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Vectors Point to Geometry and Trigonometry

Dr. Ralph deLaubenfels

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INTRODUCTION. High-school geometry, which will be abbreviated to "geometry" throughout this book, defined to be a year-long class between a first and second year of high-school algebra, has become sidelined and isolated. Very little math that precedes geometry is used in the teaching of geometry and very little that is learned in geometry is used in subsequent math classes. Sometimes it is seen as an opportunity to introduce students to rigor in mathematics; this self-consciously intellectual loftiness is usually in a stilted, mechanical style that will never be seen again, especially in mathematical circles. It gives students a false, derogatory impression of mathematics, as a set of arbitrary rules of etiquette unrelated to both the physical world and the world of ideas, unnecessarily making it seem mysterious and esoteric. At the other extreme, an "applied" or "conceptual" (adjectives that usually translate as "mediocre" or "false") approach relies on dubious visual intuition to produce large, random collections of unjustified memorized rules. This is at best risky, producing a false, insecure knowledge that is much worse than a frank admission of ignorance. Many geometry books embrace the worst of the two extremes just described, vacillating uncertainly but glibly between them, by making up their own extensive, unnatural, unpredictable, and byzantine set of axioms that students are motivated to memorize only by the force of arbitrary authority. With this style of geometry book, postulates are sprung unexpectedly throughout the book like bad science fiction bolstering a shaky plot.

The segregation of algebra and geometry limits both disciplines and is completely counter to the nature and evolution of math. The artificial break in algebra for a year, created by the unmotivated insertion of geometry between "Algebra One" and "Algebra Two," is very damaging to learning algebra; it is always better pedagogy to continually reinforce and build on past material. Geometry is virtually destroyed, as an applicable subject that can be recalled at will, by the dead end it is steered into.

Nostalgia for the classical Greeks is probably the motive for this increasingly diversionary year in the standard sequence of math classes. This nostalgia, or at least respect, is appropriate. The intellectual output of the classical Greeks was extraordinary. Euclid's "Elements" was virtually unchallenged for two thousand years, used as the textbook for geometry up to the nineteenth century.

Part of what is impressive about the mathematical creations of the classical Greeks is the lack of simplifying ideas, most immediately algebra and terminology, that we now have access to. A better sense of motion, including change and displacement, beginning as early as the fourteenth century, leads to both vectors and calculus. The Cartesian plane, especially with the additional structure of vectors and complex numbers, provides a setting where geometry can be explicitly visualized and calculated.

Except for the purposes of historical re-enactment, it does not make sense to ignore those tools in our present learning and applications of geometry. The symbiotic relationships between different areas of math need to be communicated as early as possible. Arguably the dominant theme of modern mathematics and its applications is the interaction between algebra (precision via calculations) and geometry (intuition via pictures); intertwining these gives the best of both worlds.

Another increasingly fashionable trend in high-school mathematics education is the removal of vectors and complex numbers. College students commonly do not see vectors until multi-variable calculus, and we are not cognizant of any guaranteed appearance of complex numbers; we have seen prestigious graduate-level statistics texts that severely limit the scope of their proofs (for example, of the central-limit theorem) by avoiding complex numbers. We will show in this book how these subjects make geometry and trigonometry much simpler and more intuitive and put geometry back in the mainstream of what should be a continuous, single gestalt of mathematics.

Yet another fashionable peculiarity in high-school math pedagogy is the entirely different style of presentation of geometry compared to trigonometry, which is, at least initially, virtually the same subject. While geometry, as traditionally presented, strives for some sort of mathematical rigor, trigonometry often degenerates into pages of random, byzantine formulas, neither motivated nor justified.

In this book, we have placed geometry in the mainstream of math by beginning with vectors and complex numbers, which are simple algebraic concepts easily accessible to any student competent in the traditional first year of high-school algebra. These subjects enable proofs in geometry to be a mixture of algebra, logic, and pictures, as with mathematical proofs at any level. Algebraic techniques take away much of the mystery of proofs, giving one something to do to get started either proving or deriving a result. This also places vectors and complex numbers at a more natural point in a student's intellectual development; in particular, there is no good reason to delay their introduction until after single-variable calculus.

Vectors lead, at least psychologically, to calculus, in the sense that they involve a sense of motion; this is another way in which our approach puts geometry back in the pedagogical mainstream. In addition to geometry, our book also puts vectors and complex numbers into a more natural intellectual setting in math pedagogy.

Our goal is not to belittle traditional geometry arguments, except in their exclusiveness. Our approach is meant to supplement and rearrange, not replace, the traditional Euclidean approach, with our much more complete arsenal greatly simplifying and clarifying deep and important subjects. The algebraic techniques with which we are preceding geometry and trigonometry take away much of the mystery of proofs, always providing a way to *start* a problem.

We agree with traditional pedagogy in believing that geometry is a good place to introduce mathematical proofs, because of the constant availability of pictures to guide the mind and exposition. What we are adding to this proof-writing introduction is the additional guidance of algebraic techniques, all put together into, not only the (mostly verbal) style of a mathematics proof, but the style of any persuasive reasoning: science from first principles, legal arguments, philosophy, religion, personal disagreements, all mental activities that high school students should see in traditional "English" or "Social Studies" classes. In other words, geometry should be in the mainstream, not only of mathematics, but of all analytic thought and its verbal representation. A relevant historical note: it is said that at the entrance to Plato's Academy was the written message "let no one ignorant of geometry enter here."

We have already mentioned the symbiotic relationships between different areas of math. On a more general level, there is, or should be, a symbiotic relationship between mathematical theory, applications of math, and teaching math. Even if a student's primary goal is remembering the conclusions of geometry, learning the proofs makes it much easier to remember, or derive if one forgets, those results. Someone who plans to teach geometry will do a much better job the deeper said person's understanding is.

Our approach paves the way for much deeper math and applications; the dot product, for example, leads to extensive twentieth-century research in math, physics, and statistics. We have chosen postulates that, besides being simple and believable, are an entree into calculus, further putting geometry into the mainstream of mathematics and its presentation.

See Appendix 0 for the definition of the word "postulate" and other logical terminology needed.

POSTULATES:

(1) The plane of this book may be represented as the Cartesian plane $\mathbf{R}^2 \equiv \{(x, y) \mid x, y \text{ are real numbers}\}$.

(2) Given real numbers $a < b, c < d$, the **area** of the upright rectangle

$$[a, b] \times [c, d] \equiv \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\} \quad (b > a, d > c)$$

is defined to be $(b - a)(d - c)$ and the length of the line segment from (a, c) to (b, d) is defined to be $\sqrt{(b - a)^2 + (d - c)^2}$.

An example of a surface that does *not* satisfy these postulates is the surface of the earth, an example of a *non-Euclidean space*. Many things are buried in our postulates, including Euclid's fifth postulate, the *parallel postulate* (Proposition 3.2).

See DRAWING INTRO at the end of the Introduction, for an outline of topics covered in this book.

Prerequisites for this book are the usual topics of a first year of high-school algebra, including the Cartesian plane. The content of said prerequisites is summarized in Chapter 0. This leads naturally to complex numbers and (two-dimensional) vectors, in Chapter I. Chapter II defines the relevant parameters of geometry (length, angle, and area) in a way that sets them up to be dealt with using vectors and complex numbers. The very simple idea of *parallel* vectors leads already, in Chapter III, to short, easy proofs of many interesting geometry results. The so-called *dot product* of two vectors, motivated by the Pythagorean theorem, produces an algebraic (and easily calculated) definition of two vectors' being *orthogonal* or *perpendicular*, in Chapter IV. This leads, in Chapter V, to more geometric results; in contrast to Chapter III, the Chapter V geometry involves *orthogonality* (being perpendicular) rather than being parallel. In Chapter VI, both vectors and complex numbers are used to introduce the trigonometric functions sine and cosine, and use them to calculate angles. Trigonometry is related to triangles in Chapter VII, and then used to give short, simple proofs of many fundamental results in geometry. Chapter VIII gives explicit statements of the invariance of length, angle, and area under what are called *rigid motions*: translation, rotation, and reflection. Angle is also shown to be invariant under *magnification*. Chapter IX introduces matrices, to prove the results of Chapter VIII. Chapter X uses our trigonometry results to give short, simple proofs of the popular *congruence* (meaning equal after some rigid motions) results for triangles. Chapter XI has additional parallelogram geometry, Chapter XII derives some area formulas, while Chapter XIII is about *constructions* with straight edge and compass. Chapter XIV is worked examples, preceded by a distilled short list of theoretical geometry results needed. Appendix 0 gives precise statements of functions and logic. Appendices I–IV are meant only for students who have had single-variable calculus; for those without a calculus background, the appendices may be taken as additional postulates. Appendix IV is independent of the rest of the book, giving a presentation of trigonometric and inverse trigonometric functions analogous to the presentation of exponentials and logarithms in calculus.

This book may be used in many different ways, in particular, with different degrees of rigor.

A future teacher of geometry should go through the entire book, including appendices, after single-variable calculus. A student who has not had calculus should not work through Appendices I–IV, but use their conclusions, if needed, as additional assumptions (see "Postulates" above), to be believed without proof. Chapter IX may be skipped if one is willing to take, on faith without proof, the quite plausible assertions of Chapter VIII described in the previous paragraph; in fact, in the same spirit, all but the first two and last two paragraphs of Chapter VIII may be skipped, under these articles of faith. It should be mentioned that Chapter IX also completes the introduction to the subject of *linear algebra* begun in Chapter II: Chapter II has vectors while Chapter IX has matrices. Finally, if the reader is interested only in the results, without the proofs, of geometry, said reader could read the book, especially Chapter XIV, without the proofs. The author recommends at least some informal perusal of proofs, if only to make remembering conclusions more than a random event.

Other pedagogical possibilities are to begin with Chapter XIII (constructions), providing motivation to then cover the book to see why the constructions work; or, one could begin with Chapter XIV (solving problems), providing motivation to cover the book to understand the tools just used.

Students who haven't had calculus should be encouraged to browse through the appendices, to get some of the flavor of calculus.

Other books have applied vectors to geometry, usually in a college upper-division "vector analysis" class. Other books have included the Cartesian plane in teaching geometry; we note in particular the excellent book by Serge Lang and Gene Murrow, "Geometry," second edition, 1988, where coordinates are introduced in Chapter 2 and vectors and dot product in Chapter 10, but the bulk of the book is Euclid's approach; see the last paragraph at the bottom of the first page of the introduction in said book.

See also the books by John Saxon, especially "Algebra 1. An Incremental Development," second edition, 1990 for a more applied approach to geometry and the best coverage we have seen of high-school algebra.

Our book is new in its entire reordering of the teaching of vectors, complex numbers, trigonometry, and geometry, including basic geometry definitions in terms of vectors, complex numbers, and trigonometry. The systematic application of complex numbers to geometry is, so far as we know, new with this book. The majority of proofs are original.

As with traditional geometry books, this book will introduce students to mathematical proofs, with this difference. Whereas traditional geometry books introduce students to mathematical proofs of classical Greece, our book introduces students to both classical and modern mathematical proofs, with techniques that are both easier and more powerful.

DRAWING INTRO

$$
\begin{tikzcd}
& \text{Cartesian plane} \arrow[d] & \\
& \text{vectors and complex numbers} \arrow[dl] \arrow[dr] & \\
\text{geometry} \arrow[dr] & & \text{trigonometry} \arrow[dl] \\
& \text{more geometry} &
\end{tikzcd}
$$

CHAPTER 0: Some Preliminaries.

We will need a few topics from traditional first-year high-school algebra.

Denote by $\mathbf{R}$ the set of all real numbers.

Definitions 0.1. The **Cartesian plane**, denoted $\mathbf{R}^2$ (reads "$\mathbf{R}$ two"), is the set of all ordered pairs of real numbers $\{(a,b)\,|\,a,b$ are real numbers $\}$. The ordered pair (a,b) represents the point a units to the right of the origin, b units above. See DRAWING 0.1 at the end of this chapter.

The number a is the **x coordinate** of (a,b), b is the **y coordinate**. The **x-axis** is the horizontal axis (line) $\{(x,0)\,|\,x$ is real $\}$, the **y-axis** is the vertical axis (line) $\{(0,y)\,|\,y$ is real $\}$.

