cdot 45\cdot 44/5!}{52\cdot 51\cdot 50\cdot 49\cdot 48/5!}$$

$$= \frac{48\cdot 47\cdot 46\cdot 45\cdot 44}{52\cdot 51\cdot 50\cdot 49\cdot 48} \approx 66\%$$

We could also do this as permutations or Basic Counting Principle, by ordering all poker hands: 1st card dealt first, then 2nd card, then 3rd card, etc.:

$$\frac{{}_{48}P_5}{{}_{52}P_5} = \frac{48\cdot 47\cdot 46\cdot 45\cdot 44}{52\cdot 51\cdot 50\cdot 49\cdot 48}$$

(b) For the numerator in formula (1), we have only 48 outcomes, those being the possibilities for the 5th card in your poker hand, after being given all 4 aces.

Thus our probability is

$$\frac{48}{\binom{52}{5}} = \frac{(48)(5!)}{52\cdot51\cdot50\cdot49\cdot48} = \frac{5!}{52\cdot51\cdot50\cdot49} \sim (1.8\times10^{-3})\%$$

This problem would be difficult to do with permutations, because we'd have to worry about when the non-ace was dealt.

(c) As with (b),

permutations would be difficult.

With combinations, we need to worry about $\binom{4}{2}$ ways to get two aces out of 4 available,

<ins>and</ins>, to make sure we don't get more than 2 aces,

$\binom{48}{3}$ ways to get three non-aces out of 48 available.

For the combined experiment of getting two aces and three non-aces, the Basic Counting Principle says to multiply to get the numerator in Formula 1:

our probability is

$$\left[\frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}}\right]$$

Examples 21

Suppose the Three Stooges (Moe, Larry, + Curly) are members of a club of 7 people. A cabinet, consisting of a President (Pres.), Vice President (VP) and Emperor (Emp.)

will be chosen from the club at random.

(a) What is the probability the cabinet will consist of the Three Stooges?

SOLUTION: With combinations,

$$\frac{\binom{3}{3}}{\binom{7}{3}} = \frac{1}{\left(\frac{7\cdot6\cdot5}{3\cdot2\cdot1}\right)} = \frac{1}{35}$$

With permulations,

$$\frac{{}_3P_3}{{}_7P_3} = \frac{3\cdot2\cdot1}{7\cdot6\cdot5} = \frac{1}{35}$$

With Basic Counting Principle:

denominator

	Pres.	VP	Emp.
outcomes	7	6	5

numerator

	Pres.	VP	Emp.
outcomes	3	2	1

→ probability (from Formula (1))

$$\frac{3 \cdot 2 \cdot 1}{7 \cdot 6 \cdot 5} = \frac{1}{35}$$

(b) What is the probability Moe is Pres., Larry is VP, and Curly is Emp.?

Combinations look bad,

since order matters!

$$\left.\begin{array}{ll} \text{Moe} & \text{Pres.} \\ \text{Larry} & \text{VP} \\ \text{Curly} & \text{Emp} \end{array}\right\} \neq \left\{\begin{array}{ll} \text{Larry} & \text{Pres} \\ \text{Curly} & \text{VP} \\ \text{Moe} & \text{Emp.} \end{array}\right.$$

Use permutations or Basic Counting Principle, as in (a):

$$\text{probability} = \frac{1}{7 \cdot 6 \cdot 5} = \frac{1}{210}$$

(Note that this equals $\dfrac{1/35}{6}$ & 6 is the number of permutations of Moe, Larry & Curly)

(c) What is the probability the cabinet contains no stooges?

Combinations: (4 non-Stooges)

$$\frac{\binom{4}{3}}{\binom{7}{3}} = \frac{\frac{4\cdot 3\cdot 2}{3!}}{\frac{7\cdot 6\cdot 5}{3!}} = \frac{4\cdot 3\cdot 2}{7\cdot 6\cdot 5}$$

Permutations:

$$\frac{{}_4P_3}{{}_7P_3} = \frac{4\cdot 3\cdot 2}{7\cdot 6\cdot 5} = \frac{4}{35} \sim 11\%.$$

Should the 3 Stooges then sue for bias?

See more examples in [2, Ch. III]

Remarks 22

There is an interesting relationship between combinations and Pascal's triangle. Look, for example, at

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$= 1 \quad 4 \quad 6 \quad 4 \quad 1,$$

the row of Pascal's triangle corresponding to 4 flips of a coin.

Pascal's triangle turns out to consist entirely of combinations counting

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\vdots \qquad \vdots \qquad \vdots$$

We've come full circle since Examples 2(b) and (c), that motivated Pascal's triangle to deal with coin flipping and other binomial behavior. See [2, Ch. VII, 7.2] for more on the relationship between combinations and binomial random variables.

p. 55

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