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Probability: When to Add and When to Multiply
MATHematics MAGnification™

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PROBABILITY: WHEN to ADD and WHEN to MULTIPLY MAGNIFICATION

This is one of a series of very short books on math, statistics, and physics called "Math Magnifications." The "magnification" refers to focusing on a particular topic that is pivotal in or emblematic of mathematics.

OUTLINE

This magnification is one of four intuitive introductory magnifications about probability. See [2] for more topics and details, especially extensive examples and homework, or see [1, Chapter I] for an informal introduction to probability.

This magnification focuses on a common mistake of students taking an introductory probability class, adding probabilities of events when they should be multiplying, and vice versa. For those who have the vocabulary, here is a spoiler for this magnification, for events $A_1, A_2, A_3, \ldots$.

If $A_1, A_2, A_3, \ldots$ are mutually exclusive (also known as disjoint), then

$$P(A_1 \text{ or } A_2 \text{ or } A_3 \ldots) = P(A_1) + P(A_2) + P(A_3) + \ldots;$$

if $A_1, A_2, A_3, \ldots$ are mutually independent, then

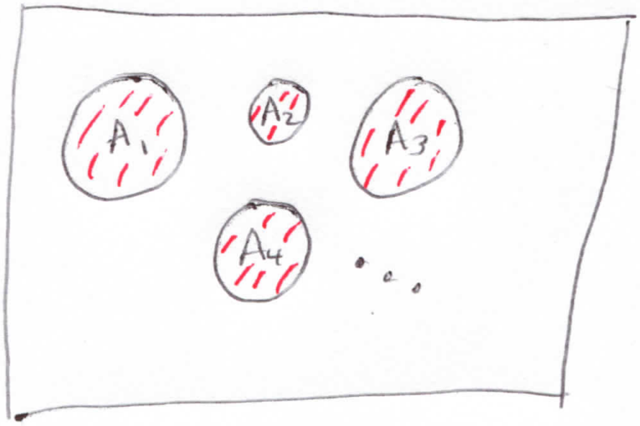
$$P(A_1 \text{ and } A_2 \text{ and } A_3 \ldots) = [P(A_1)]\,[P(A_2)]\,[P(A_3)] \ldots.$$

Examples include coin flipping and outer space bugs invading earth.

The only prerequisites for this magnification are arithmetic, including fractions and percents.

See [1] or [2] for basic probability terminology and pictures. All we need at this moment is the fact that *events* are the entities we take probabilities of; $P(A)$ denotes the probability of the event A.

Definition 1. A sequence of events $\{A_1, A_2, A_3, \dots\}$ is **disjoint** or **mutually exclusive** if no pair of events has any outcomes in common.



Example 2. If we are firing a missile and keeping track of where it lands, the events "missile lands in ocean" and "missile lands in Ohio" are mutually exclusive.

Theorem 3. If $A_1, A_2, A_3, \dots$ are mutually exclusive, then

$$P(A_1 \text{ or } A_2 \text{ or } A_3 \dots) = P(A_1) + P(A_2) + P(A_3) + \dots.$$

Examples 4. (a) Suppose thirty percent of the population is more than 6 feet tall and ten percent is less than 5 feet tall. What percent of the population is more than 6 feet or less than 5 feet tall?

(b) Suppose you are parachuting onto a part of the ocean that contains five islands, named John, Jim, Ellen, Nancy, and Florence. The probability you will land on John is 0.1, the probability you will land on Jim is 0.25, on Ellen, 0.05, on Nancy, 0.15, on Florence 0.2. What is the probability you will not land in water?

Solutions. (a) Let A be the event "(a randomly chosen person is) more than 6 feet tall" and B the event "(a randomly chosen person is) less than 5 feet tall." A and B are disjoint, since nothing can be simultaneously more than 6 and less than 5, thus

$$P(A \text{ or } B) = P(A) + P(B) = 0.3 + 0.1 = 0.4,$$

or forty percent.