The **upper half plane** is $\{(x,y)\,|\,y>0\}$, the **right half plane** is $\{(x,y)\,|\,x>0\}$; we leave it to the reader to write definitions for **lower half plane** and **left half plane**.

Definition 0.2. The **distance** between (x_1,y_1) and (x_2,y_2) is

$$d\left((x_1,y_1),(x_2,y_2)\right)\equiv\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}.$$

See DRAWING 0.3 at the end of this chapter.

Definition 0.3. If a,b, and c are real numbers then

$$\{(x,y)\,|\,ax+by=c\}$$

is a **line** in $\mathbf{R}^2$. The formula $ax+by=c$ is an **equation** of the line. See DRAWING 0.2 at the end of this chapter.

Definitions 0.4. The **slope** of a nonvertical line is

$$m\equiv\frac{(y_2-y_1)}{(x_2-x_1)},$$

for *any* pair of different points $(x_1,y_1),(x_2,y_2)$ on the line. We will define the slope of a vertical line $x=c$ to be $m=\infty$.

Note that we could just as naturally think of the slope of a vertical line as $-\infty$, in the following way. The vertical line $x=c$ could be thought of as a limit, as n gets arbitrarily large, of the lines of slope n: $\frac{y}{n}=(x-c)$; or as the limit of the lines of slope $-n$: $\frac{y}{-n}=(x-c)$. See DRAWING 0.3 at the end of this chapter.

Assertion and Definition 0.5. If a line has finite slope m and crosses the y-axis at the point $(0,b)$, then an equation of the line is

$$y=mx+b.$$

b is called the **y intercept** of the line.

Lines of infinite slope always have the form $x=c$, for some fixed real number c.

Definitions 0.6. Completing the square means writing the quadratic expression

$$y=ax^2+bx+c\quad(a,b,c\text{ are numbers})$$

in the form

$$y=a(x+\beta)^2+\gamma.$$

Thinking backwards from the expansion

$$(x+\beta)^2 = x^2 + 2\beta x + \beta^2$$

leads to

$$\frac{y-c}{a} = \left(x^2 + \frac{b}{a}x\right) = \left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) = \left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2$$

so that

$$y = a\left(x + \frac{b}{2a}\right)^2 + c - a\left(\frac{b}{2a}\right)^2 = a\left(x + \frac{b}{2a}\right)^2 + \left[\frac{4ac-b^2}{4a}\right] \text{ (completed square).}$$

Setting $y = 0$ produces the **quadratic formula** giving the following solutions to the **quadratic equation** $ax^2 + bx + c = 0$:

$$x = \frac{1}{2a}\left(-b \pm \sqrt{b^2 - 4ac}\right) \quad (a \neq 0).$$

Examples 0.7. (a) Find the slope of the line whose equation is $6x + 2y = 5$.

(b) Find the slope of the line thru $(0, 5)$ and $(3, 2)$.

(c) Find the equation of the line thru $(0, 5)$ and $(3, 2)$.

(d) Complete the square: $y = -2x^2 + 6x - 3$.

(e) Use completing the square to solve $x^2 = 10 - 4x$.

(f) Use the quadratic formula to find where $y = 3x - 2$ intersects $y = x^2 + x - 8$.

Solutions. (a) $y = -3x + \frac{5}{2} \to$ slope is -3.

(b) $\frac{2-5}{3-0} = -1$.

(c) $\frac{y-5}{x-0} = -1 \to y = 5 - x$.

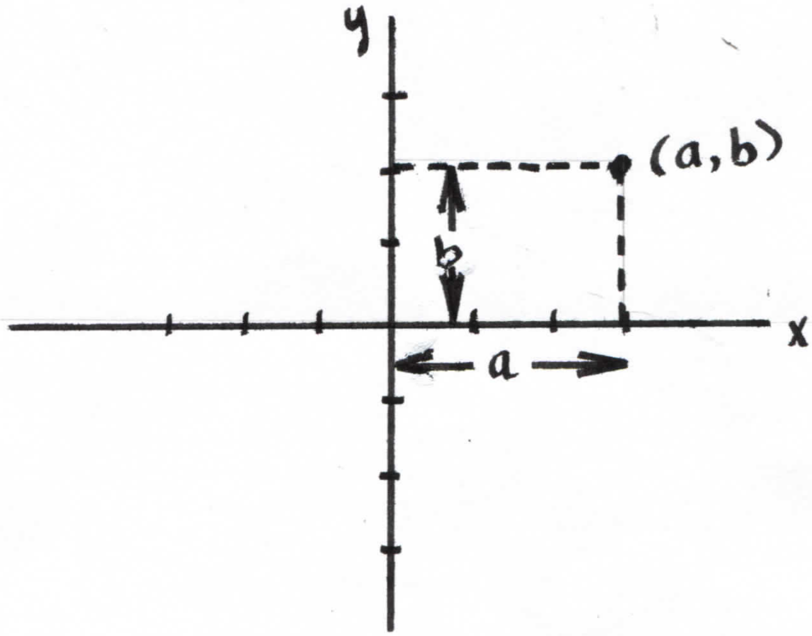
(d) $\frac{-y}{2} = x^2 - 3x + \frac{3}{2} \to (\frac{-y}{2} - \frac{3}{2}) = x^2 - 3x + (\frac{3}{2})^2 - (\frac{3}{2})^2 = (x - \frac{3}{2})^2 - \frac{9}{4} \to \frac{-y}{2} = (x - \frac{3}{2})^2 - \frac{3}{4} \to$ $y = -2(x - \frac{3}{2})^2 + \frac{3}{2}$.

(e) $x^2 + 4x = 10 \to x^2 + 4x + (\frac{4}{2})^2 = 10 + (\frac{4}{2})^2 \to (x+2)^2 = 14 \to (x+2) = \pm\sqrt{14} \to x = -2 \pm \sqrt{14}$.

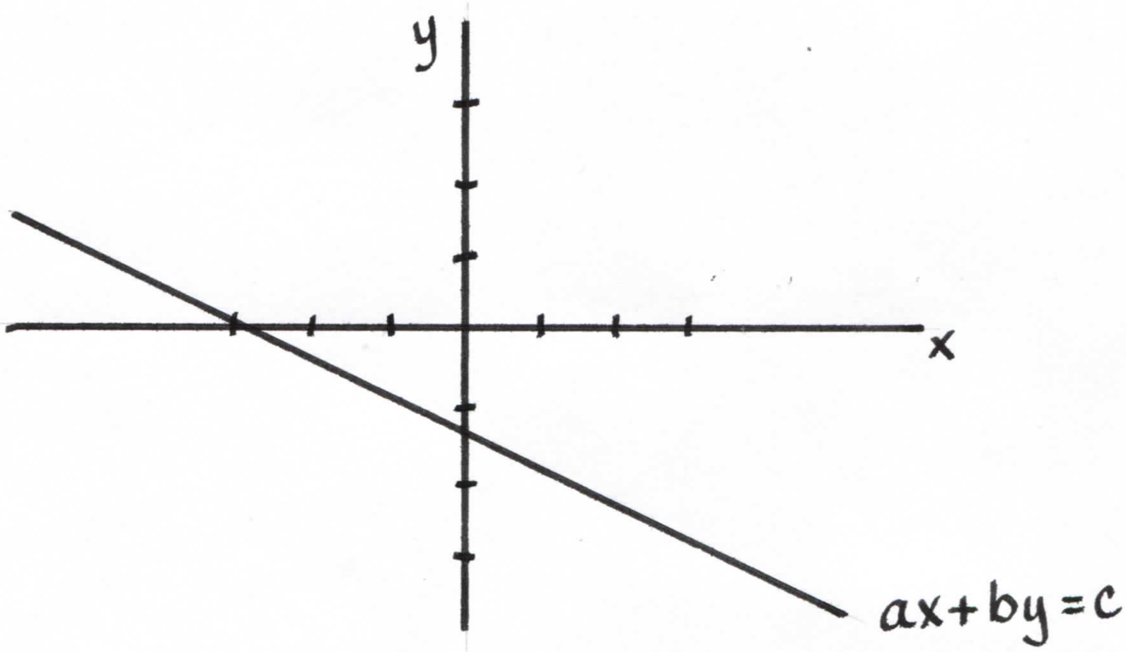
(f) $3x - 2 = x^2 + x - 8 \to x^2 - 2x - 6 = 0 \to$

$$x = \frac{1}{2}(2 \pm \sqrt{(-2)^2 - 4(1)(-6)}) = \frac{1}{2}(2 \pm \sqrt{28}) = 1 \pm \sqrt{7}.$$

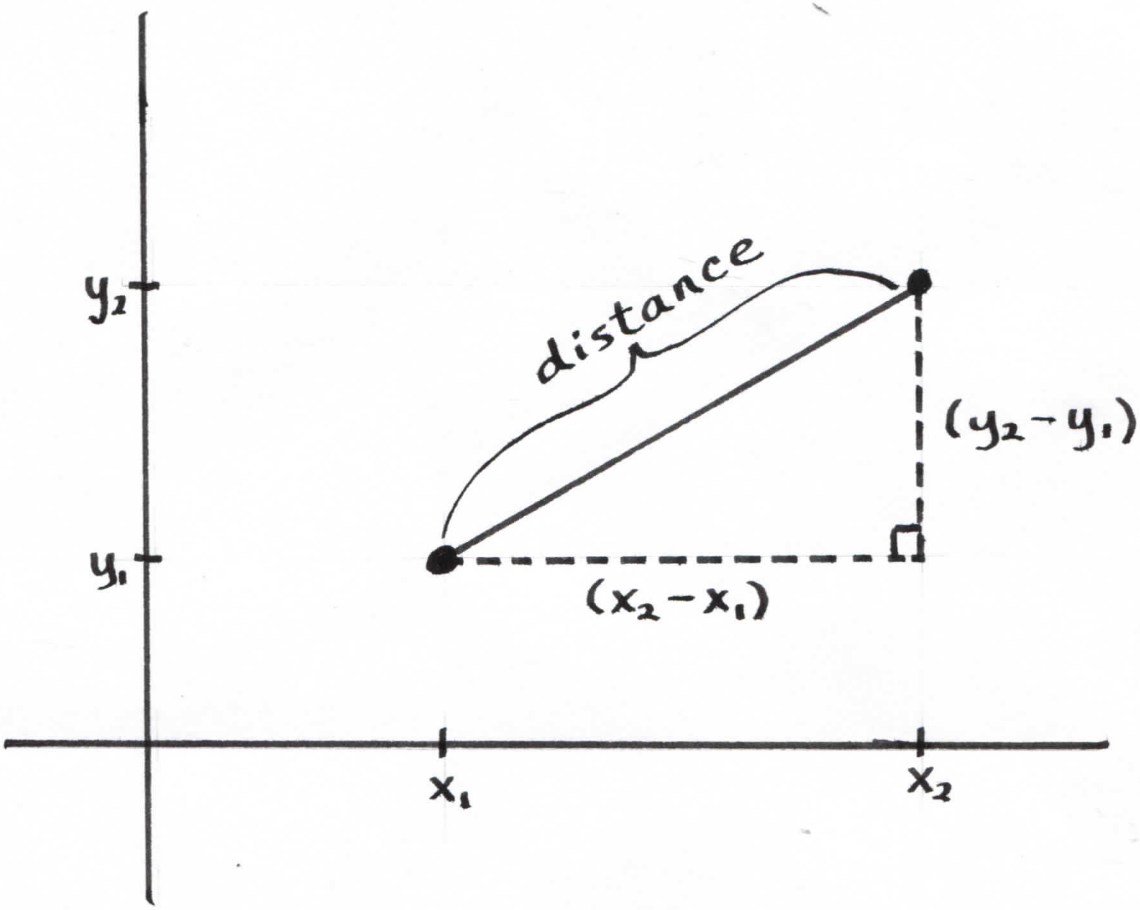
DRAWING 0.1



DRAWING 0.2



DRAWING 0.3



The <u>slope</u> of the line through (x_1, y_1) and (x_2, y_2) is

$$m \equiv \frac{(y_2-y_1)}{(x_2-x_1)}$$

CHAPTER I: Vectors and Complex Numbers: Two New Ways to View the Plane.

Our goal in this chapter is to describe points in the Cartesian plane $\mathbf{R}^2$ both as *vectors* (Definition 1.2) and as *complex numbers* (Definitions 1.8).

Vectors will provide a sense of motion or displacement; complex numbers put an algebraic structure (addition, subtraction, multiplication, and division) onto $\mathbf{R}^2$.

The definition of a vector requires a *pair* of points, connected by an arrow.

Definitions 1.1. If I and T are points, denote by $\overrightarrow{IT}$ the arrow or **directed line segment** that begins at I and ends at T. I is the **initial point**, T the **terminal point**; $\overrightarrow{IT}$ is traditionally drawn with the arrow tip or arrowhead at T and a fat dot, in lieu of feathers, at I. See DRAWING 1.1 at the end of this chapter.

If $I = (x_1, y_1)$ and $T = (x_2, y_2)$, then the **components** of $\overrightarrow{IT}$ are $(x_2 - x_1)$ (the **x component**) and $(y_2 - y_1)$ (the **y component**). See DRAWING 0.3 at the end of Chapter 0 and DRAWING 1.1 at the end of this chapter.

As a set of points in $\mathbf{R}^2$,

$$\overrightarrow{IT} = \{(x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1) \,|\, 0 \le t \le 1\}. \quad (*)$$

But there is more to a directed line segment than a set of points; there is a sense of *motion*; you imagine yourself traveling from the initial point to the terminal point. In (*), we think of the variable t as being *time*: at $t = 0$, we are at the initial point $I \equiv (x_1, y_1)$, at $t = 1$, we are at the terminal point $T \equiv (x_2, y_2)$, and in general, as t increases, we are moving away from I and towards T. See DRAWING 1.2 at the end of this chapter.

Definition 1.2. A **vector** (in two dimensions) is an object represented by a directed line segment, with the understanding that two directed line segments represent the same vector if they have the same x components and the same y components. Intuitively, two directed line segments with the same length and direction represent the same vector.

We will denote

$$\overrightarrow{v} = < v_1, v_2 >$$

to mean the vector represented by any directed line segment with x component v_1 and y component v_2. See DRAWING 1.3 at the end of this chapter.

Vectors should be considered *displacement* (note that components are displacements: the x component is the displacement, that is, the change, in the x coordinate, the y component the displacement in the y coordinate). What matters is not where you began or ended, but what change occurred from your motion in passing from the initial point to the terminal point.

In any physical model, the first distinction that must be made is whether a parameter is a number or a vector. For example, a wind speed of 20 mph (twenty miles per hour) is not complete information; a wind *velocity*, say, of 20 mph North by Northeast, tells me I'll work harder when I travel South by Southwest, against the wind, than when I travel North by Northeast, with the wind. Speed is a number, giving only *magnitude*, while velocity is a vector, with the additional information of *direction* (see Definitions 2.4).

Force is a vector, pushing you in a certain direction (e.g., towards the ground if you're in the air); work is a number.

Definitions 1.3. Denote the **origin** $(0,0)$ by O. Given a vector $\overrightarrow{v} =< v_1, v_2 >$, let T be the point (v_1, v_2). The **standard position** of $\overrightarrow{v}$ is the directed line segment $\overrightarrow{OT}$. Conversely, given a point T, its **position vector** is the vector represented by the directed line segment $\overrightarrow{OT}$; if $T = (v_1, v_2)$, then its position vector is $< v_1, v_2 >$. See DRAWING 1.4 at the end of this chapter.

There will be times when we want you to leap nimbly between point and vector, via the correspondences just described; see 1.16.

Definitions 1.4. We would like to have algebra with vectors; in this chapter, we will define only addition of vectors and multiplication of vectors by real numbers. See Chapter IV for a sort of multiplication of vectors.

It is always desirable to correlate algebra and geometry; algebra (calculations) gives precision, while geometry (pictures) provides intuition.

If $\overrightarrow{v} \equiv< v_1, v_2 >$ and $\overrightarrow{w} \equiv< w_1, w_2 >$, then

$$(\overrightarrow{v} + \overrightarrow{w}) \equiv< v_1 + w_1, v_2 + w_2 > .$$

That's an algebraic definition of vector addition; see DRAWING 1.5 at the end of this chapter for the geometric picture.

DRAWING 1.5 makes sense when you think of a vector as traveling or displacement; $(\overrightarrow{v} + \overrightarrow{w})$ is the *net* displacement, after being displaced by $\overrightarrow{v}$, then by $\overrightarrow{w}$. Think of the two vectors as a pair of professional wrestlers, each contributing a different force.

The **zero**, or **trivial**, vector is $\overrightarrow{0} \equiv< 0, 0 >$. Note that $\overrightarrow{0} + \overrightarrow{a} = \overrightarrow{a}$, for any vector $\overrightarrow{a}$. Vectors in general represent displacement; the trivial vector means no displacement, no change or motion.

If $\overrightarrow{v} \equiv< v_1, v_2 >$ and c is a real number, then the vector $c\overrightarrow{v}$ is defined by

$$c\overrightarrow{v} \equiv< cv_1, cv_2 > .$$

In DRAWING 1.6 at the end of this chapter we drew pictures of $2\overrightarrow{v}$ and $(-\overrightarrow{v})$. Note that $(-\overrightarrow{v}) + \overrightarrow{v} = \overrightarrow{0}$; displacement by the vector $(-\overrightarrow{v})$ cancels out the displacement of $\overrightarrow{v}$, leaving the would-be traveler at the beginning of the attempted net motion.

The pictures in DRAWING 1.6, of multiplication of a vector by a real number, suggest the following definitions.

Definitions 1.5. Two vectors $\overrightarrow{v}$ and $\overrightarrow{w}$ are **parallel** if one is a real multiple of the other: $\overrightarrow{v} = c\overrightarrow{w}$ or $\overrightarrow{w} = c\overrightarrow{v}$, for some real number c. If $c > 0$, $\overrightarrow{v}$ and $\overrightarrow{w}$ **point in the same direction**; if $c < 0$, then $\overrightarrow{v}$ and $\overrightarrow{w}$ **point in opposite directions**.

In DRAWING 1.6, $2\overrightarrow{v}$ points in the same direction as $\overrightarrow{v}$, while $(-\overrightarrow{v})$ points in the opposite direction.

Finally, we have the peculiar algebra of adding a point to a vector, to produce another point.

Definition 1.6.

$$I + \overrightarrow{IT} = T; \quad \text{OR} \quad (a, b)+ < v_1, v_2 >= (a + v_1, b + v_2).$$

See DRAWING 1.7 at the end of this chapter.

Intuitively, we start at the point I, then the vector $\overrightarrow{IT}$ moves us to the point T.

Examples 1.7. $(2, 0) + 3 < 1, 2 >= (2, 0)+ < 3, 6 >= (5, 6)$.

$< 2, 0 > +3 < 1, 2 >=< 2, 0 > + < 3, 6 >=< 5, 6 >$.

See DRAWING 1.8 at the end of this chapter.

Definitions 1.8. For any real number t, $t^2 \geq 0$. Thus there is no real solution of the equation $t^2 + 1 = 0$; that is, no real square root of (-1).

When something does not exist, it is good strategy to give it a name. This creates the comfortable illusion, not only of existence, but of understanding.

By definition, "i" (short for "imaginary") is a number whose square is (-1):

$$i^2 = (-1) \quad \text{or} \quad i \equiv \sqrt{(-1)}.$$

Imaginary numbers are real multiples of i; **complex numbers** are sums of real numbers and imaginary numbers $(x + iy)$, where x and y are real numbers:

$$\mathbf{C} \equiv \{\text{complex numbers}\} \equiv \{(x + iy) \,|\, x, y \text{ are real}\}.$$

The **real part** of $z \equiv (x + iy)$, denoted $\mathrm{Re}(z)$, is x; the **imaginary part**, denoted $\mathrm{Im}(z)$, is y.

We may add complex numbers and multiply complex numbers by real numbers:

$$(a + ib) + (c + id) = (a + c) + i(b + d); \quad c(a + ib) = ca + i(cb). \quad (a, b, c, d \text{ real}).$$

These operations are not really new (see Definitions 1.4), when we think of a complex number as a point or vector (in standard position) in the plane:

$$(a + ib) \sim (a, b) \quad \text{or} \quad < a, b >,$$

where "$\sim$" refers to not-necessarily-rigorous equating. See DRAWING 1.9 at the end of this chapter.

In this setting, the x-axis is called the **real axis R**, the y-axis is called the **imaginary axis** $i\mathbf{R}$, and **C** is called the **complex plane**.

What *is* new is that we may multiply two complex numbers together or divide one complex number by another:

$$(a + ib)(c + id) = (ac + iad) + (icb + dbi^2) = (ac - db) + i(ad + cb) \quad (a, b, c, d \text{ real});$$

$$\frac{(a + ib)}{(c + id)} = \frac{(a + ib)(c - id)}{(c + id)(c - id)} = \frac{(ac + bd) + i(bc - ad)}{(c^2 + d^2)} = \left(\frac{(ac + bd)}{(c^2 + d^2)}\right) + i\left(\frac{(bc - ad)}{(c^2 + d^2)}\right) \quad (a, b, c, d \text{ real}, \ c^2 + d^2 \neq 0).$$

Definitions 1.9. Suppose a and b are real and $z \equiv (a + ib)$. The **conjugate** of z is

$$\overline{z} \equiv (a - ib).$$

The **absolute value** of z is

$$|z| \equiv \sqrt{a^2 + b^2}.$$

Note that $|z|$ is the length of the directed line segment representing the vector $< a, b >$ and conjugation is reflection through the x axis. See DRAWING 1.10 at the end of this chapter.

Conjugation Lemma 1.10. Suppose z and w are complex numbers.

(1) $|z|^2 = z\overline{z}$.

(2) $\overline{(z + w)} = \overline{z} + \overline{w}$.

(3) $\overline{(zw)} = (\overline{z})(\overline{w})$.

(4) $(z + \overline{z}) = 2 \ \mathrm{Re}(z)$.

(5) $(z - \overline{z}) = 2i \ \mathrm{Im}(z)$.

Proof: Write $z = x + iy, w = u + iv$, with x, y, u, v real.

(1) $z\overline{z} = (x + iy)(x - iy) = x^2 - (iy)^2 = x^2 + y^2 = |z|^2$.

(2) $\overline{(z + w)} = \overline{(x + u) + i(y + v)} = (x + u) - i(y + v) = (x - iy) + (u - iv) = \overline{z} + \overline{w}$.

(3) $\overline{(zw)} = \overline{(xu - yv) + i(xv + yu)} = (xu - yv) - i(xv + yu) = (x - iy)(u - iv) = (\overline{z})(\overline{w})$.

(4) $(z + \overline{z}) = (x + iy) + (x - iy) = 2x = 2\ \mathrm{Re}(z)$.

(5) $(z - \overline{z}) = (x + iy) - (x - iy) = 2yi = 2i\ \mathrm{Im}(z)$. □

Examples 1.11. $(2 - i) + 3(1 + 2i) = (2 - i) + (3 + 6i) = (5 + 5i)$.

$(2 - i)(1 + 2i) = 2(1 + 2i) - i(1 + 2i) = (2 + 4i) - (i + 2(-1)) = (4 + 3i)$.

$|(2 - i)| = \sqrt{4 + 1} = \sqrt{5}, |(1 + 2i)| = \sqrt{1 + 4} = \sqrt{5}, |(4 + 3i)| = \sqrt{16 + 9} = 5 = |(2 - i)||(1 + 2i)|$.

$\overline{(2 - i)} = (2 + i)$; NOTE that $\overline{(2 - i)}(2 - i) = 5 = |(2 - i)|^2$.

$\frac{1}{(2-i)} = \frac{(2+i)}{(2-i)(2+i)} = \frac{(2+i)}{5} = \frac{2}{5} + \frac{1}{5}i$.

We will now present a representation of complex numbers (Definition 1.15) very useful in understanding angles (Definitions 2.10). This will involve a function ("exp," see Theorem 1.12; see Definitions APP0.3 for the precise definition of "function") fundamental to many applications of calculus, whose restriction to the unit circle $x^2 + y^2 = 1$ defines trigonometry (Definition 6.1).

Theorem 1.12. There exists a function $\exp : \mathbf{C} \to \mathbf{C}$ such that, for any complex z, w,

(1)

$$\exp(z + w) = \exp(z)\exp(w)$$

and

(2)

$$\exp(\overline{z}) = \overline{\exp(z)}.$$

Proof: Appendix Three, Theorem APP3.2. □

Motivated by Property (1) of Theorem 1.12, the function $\exp(z)$ is traditionally written e^z, where $e \equiv \exp(1)$. Because of its derivative properties (see Appendix Three, Theorem APP3.2(3)) exp is called **the** (as opposed to "an") **exponential function**.