(b) Islands, by definition, do not overlap, so

$$P(\text{ John or Jim or Ellen or Nancy or Florence }) = P(\text{ John }) + P(\text{ Jim }) + P(\text{ Ellen }) + P(\text{ Nancy }) + P(\text{ Florence })$$
$$= 0.1 + 0.25 + 0.05 + 0.15 + 0.2 = 0.75.$$

See [2, Chapter III], for more examples similar to Examples 4.

Definition 5. A sequence of events $\{A_1, A_2, A_3, \dots\}$ is **mutually independent** if

$$P\left(A_{j_1} \text{ and } A_{j_2} \text{ and } A_{j_3} \text{ and } \dots\right) = [P(A_{j_1})]\,[P(A_{j_2})]\,[P(A_{j_3})]\cdots,$$

for any sequence of natural numbers $j_1, j_2, j_3, \dots$.

The following is a sufficient special case for us.

Proposition 6. If $A_1, A_2, A_3, \dots$ are mutually independent, then

$$P(A_1 \text{ and } A_2 \text{ and } A_3 \dots) = [P(A_1)]\,[P(A_2)]\,[P(A_3)]\dots.$$

Remarks 7. The intuition of independence of events is that one event has no influence on the others. More precisely, define (see [2] and [3]), for any events A and B, $P(A|B)$, the **probability of A given B**, to be the probability of A under the assumption of B.

In general, $P(A \text{ and } B) = P(A|B)P(B)$. Thus A and B are (mutually) independent if and only if

$$P(A|B) = P(A);$$

this is saying that A does not care if B has occurred or not.

Similar equivalent definitions of mutually independent events, in terms of conditional probability, are possible. Again, the intuition is that none of the events are influenced by the occurrence of other events in the independent collection of events.

Examples 8. (a) An example of (probably) independent events is A defined to be "it is raining here" and B defined to be "it is warm on a planet orbiting Alpha Centauri" (more than four light years away).

(b) Toss a fair coin infinitely often. Let A_1 be "heads on the first flip," A_2 "heads on the second flip," etc. Then $\{A_1, A_2, A_3, \dots\}$ is mutually independent; the coin retains no memory of its previous flips, nor, if it did, could it alter its probability of coming up heads in future flips.

For example, after getting tails ten times in a row, the probability that a fair coin will come up heads on the next flip is still $\frac{1}{2}$, contrary to popular intuition.

(c) Here is a scenario to motivate Proposition 6. Imagine bugs from outer space are invading earth. We put up screens to keep (most of) them out. Let's say that Screen One only lets ten percent of bugs through, Screen Two lets twenty percent through and Screen Three lets five percent through.

To reassure earthlings, we'd like to know what percent of bugs get through all three screens, if we assume the screens act independently.

We could imagine a squadron of 1,000 bugs passing through all three screens. Screen One lets ten percent of the 1,000 bugs through; that's 100 bugs getting through Screen One. Twenty percent of those 100 bugs get through Screen Two; that's 20 bugs that get through Screens One and Two. Finally, five percent of those remaining 20 bugs get through Screen Three. That means only one bug, or 0.1% of the original 1,000, get through all three screens.

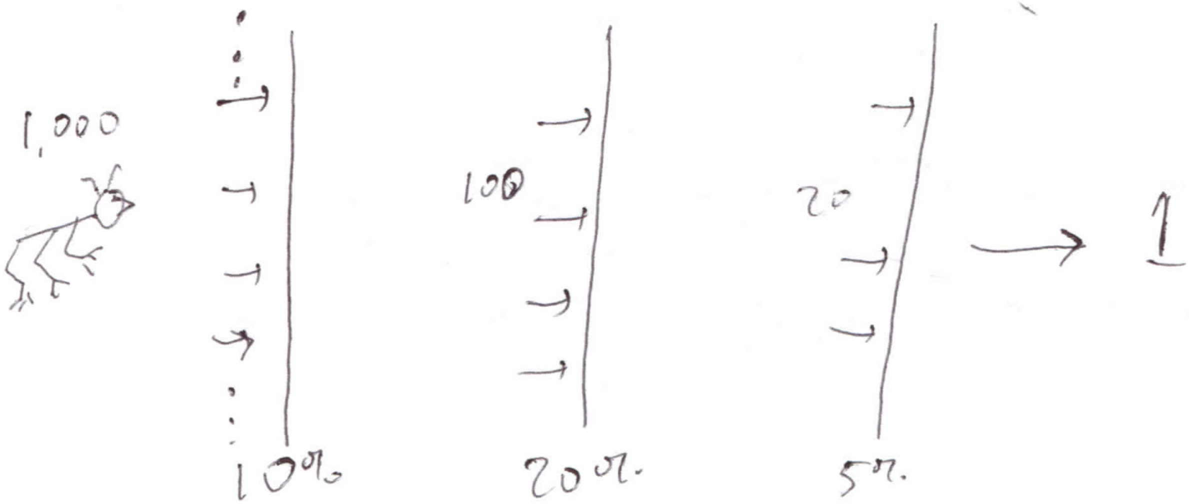
Alternatively, we could have multiplied the probabilities of getting through sequential screens:

$$0.1 \times 0.2 \times 0.05 = 0.001,$$

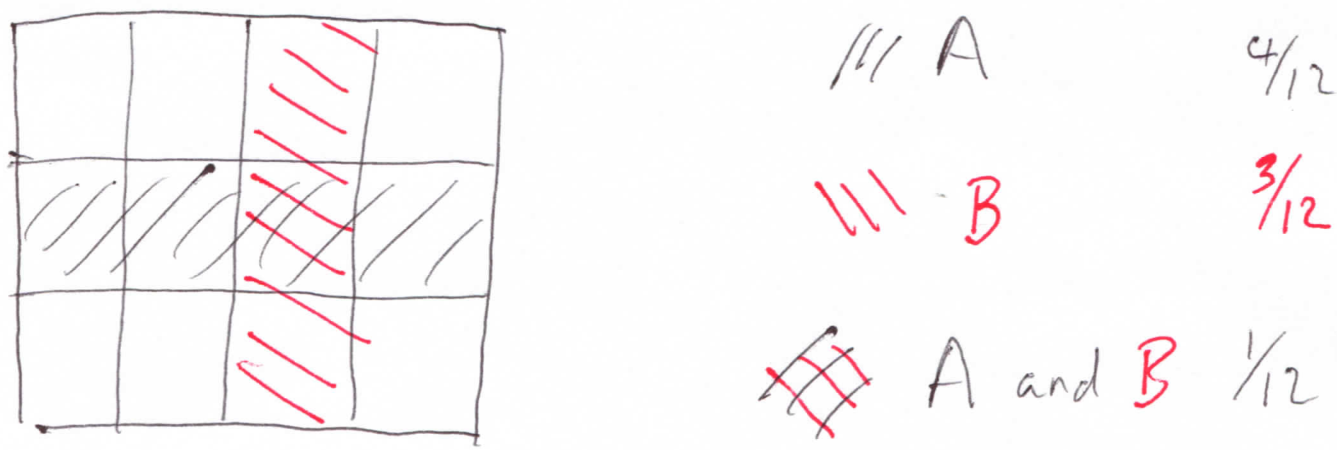
or 0.1% of bugs, get through all three screens. Very informally,

$$(\text{ten percent of twenty percent of five percent}) \text{ equals } (0.1 \text{ percent}).$$

ONE THOUSAND OUTER SPACE BUGS INVADE EARTH



A picture of independence, even for two events, analogous to the picture of mutually exclusive, is more challenging. Rectangles are the best way to go.



Example 9. Suppose that, every time I step outdoors, there is a ten percent chance I will slip on ice. If I step outdoors twenty times, what is the probability I will slip on ice at least once, assuming independence?

For this, we need the **Law of the Complement**, which follows from Theorem 3:

$$P(A^c) = 1 - P(A),$$

where A^c is the **complement** of A, meaning all outcomes that are not in A.

Solution. Let A_1 be the event "I don't slip on the ice the first time I step outdoors," A_2 the event "I don't slip on the ice the second time I step outdoors," etc.