Corollary 1.13. For any complex z, $|\exp(z)| = \exp(\mathrm{Re}(z))$.

Proof:

$$|\exp(z)|^2 = \exp(z)\overline{\exp(z)} = \exp(z)\exp(\overline{z}) = \exp(z + \overline{z}) = \exp(2\mathrm{Re}(z)) = (\exp(\mathrm{Re}(z)))^2\,.$$

We will be done if we can show that $\exp(\mathrm{Re}(z))$ is real and nonnegative. Let $x \equiv (\mathrm{Re}(z))$. Since x is real, Property (2) of Theorem 1.12 implies that $\exp(\frac{x}{2})$ is real. Writing, by Property (1) of Theorem 1.12,

$$\exp(x) = \left(\exp(\frac{x}{2})\right)^2$$

implies that $\exp(x)$ is nonnegative. □

Corollary 1.14. For any $R \geq 0$, real $\theta, Re^{i\theta}$ is a point on the circle $x^2 + y^2 = R^2$.

Proof: Since $\mathrm{Re}(i\theta) = 0$, Corollary 1.13 implies that

$$|Re^{i\theta}| = R|e^{i\theta}| = R.$$

□

See DRAWING 1.11 at the end of this chapter.

Definition 1.15. Writing a complex number z as $re^{i\theta}, r \equiv |z|, \theta$ real, is called the **polar form** for z.

Lemma 2.8 shows that every complex number has a polar form. Also see Lemma 2.8 and DRAWING 2.11 at the end of Chapter II for a picture of what θ means. The parameter θ will, in addition, give us the measure of angles between vectors (see Definitions 2.10).

Glib Equivalences 1.16. For x, y real numbers, we have already mentioned that we will want to associate the point (x, y) in $\mathbf{R}^2$, as in Definitions 0.1, with the vector $< x, y >$ in standard position (see Definitions 1.3 and DRAWING 1.4 at the end of this chapter). We will also find it very convenient to, in addition, associate (x, y) with the complex number $(x + iy)$; see DRAWING 1.9 at the end of this chapter and Definitions 1.8.

Throughout this book, we might make these associations implicitly; for example, we might say "i is on the unit circle" as a shorthand for "the point $(0, 1)$ representing the complex number i is on the unit circle."

Examples 1.17. Which of the following pairs of vectors are parallel? Which point in the same direction? Which point in opposite directions?

(a) $\{< 1, 2 >, < -2, 1 >\}$.

(b) $\{< 4, -2 >, < -2, 1 >\}$.

(c) $\{< 4, -2 >, < 2, -1 >\}$.

(d) $\{< 5, 17 >, < 0, 0 >\}$.

Solutions. See Definitions 1.5.

(a) If $< 1, 2 >= c < -2, 1 >$, then $1 = -2c$, so that $c = -\frac{1}{2}$, while from the second component $c = 2$. Similar problems arise when we set $< -2, 1 >= c < 1, 2 >$. Thus this pair of vectors is not parallel.

(b) Setting $4 = c(-2) \to c = -2$; since $< 4, -2 >= (-2) < -2, 1 >$, we conclude that the vectors are parallel, pointing in opposite directions.

(c) The same calculations show that $< 4, -2 >= 2 < 2, -1 >$, thus the vectors are parallel, pointing in the same direction.

(d) Since $< 0, 0 >= 0 < 5, 17 >$, the vectors are parallel (in Definitions 1.5, $c = 0$). Since c is neither greater than 0 nor less than 0, the two vectors neither point in the same direction, nor in opposite directions.

Notice also that $< 5, 17 >$ does not equal $c < 0, 0 >$, for any real c.

People are sometimes reluctant to talk about being parallel to $< 0, 0 >$, because it implies that $< 0, 0 >$ has a direction, which is problematic: what is the direction of the wind when there is no wind?

HOMEWORK

HWI.1. If $I = (1, 2)$ and $T = (-3, 1)$, get (the components of) $\overrightarrow{IT}$, and draw I, T, and $\overrightarrow{IT}$.

HWI.2. Get and draw $(-1, 2) + 2 < 1, 3 >$ and $< -1, 2 > +2 < 1, 3 >$.

HWI.3. Simplify each of the following.

(a) $(-1 + 2i) + 2(1 + 3i)$.

(b) $\overline{(\sqrt{7} - \frac{\sqrt{3}}{2} i)}$.

(c) $|1 - \sqrt{2}i|^2$.

(d) $(1 - 2i)(3 + i)$.

(e) $\frac{(1-2i)}{(3+i)}$.

(f) $|3 + 4i|$.

HWI.4. Which of the following pairs of vectors are parallel? Which point in the same direction? Which point in opposite directions?

(a) $< 1, -3 >, < 3, -9 >$.

(b) $< 1, -3 >, < -3, 9 >$.

(c) $< 1, -3 >, < 0, 0 >$.

(d) $< 1, -3 >, < 6, 2 >$.

HWI.5. Show that, for any complex $z \neq 1$,

$$\left(1 + z + z^2 + \cdots + z^{n-1}\right) = \frac{1 - z^n}{1 - z},$$

for $n = 1, 2, 3, \ldots$.

HWI.6. Use conjugation to show that

$$|zw| = |z||w|,$$

for any complex z and w.

HOMEWORK ANSWERS

HWI.1. $\overrightarrow{IT} =< -4, -1 >$. See DRAWING 1.12 at the end of this chapter.

HWI.2. $(-1, 2) + 2 < 1, 3 >= (1, 8)$ and $< -1, 2 > +2 < 1, 3 >=< 1, 8 >$. See DRAWINGS 1.13 at the end of this chapter.

HWI.3. (a) $(-1 + 2i) + 2(1 + 3i) = (1 + 8i)$.

(b) $\overline{(\sqrt{7} - \frac{\sqrt{3}}{2}i)} = (\sqrt{7} + \frac{\sqrt{3}}{2}i)$.

(c) $|1 - \sqrt{2}i|^2 = 3$.

(d) $(1 - 2i)(3 + i) = (5 - 5i)$.

(e) $\frac{(1-2i)}{(3+i)} = (\frac{1}{10} - \frac{7}{10}i)$.

(f) $|3 + 4i| = 5$.

HWI.4. (a) $< 1, -3 >, < 3, -9 >$ are parallel and point in the same direction.

(b) $< 1, -3 >, < -3, 9 >$ are parallel and point in opposite directions.

(c) $< 1, -3 >, < 0, 0 >$ are parallel.

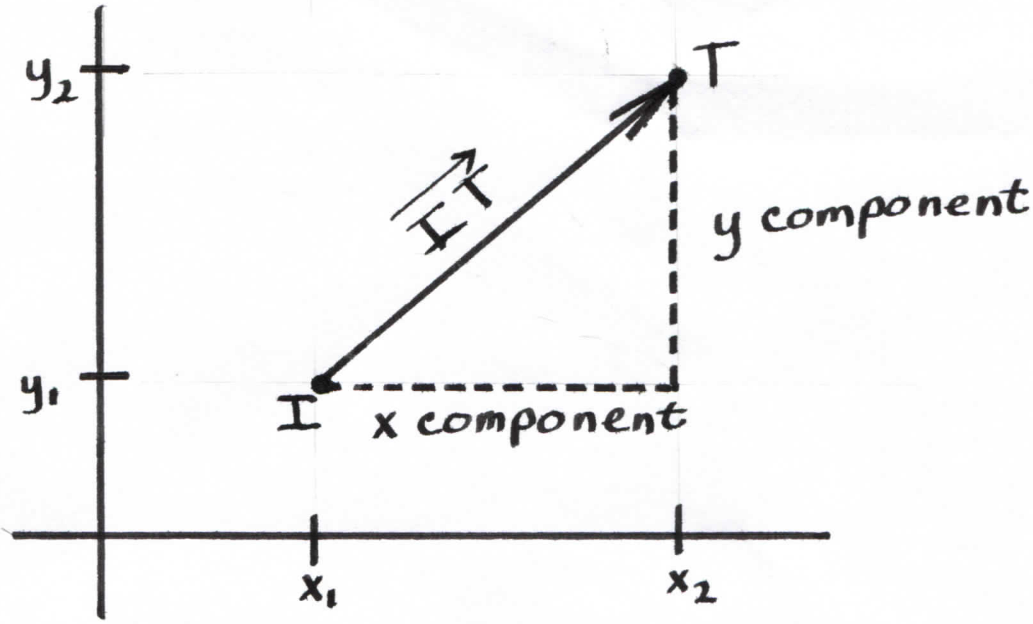
(d) $< 1, -3 >, < 6, 2 >$ are not parallel.

HWI.5.
$(1-z)\left(1 + z + z^2 + \cdots + z^{n-1}\right) = (1+z+z^2+\cdots+z^{n-2}+z^{n-1}) - (z+z^2+z^3+\cdots+z^{n-1}+z^n) = (1-z^n)$, after extensive cancellation.

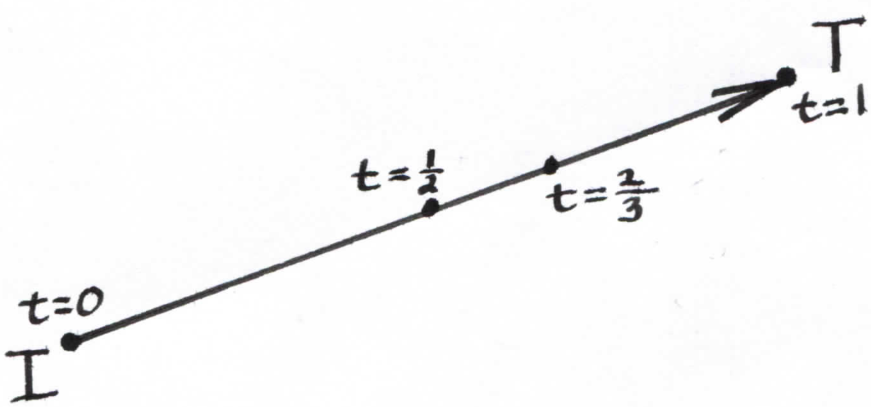
HWI.6.

$$|zw|^2 = (zw)(\overline{zw}) = (zw)(\overline{z})(\overline{w}) = (z\overline{z})(w\overline{w}) = |z|^2|w|^2 = (|z||w|)^2.$$

DRAWING 1.1

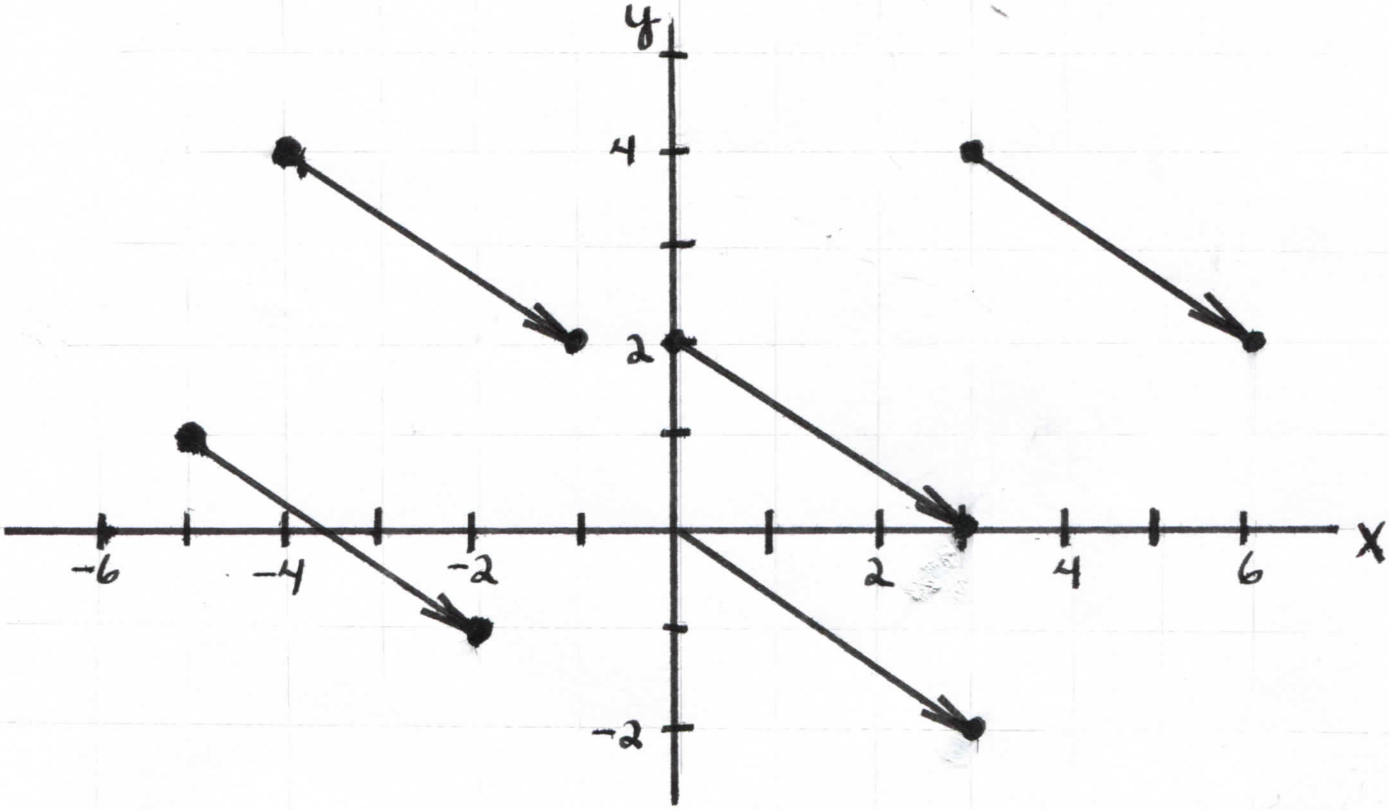


DRAWING 1.2

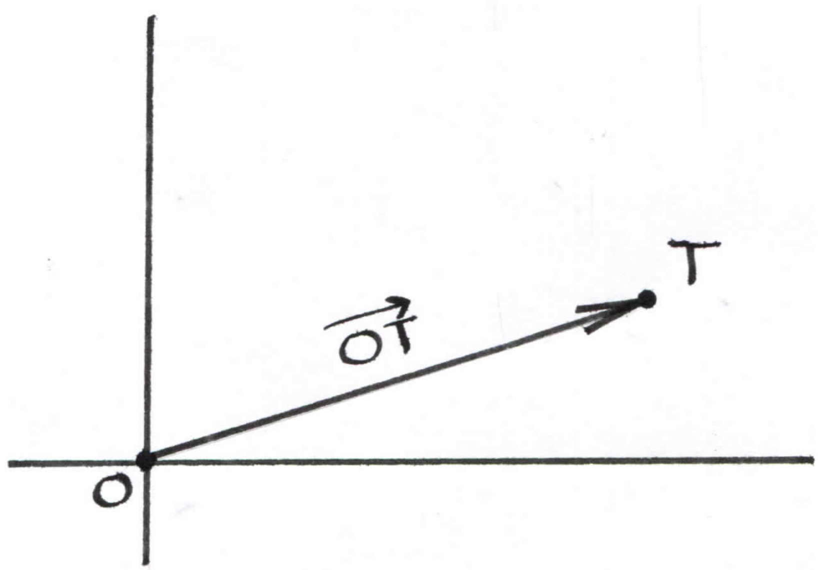


DRAWING 1.3

(SAME vector $\vec{v} \equiv \langle 3, -2 \rangle$)