By the Law of the Complement,

$$P(A_1) = P(A_2) = \cdots = P(A_{20}) = 1 - 0.1 = (0.9);$$

by independence,

$$P(A_1 \text{ and } A_2 \text{ and } A_3 \ldots \text{ and } A_{20}) = (0.9)^{20},$$

thus, by the Law of the Complement,

$$P(A_1^c \text{ or } A_2^c \text{ or } \ldots) = 1 - (0.9)^{20} \sim 88\%.$$

Notice that, if I had added instead of multiplying:

$$0.1 + 0.1 + \ldots 0.1 = 20 \times 0.1,$$

we would get a probability of 200%, which is impossible, since it is more than 100%.

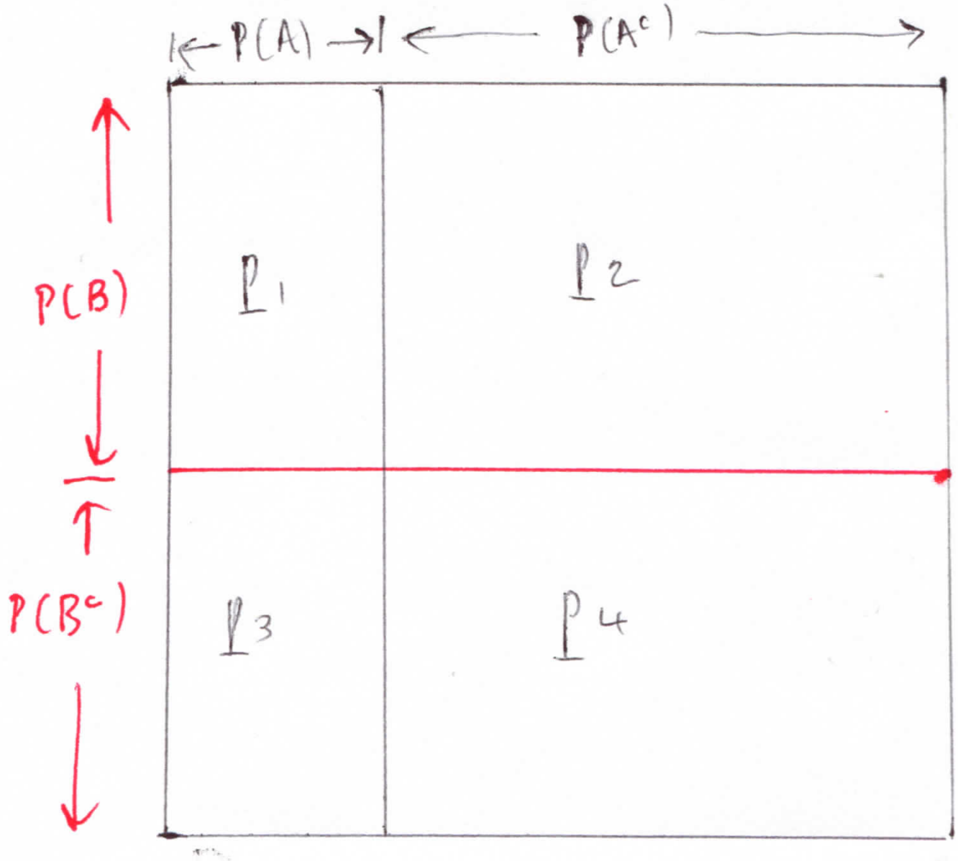
See [2, Chapter IV], for more examples similar to Example 9.

Discussion 10. As in [3, Remark 1.8], the following table describes all relationships between two events A and B, where

$$p_1 = P(A \text{ and } B), p_2 = P(A^c \text{ and } B), p_3 = P(A \text{ and } B^c), p_4 = P(A^c \text{ and } B^c):$$

	A	A^c
B	p_1	p_2
B^c	p_3	p_4

When A and B are independent, the following rectangles represent p_1, p_2, p_3, and p_4 as areas; notice that the largest rectangle is a square of side one:



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3. Ralph deLaubenfels, "Probability: Multiple Events Magnification," http://teacherscholarinstitute.com/MathMagnificationsReadyToUse.html.