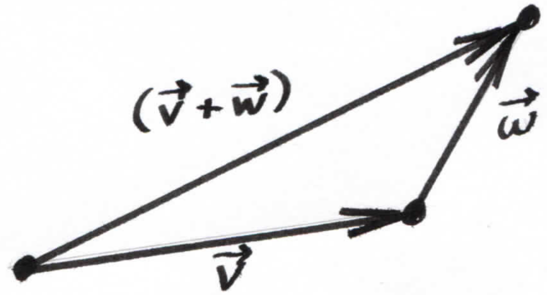


DRAWING 1.4

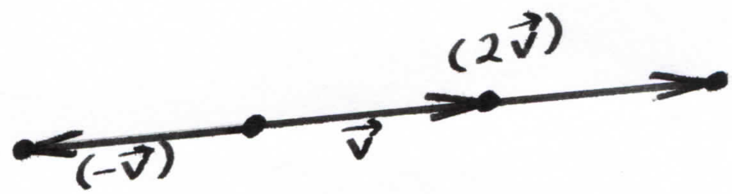


$\vec{OT}$ is the position vector for T, in standard position

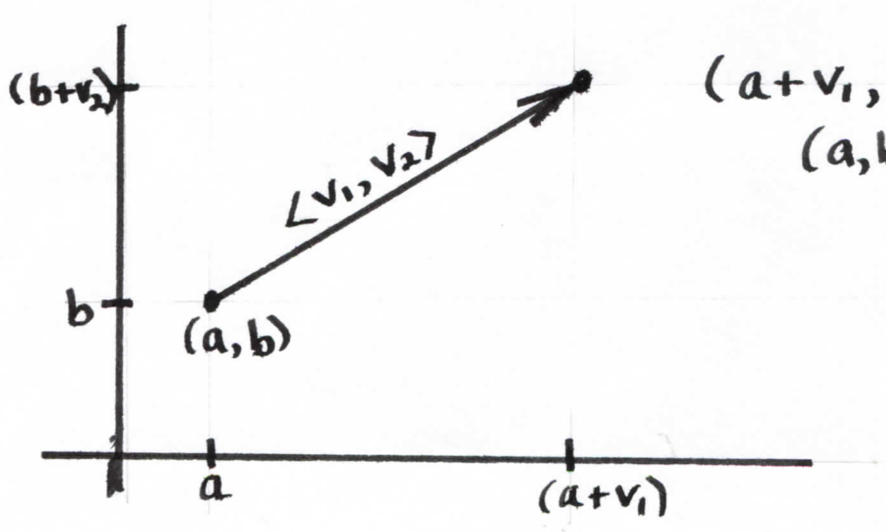
DRAWING 1.5



DRAWING 1.6

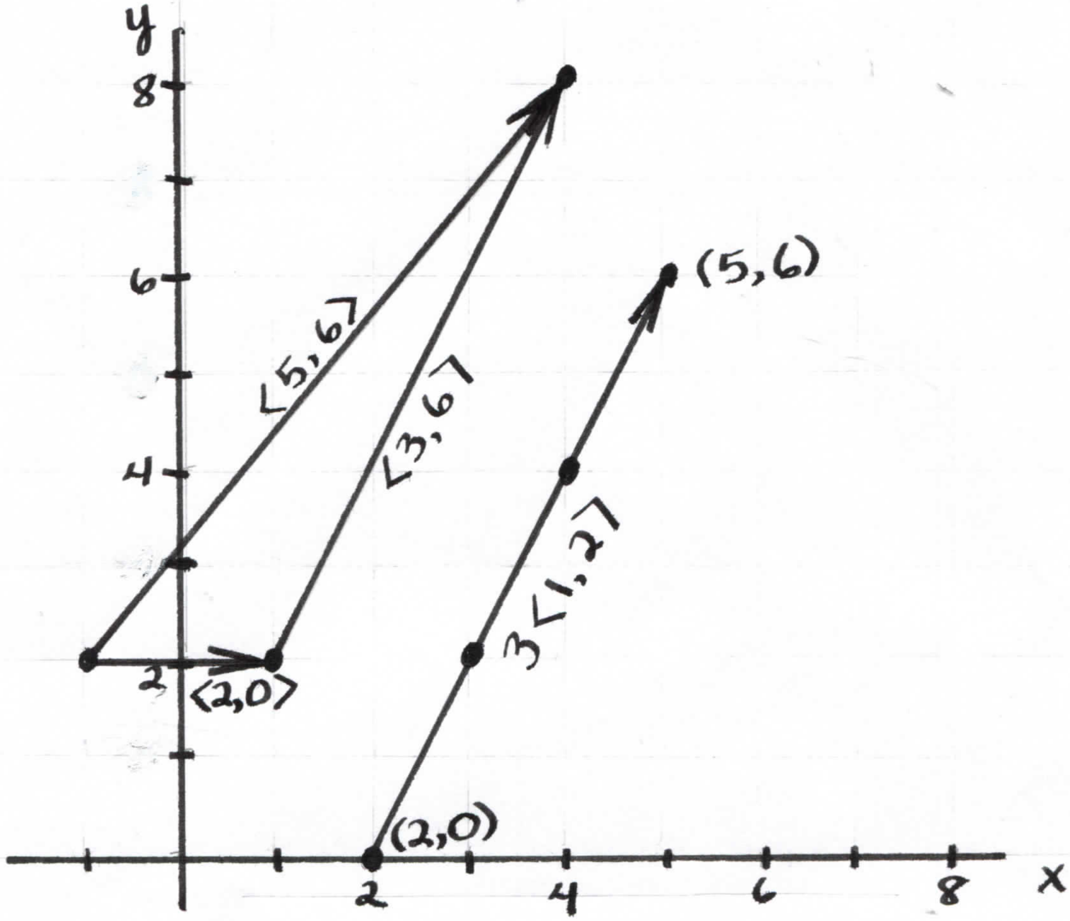


DRAWINGS 1.7

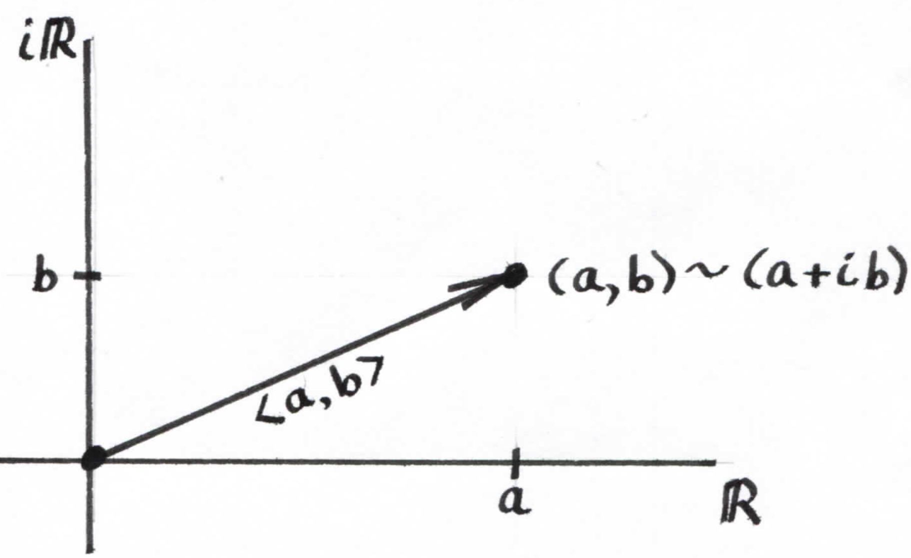


$(a+v_1, b+v_2) =$
$(a,b) + \langle v_1, v_2 \rangle$

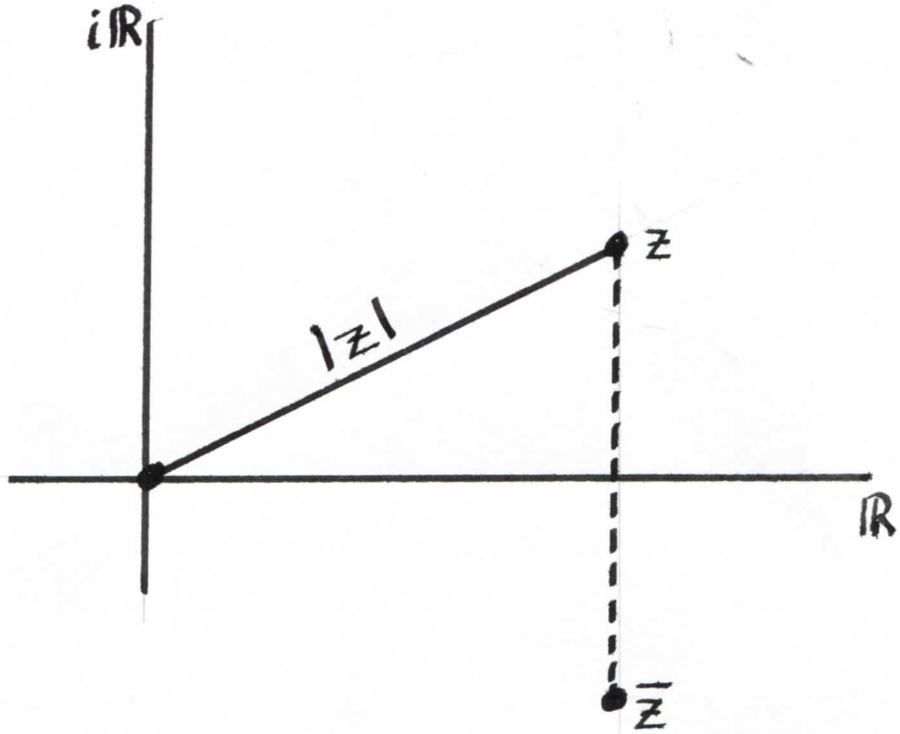
DRAWING 1.8



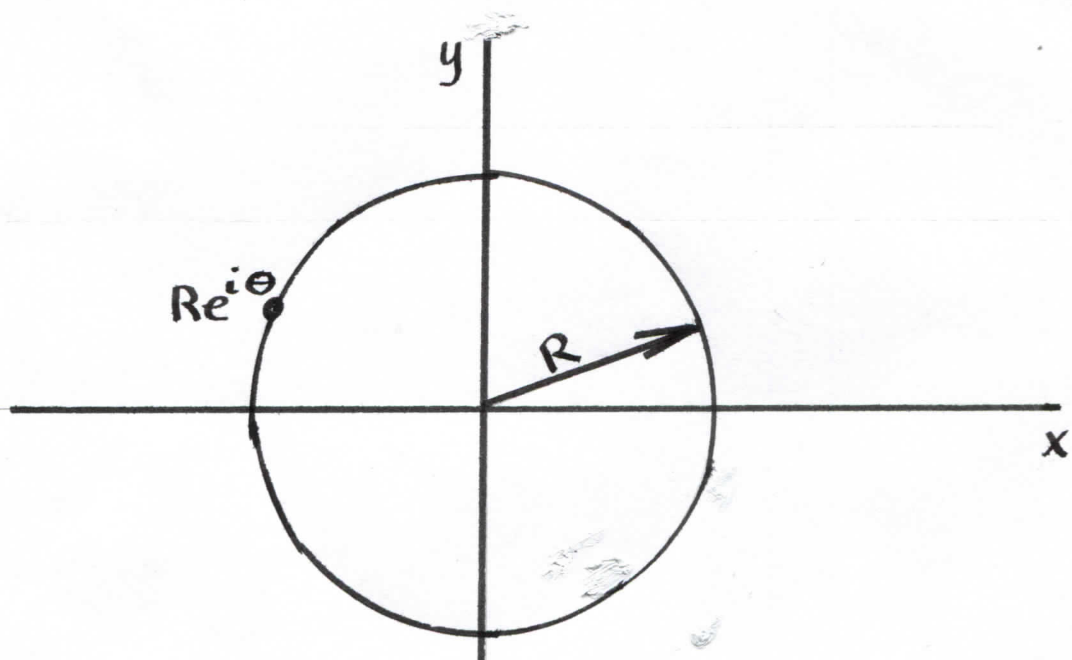
DRAWING 1.9



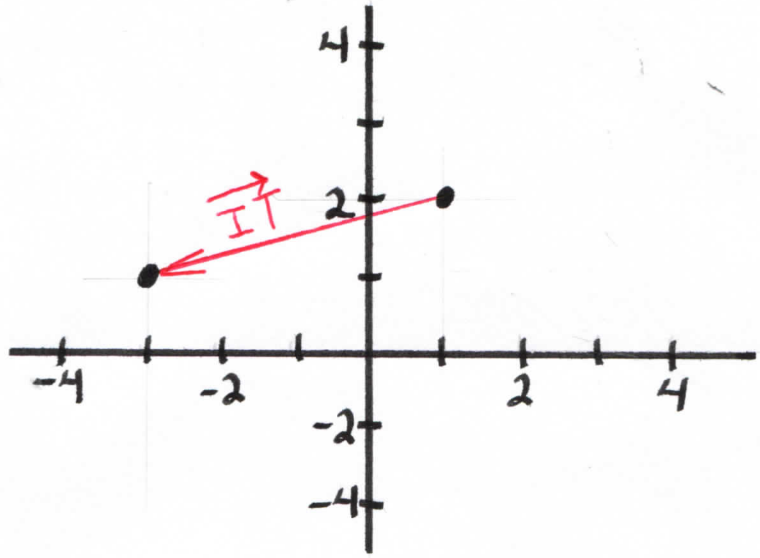
DRAWING 1.10



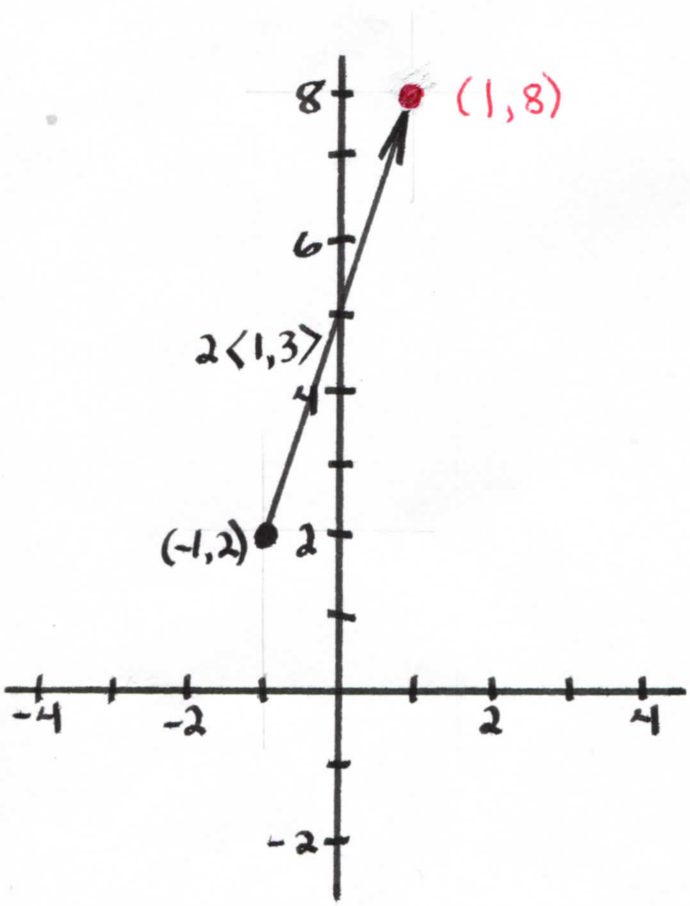
DRAWING 1.11

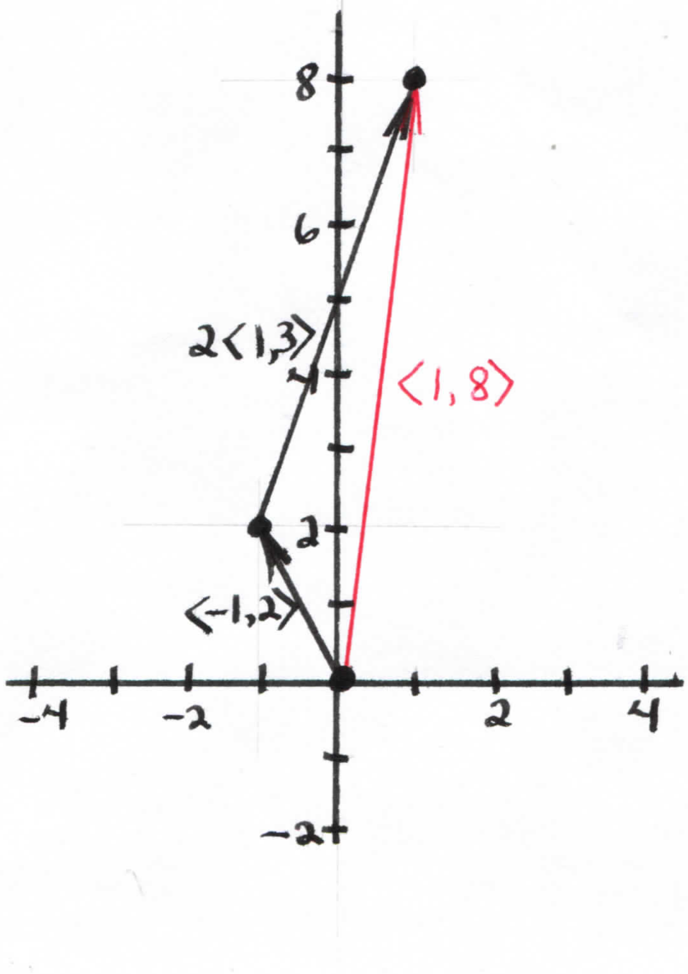


DRAWING 1.12



DRAWINGS 1.13





CHAPTER II: Objects and Parameters of Interest.

The objects of interest in geometry are angles (Definitions 2.10) and (pieces of) lines (Definition 2.1) and circles (Definitions 2.6) and the subsets of $\mathbf{R}^2$ they enclose (Definitions 2.3 and 2.6).

It is an easy calculation to see that our definitions of length and area (Definitions 2.4 and postulates of Introduction) are unaffected by *translation*, that is, by adding a fixed vector to every point in a set. Since two directed line segments representing the same vector differ only by a translation (see Definitions 1.1 and 1.2 and DRAWING 1.3 at the end of Chapter I), this means that we may state all results in terms of vectors; a statement involving vectors is true for any directed line segment representing a vector (see Definitions 1.1 and 1.2). Standard position (Definitions 1.3) is often the most convenient representation of a vector.

At first glance, this chapter may seem too long and detailed, for matters that we have a picture or hand gestures for. We encourage the reader to refer often to the drawings, to hold onto the visual intuition.

The details are desirable for getting vectors, with their own dynamic intuition and computational possibilities, into the picture, along with the beginnings of rigor.

Because we will use curves, which may be considered twisted line segments (see Remarks 2.5), to define angle, so that length of a curve will define the measurement of an angle, we will define both lines and polygons, and their measures, before we address angles in Definitions 2.10.

Definitions 2.1. All we need to describe a line is a point on the line and a direction.

Given a point P_0 and a nontrivial vector $\vec{v}$, the **line** thru P_0 in the direction $\vec{v}$ is the set of all points of the form

$$P = P_0 + t\vec{v}$$

for some real t. The vector $\vec{v}$ is a **direction vector** for the line. See DRAWING 2.1 at the end of this chapter.

This form of description of a line (sometimes called *parametric*) actually contains more information than Definition 0.3. As in the paragraph at the end of Definitions 1.1, we intuitively think of t as time, and visualize ourselves traveling along the line, with the clock starting when we are at P_0; positive t corresponds to where we will be in the future, negative t to where we were in the past. In this visualization, P_0 is called the **initial point**.

Note that

$$P = P_0 + (t - t_0)\vec{v} \ \ (t \text{ real})$$

and

$$P = P_0 + ts_0\vec{v} \ \ (t \text{ real})$$

describe the same line, for any real t_0, nonzero s_0. It is only the implied sense of *motion* traveling along the line that changes; specifically, starting the clock at $t = t_0$ rather than $t = 0$, or changing your speed of travel by a factor of $|s_0|$; also, if $s_0 < 0$, the direction of the implied motion is reversed.

The line itself, which may be thought of as a trail left behind from your motion (think of Hansel and Gretel with bread crumbs, or a slug crawling through your kitchen), does not change.

The **ray** or **half line** with initial point P_0 is the set of all points of the form

$$P = P_0 + t\vec{v}$$

for t nonnegative. As with a line, $\vec{v}$ is a **direction vector** for the half line and

$$P = P_0 + ts_0\vec{v} \ \ (t \geq 0)$$

describes the same half line, for any positive s_0. See DRAWING 2.2 at the end of this chapter.

Given two points A, B, the **line segment** from A to B, denoted AB, is the set of all points of the form

$$P = A + t(\overrightarrow{AB}), \quad 0 \leq t \leq 1.$$

Intuitively, in this formulation, we are starting (at time $t = 0$) at A and ending (at time $t = 1$) at B. As with a line, $\overrightarrow{AB}$ is a **direction vector** for the line segment.

The points A and B are the **endpoints** of the line segment from A to B. See DRAWING 2.3 at the end of this chapter.

Note that

$$P = B + t(\overrightarrow{BA}), \quad 0 \leq t \leq 1$$

describes the same line segment, that is, the same set of points, although the implied motion is different (mainly, in the opposite direction).

In our symbols, $AB = BA$ but $\overrightarrow{AB} \neq \overrightarrow{BA}$; AB and BA represent the same line segment, but $\overrightarrow{AB}$ and $\overrightarrow{BA}$ represent different *directed* line segments (see Definitions 1.1). The only difference between the line segment AB and the directed line segment $\overrightarrow{AB}$ or $\overrightarrow{BA}$ is the removal of direction.

Two lines or rays or line segments are **parallel** if they have parallel direction vectors (see Definitions 1.5). Equivalently (HWII.3), the set of direction vectors for one line equals the set of direction vectors for the other line.

Example 2.2. If we want to describe the line through $A \equiv (1, 3)$ and $B \equiv (-2, 4)$, we first need a direction vector; the easiest choice is

$$\overrightarrow{AB} =< -3, 1 >,$$

giving us the set of all points

$$P = (1, 3) + t < -3, 1 >= (1 - 3t, 3 + t) \quad (t \text{ real}).$$

The half line with initial point $(1, 3)$ is described by

$$P = (1, 3) + t < -3, 1 >= (1 - 3t, 3 + t) \quad (t \geq 0).$$

and the line segment from $(1, 3)$ to $(-2, 4)$ is

$$P = (1, 3) + t < -3, 1 >= (1 - 3t, 3 + t) \quad (0 \leq t \leq 1).$$

See DRAWINGS 2.4 at the end of this chapter.

The line above is also described by

$$P = (-2, 4) + t < -3, 1 >= (-2 - 3t, 4 + t) \quad (t \text{ real}),$$

intuitively starting at $(-2, 4)$ instead of $(1, 3)$, or

$$P = (1, 3) + t < 3, -1 >= (1 + 3t, 3 - t) \quad (t \text{ real}),$$

intuitively travelling the same line in the opposite direction; or

$$P = (1, 3) + 2t < -3, 1 >= (1 - 6t, 3 + 2t) \quad (t \text{ real}),$$

intuitively travelling twice as fast.

Let's describe the same line in the language of Chapter 0. Use the direction vector $< -3, 1 >$ to get the slope $m = \frac{1}{-3} = -\frac{1}{3}$ (see DRAWINGS 2.4 at the end of this chapter and Definition 0.4). For arbitrary (x, y) on the line, using Definition 0.4 and the point $(1, 3)$ on the line,

$$-\frac{1}{3} = \frac{y-3}{x-1} \to (y-3) = -\frac{1}{3}(x-1) \to y = 3 - \frac{1}{3}x + \frac{1}{3} = -\frac{1}{3}x + \frac{10}{3} \quad \text{OR} \quad 3(y-3) = -(x-1) \to x + 3y = 10.$$

Using the point $(-2, 4)$ instead of $(1, 3)$:

$$-\frac{1}{3} = \frac{y-4}{x+2}$$

leads to the same equations describing the line.

Definitions 2.3. For $n = 3, 4, 5, ...$, the n-sided **polygon**, or n-gon, is defined by n (different) points $P_1, P_2, \ldots, P_n = P_0$ and the connected sequence of line segments they determine, P_1P_2 (from P_1 to P_2), P_2P_3 (from P_2 to P_3), $P_3P_4, \ldots,$ $P_{n-1}P_n$, and $P_nP_1 = P_0P_1$, with no line segments intersecting except at shared endpoints. Consecutive sides are assumed to be not parallel.

Not all sequences $P_1, P_2, \ldots$ will produce such a polygon; see HWII.4.

The points $P_1, P_2, \ldots, P_n$ are **vertices** (singular: vertex) of the polygon. The connected sequence of line segments is the **boundary** or **edge** or **sides** of the polygon and the area enclosed by those line segments is the **interior** or **inside** of the polygon; that area is also sometimes called a **polygonal region**. See DRAWING 2.5 at the end of this chapter.

For $k = 0, 1, 2, \ldots, (n-1)$, let
$$\overrightarrow{S_k} \equiv \overrightarrow{P_kP_{k+1}}.$$
An n-gon is best described by the sequence of vectors $\overrightarrow{S_k}, k = 0, 1, 2, \ldots, (n-1)$, with
$$\sum_{k=0}^{n-1} \overrightarrow{S_k} \equiv \left(\overrightarrow{S_0} + \overrightarrow{S_1} + \overrightarrow{S_2} + \cdots + \overrightarrow{S}_{n-1}\right) = \vec{0}.$$
See DRAWING 2.6 at the end of this chapter.

A polygon (to be precise, the union of its interior and boundary) is **convex** if the line segment between any two points in the interior or boundary of the polygon is contained in the interior or boundary of the polygon. See DRAWING 2.7 at the end of this chapter.

A **triangle** is a 3-gon. A **quadrilateral** is a 4-gon. A **parallelogram** is a quadrilateral whose nonconsecutive sides are parallel.

A **triangle formed by $\vec{a}$ and $\vec{b}$** will mean a triangle with vertices $I, I + \vec{a}$ and $I + \vec{b}$, for some point I. See DRAWING 2.8(a) at the end of this chapter.

A **parallelogram formed by $\vec{a}$ and $\vec{b}$** will similarly mean a parallelogram with vertices $I, I + \vec{a}, I + \vec{b}, I + \vec{a} + \vec{b}$, for some point I.

We will see later (Proposition 3.3 and Corollary 3.4) that all parallelograms have this form, because nonconsecutive sides of a parallelogram automatically have equal length. See DRAWING 2.8(b) at the end of this chapter.

We will assume, without proof, throughout this book, that any n-gon is the union of $(n-2)$ triangles whose interiors do not overlap.

The parameters, that is, measurements, of interest are length and area (as we mentioned before, measurement of angle will also be a certain length).

Definitions 2.4. Recall that our postulates, in the Introduction, require that only three assumptions (at least, when armed with calculus), one for length and one for area, along with placement in the Cartesian plane, must be made.

Here we place the length postulate in the language of vectors.

The **length** or **norm** or **magnitude** of the vector $< v_1, v_2 >$ is (see DRAWINGS 2.9 at the end of this chapter)
$$\| < v_1, v_2 > \| \equiv \sqrt{v_1^2 + v_2^2}.$$

Note that this is the distance (Definition 0.2) from the initial point of a directed line segment representing $< v_1, v_2 >$ to its terminal point; see DRAWING 0.3 at the end of Chapter 0 and DRAWING 1.1 at the end of Chapter I. For the complex number analogue, note also that $\|\vec{v}\| = |v_1 + iv_2|$ (see Definitions 1.9).

The length of the line segment from A to B is the length of the directed line segment $\overrightarrow{AB}$.

A **unit** vector is a vector of norm one. Intuitively, a vector has direction and magnitude; if we wish to focus on direction, it is natural to use unit vectors.

The **perimeter** of a polygon is the sum of the lengths of the sides.

Remarks 2.5. Calculus, which should always be viewed as a doctor-prescribed wonder drug, uses approximations followed by limiting processes to extend length and area from the Postulates of the Introduction to a large class of curves and subsets of $\mathbf{R}^2$ (see Appendices One and Two).

Of particular interest will be the lengths of arcs of circles (see Definitions 2.6 and 2.18), since angle (see Definitions 2.10) will be defined to be such an arc, and the measure of that angle will be its length.

The invariance under translation, for length and area, as mentioned at the beginning of this chapter, extends to arbitrary curves and subsets of $\mathbf{R}^2$; see Propositions APP1.3 and APP2.2.

The reader should assume that length and area are *additive* (see Appendices One and Two). For length, this means that, if C_1 and C_2 are two curves that intersect at only finitely many points and C is the union of C_1 and C_2, then

$$\text{length of } C = (\text{length of } C_1) + (\text{ length of } C_2).$$

For area, additivity means that, if Ω_1 and Ω_2 are two subsets of $\mathbf{R}^2$ that intersect at most only on a curve and Ω is the union of Ω_1 and Ω_2, then

$$\text{area of } \Omega = (\text{ area of } \Omega_1) + (\text{ area of } \Omega_2).$$

See DRAWINGS 2.9 at the end of this chapter.

Physically, a curve may be thought of as a piece of string arranged in a particular way in $\mathbf{R}^2$. The length is realized by picking up the string, pulling it straight, and aligning it with a ruler marked with inches.

The area of a subset of $\mathbf{R}^2$ may be approximated by covering it with as few as possible one inch by one inch squares.

Definitions 2.6. The **circle** of radius R, centered at (h, k) is the set of points (x, y) that satisfy

$$(x - h)^2 + (y - k)^2 = R^2.$$

Note that these are points whose distance (Definition 0.2) to (h, k) equals R.

When we think of $\mathbf{R}^2$ as the complex plane (Definitions 1.8), then said circle is the set of all complex z such that

$$|z - (h + ik)| = R.$$

See DRAWINGS 2.10 at the end of this chapter.

The **open disc** of radius R, centered at (h, k) is the inside of the circle: points (x, y) that satisfy

$$(x - h)^2 + (y - k)^2 < R^2.$$

In complex language, this is the set of all z such that

$$|z - (h + ik)| < R.$$

The **closed disc** of radius R is the same as the open disc, with "$\leq$" replacing "$<$."

Consistent with the definition of boundary of a polygon (Definitions 2.3), the circle defined above is the **boundary** of the (open or closed) disc defined above. More generally, if Ω is a subset of $\mathbf{R}^2$ enclosed by a curve C, then C is the **boundary** of Ω.

The **circumference** of a circle or disc is the length of the circle. A consequence of Lemma 2.8 will be that the circumference of a circle of radius R is $2\pi R$, where π is defined in Definition 2.7.

A **chord** in a disc is a line segment between two points on the boundary. A **diameter** is a chord that goes thru the center of this disc.

See DRAWINGS 2.10 at the end of this chapter.

Definition 2.7. The number π (written **"pi"** and pronounced "pie") is the length of the upper half of the unit circle

$$\{(x, y) \,|\, x^2 + y^2 = 1,\, y \geq 0\}.$$

See Proposition APP3.3 for a rigorous definition.

By Proposition APP2.3(b), π also equals the length of the *lower* half of the unit circle

$$\{(x, y) \,|\, x^2 + y^2 = 1,\, y \leq 0\},$$

thus 2π equals the length of the entire unit circle

$$\{(x, y) \,|\, x^2 + y^2 = 1\}.$$

We introduced the polar form of a complex number in Definition 1.15. Armed with the idea of *length*, we may now assert that every nonzero complex number has a polar form $re^{i\theta}$, with θ being a particular length.

Lemma 2.8. Every nonzero complex number z has a polar form

$$z = Re^{i\theta},$$

with $R = |z|, 0 \leq \theta \leq 2\pi$, and $R\theta$ equal to the length of $C_{\theta,R}$, where

$$C_{\theta,R} \equiv \{Re^{it} \,|\, 0 \leq t \leq \theta\}.$$

Proof: Appendix Three, APP3.5. See DRAWING 2.11 at the end of this chapter. $\square$

Examples 2.9. Let's apply Lemma 2.8 for some particular choices of θ.

By Definition 2.7, $e^{i\pi} = -1$ and $e^{2\pi i} = 1$; much more generally, by Theorem 1.12(1),

$$e^{i(\pi+\theta)} = e^{i\pi}e^{i\theta} = -e^{i\theta} \quad \text{and} \quad e^{i(2\pi+\theta)} = e^{2\pi i}e^{i\theta} = e^{i\theta}$$

for any real θ. As θ increases, we move counterclockwise around the unit circle $x^2 + y^2 = 1$, thus it is not surprising that we eventually come back to where we started. See DRAWING 2.12 at the end of this chapter.

Since $-1 = e^{i\pi} = \left(e^{i\frac{\pi}{2}}\right)^2$, we have $e^{i\frac{\pi}{2}} = i$.

Similarly, $\left(e^{i\frac{3\pi}{2}}\right)^2 = -1$, but we did *not* choose $e^{i\frac{3\pi}{2}}$ as i, because the length of $C_{\frac{3\pi}{2},1}$ is $\frac{3\pi}{2}$, which is greater than π, the length of $C_{\pi,1}$ (see Lemma 2.8 and DRAWING 2.11 at the end of this chapter); $e^{i\frac{3\pi}{2}}$ is the other square root of -1, $-i$. See DRAWING 2.13 at the end of this chapter.

Since the real numbers were enlarged by throwing in $i \equiv \sqrt{-1}$, it might appear (a famous science fiction writer asserted this) that the complex numbers could be enlarged by adding on $\sqrt{i}$. This is not true: both square roots of i are complex numbers.

We can calculate $\sqrt{i}$ directly. We want $a+bi$, with a and b real, so that

$$i = (a+bi)^2 = (a^2-b^2)+i(2ba),$$

so that

$$(a^2-b^2)=0 \quad \text{and} \quad 2ba = 1,$$

which leads to $(a,b) = \pm\left[(\frac{1}{\sqrt{2}})+i(\frac{1}{\sqrt{2}})\right]$.

Since $e^{i\frac{\pi}{2}} = i$, Theorem 1.12(1) and Lemma 2.8 imply that

$$e^{i\frac{\pi}{4}} = \left[(\frac{1}{\sqrt{2}})+i(\frac{1}{\sqrt{2}})\right] \quad \text{and} \quad e^{i\frac{5\pi}{4}} = -\left[(\frac{1}{\sqrt{2}})+i(\frac{1}{\sqrt{2}})\right].$$

Note that $e^{i\frac{\pi}{4}}$ is on the line $y=x$, bisecting the first quadrant of the xy plane. See DRAWING 2.13 at the end of this chapter.

Definitions 2.10. (a) Suppose $\vec{a}$ and $\vec{b}$ are nontrivial vectors that do not point in the same direction. Define the two angles between $\vec{a}$ and $\vec{b}$ as follows (see DRAWINGS 2.14 at the end of this chapter).

1. Represent $\vec{a}$ and $\vec{b}$ by directed line segments in standard position.
2. Use Lemma 2.8 and Definition 2.7 to choose real θ_1, θ_2 such that $0 \le \theta_1 < \theta_2 < \theta_1 + 2\pi$,

$$\frac{\vec{a}}{\|\vec{a}\|} = e^{i\theta_1} \quad \text{and} \quad \frac{\vec{b}}{\|\vec{b}\|} = e^{i\theta_2}.$$

Here we are, for any real θ, thinking of $e^{i\theta}$ as being a vector, as in 1.16.

The **counterclockwise angle** from $\vec{a}$ to $\vec{b}$ is

$$\{e^{i\theta} \,|\, \theta_1 \le \theta \le \theta_2\}.$$

The **measure** of this angle is $(\theta_2 - \theta_1)$ radians.

The **clockwise angle** from $\vec{a}$ to $\vec{b}$ is the counterclockwise angle from $\vec{b}$ to $\vec{a}$:

$$\{e^{i\theta} \,|\, \theta_2 \le \theta \le \theta_1 + 2\pi\}.$$

The **measure** of this angle is $(\theta_1 + 2\pi - \theta_2) = 2\pi - (\theta_2 - \theta_1)$ radians.

See DRAWINGS 2.14 at the end of this chapter.

(b) If $\vec{a}$ and $\vec{b}$ are not parallel, then the **angle between** $\vec{a}$ and $\vec{b}$ is the angle of smallest measure from $\vec{a}$ to $\vec{b}$.

If $\vec{a}$ and $\vec{b}$ point in the same direction, then use Lemma 2.8 and Definition 2.7 to choose θ_0, with $0 \le \theta_0 < 2\pi$, so that

$$\frac{\vec{a}}{\|\vec{a}\|} = e^{i\theta_0} = \frac{\vec{b}}{\|\vec{b}\|}.$$

Then the **angle between** $\vec{a}$ and $\vec{b}$ is $\{e^{i\theta_0}\}$.

The **measure** of this angle is 0 radians.

See Corollary 2.14(b), to see why we can't have "angle between $\vec{a}$ and $\vec{b}$" when $\vec{a}$ and $\vec{b}$ point in opposite directions.

Notice that measuring angle is reduced to measuring length: by Lemma 2.8, the measure of an angle between two vectors is the length of a piece of the circle of radius one centered at the origin $x^2+y^2=1$ (see DRAWINGS 2.14 at the end of this chapter). The already-stated invariance of length under translation (Proposition APP2.3(a)) implies that we could translate the shared initial point of $\vec{a}$ and $\vec{b}$ to some point C, then translate an angle from $\vec{a}$ to $\vec{b}$ by the same point C without affecting the measure of the angle.

See Definitions APP2.1 for a rigorous (calculus) definition of length of a curve; for circular length, hence angle measure, see Lemma 2.8 and APP3.5.

The angles between two intersecting lines, rays, or line segments are defined to be the angles between direction vectors for the lines, rays, or line segments.

Throughout this book, if no units are mentioned, it is assumed that angles are measured in radians.

Since angle measure is determined by length, it is additive: that is, if the counterclockwise angle from $\vec{a}$ to $\vec{b}$ has measure θ_1 and the counterclockwise angle from $\vec{b}$ to $\vec{c}$ has measure θ_2, then the counterclockwise angle from $\vec{a}$ to $\vec{c}$ has measure $(\theta_1 + \theta_2)$.

Examples 2.11. By Examples 2.9, $\frac{<1,1>}{\|<1,1>\|} = e^{i\frac{\pi}{4}}$ and $< 0,1 >= e^{i\frac{\pi}{2}}$ (see DRAWINGS 2.15(1) at the end of this chapter), thus the counterclockwise angle from $< 1,1 >$ to $< 0,1 >$ is (see DRAWINGS 2.15(2) at the end of this chapter),

$$\{e^{i\theta} \mid \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}\},$$

with measure $(\frac{\pi}{2} - \frac{\pi}{4}) = \frac{\pi}{4}$.

The clockwise angle from $< 1,1 >$ to $< 0,1 >$ is (note that $2\pi + \frac{\pi}{4} = \frac{9\pi}{4}$)

$$\{e^{i\theta} \mid \frac{\pi}{2} \leq \theta \leq \frac{9\pi}{4}\},$$

with measure $(\frac{9\pi}{4} - \frac{\pi}{2}) = (2\pi - \frac{\pi}{4}) = \frac{7\pi}{4}$ (see DRAWINGS 2.15(3) at the end of this chapter).

Since $\frac{\pi}{4} < \frac{7\pi}{4}$, the angle between $< 0,1 >$ and $< 1,1 >$ is

$$\{e^{i\theta} \mid \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}\},$$

with measure $\frac{\pi}{4}$.

Again using Examples 2.9, since $< -1,0 >= e^{i\pi}$, the counterclockwise angle from $< 0,1 >$ to $< -1,0 >$ is

$$\{e^{i\theta} \mid \frac{\pi}{2} \leq \theta \leq \pi\},$$

with measure $(\pi - \frac{\pi}{2}) = \frac{\pi}{2}$ (see DRAWINGS 2.15(4) at the end of this chapter).

By additivity of angles, the counterclockwise angle from $< 1,1 >$ to $< -1,0 >$ is

$$\{e^{i\theta} \mid \frac{\pi}{4} \leq \theta \leq \pi\},$$

with measure (from previous angle calculations in this example) $(\pi - \frac{\pi}{4}) = \frac{3\pi}{4}$ (see DRAWINGS 2.15(5) at the end of this chapter).

Notice that we are adding two counterclockwise measures: the measure of $\frac{\pi}{4}$ from $< 1,1 >$ to $< 0,1 >$ added to the measure of $\frac{\pi}{2}$ from $< 0,1 >$ to $< -1,0 >$.

Intuitively, angle is like opening a door or a crocodile's mouth. With the door, the angle is between the door and the wall it's hinged to. With the crocodilian mouth, the angle is between the upper and lower jaws; the measure of the angle begins at zero, as the crocodile feigns unconsciousness, then increases as the mouth opens more.

It should be noted here that, since the definition of angle (Definitions 2.10) between vectors $\vec{a}$ and $\vec{b}$ involve the *unit* vectors (Definitions 2.4) $\frac{\vec{a}}{\|\vec{a}\|}$ and $\frac{\vec{b}}{\|\vec{b}\|}$, angle is unchanged by replacing a vector with another vector that points in the same direction.

Proposition 2.12. Suppose $\vec{v}_1$ points in the same direction as $\vec{v}_2$ and $\vec{v}_3$ points in the same direction as $\vec{v}_4$. Then, the counterclockwise angle from $\vec{v}_1$ to $\vec{v}_3$ equals the counterclockwise angle from $\vec{v}_2$ to $\vec{v}_4$.

The following theorem emphasizes further the role of complex exponentials in Definitions 2.10 and DRAWING 2.14 at the end of this chapter: when vectors are placed in standard position, multiplication by $e^{i\theta}$ rotates a vector θ radians counterclockwise. See DRAWING 2.16 at the end of this chapter.

In the following, as with Definitions 2.10, see 1.16 for the equating of point, vector and complex number.

Theorem 2.13. For any nontrivial vector $\vec{v}$ in standard position and $0 \leq \theta < 2\pi$, the measure of the counterclockwise angle from $\vec{v}$ to $(e^{i\theta}\vec{v})$ is θ.

Proof: By Lemma 2.8, there exists ϕ, with $0 \leq \phi \leq 2\pi$, so that $\frac{\vec{v}}{\|\vec{v}\|} = e^{i\phi}$. Then

$$\frac{e^{i\theta}\vec{v}}{\|e^{i\theta}\vec{v}\|} = e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)},$$

so that, by Definitions 2.10, the measure of the counterclockwise angle from $\vec{v}$ to $(e^{i\theta}\vec{v})$ is θ.

See DRAWING 2.16 at the end of this chapter. □

Corollary 2.14. All vectors in the following are nontrivial.

(a) A vector $\vec{w}$ makes an angle of measure $\frac{\pi}{2}$ with another vector $\vec{v}$ if and only if $\vec{w}$ makes an angle with $\vec{v}$ of the same measure as with $(-\vec{v})$. (This is what Euclid called a **right angle**.)

(b) Suppose $\vec{a}$ and $\vec{b}$ point in opposite directions. Then π equals the measure of both the clockwise and counterclockwise angles from $\vec{a}$ to $\vec{b}$.

(c) For any real numbers x_1, y_1, the measure of the counterclockwise angle from $< x_1, y_1 >$ to $< -y_1, x_1 >$ is $\frac{\pi}{2}$.

See DRAWINGS 2.17 at the end of this chapter.

Proof: By Proposition 2.12, we may assume $\vec{b} = \vec{a}$ in (a) and $\vec{b} = -\vec{a}$ in (b).

(b) is Theorem 2.13, with $\vec{v} \equiv \vec{a}$ and $\theta \equiv \pi$, since $(-1) = e^{i\pi}$.

(c) Note that $i(x_1 + iy_1) = (-y_1 + ix_1)$; in the equating of vectors and complex numbers of 1.16, this is

$$e^{i\frac{\pi}{2}} < x_1, y_1 >= i < x_1, y_1 >=< -y_1, x_1 >,$$

so that (c) follows from Theorem 2.13, with $\vec{v} \equiv < x_1, y_1 >, \theta \equiv \frac{\pi}{2}$.

(a) follows from (b) and the additivity of angle measure. □

Remarks 2.15. We shall see (beginning with Chapter IV) that angles of measurement $\frac{\pi}{2}$ play a distinguished role in the physical and mathematical world.

The relationship between $< x_1, y_1 >$ and $< -y_1, x_1 >$ in (c) of Corollary 2.14 will be clarified in Chapter IV; they are *orthogonal* (Definition 4.3), meaning they have *dot product* (Definition 4.1) zero.

Definitions 2.16. In a convex polygon (see Definitions 2.3), the **interior angle** at a vertex is the angle within the area enclosed by the sides. An **exterior angle** at a vertex is the angle between one side meeting at the vertex and the extension of the other side meeting at the vertex. See DRAWINGS 2.18 at the end of this chapter, where, in (b), we have also calculated the measures of the interior angles of the triangle formed by $< -1, 0 >$ and $< 0, 1 >$, similarly to Examples 2.11.

If θ is the measure of the interior angle, then $(\pi - \theta)$ is the measure of each of the two exterior angles, by Corollary 2.14(b) and additivity of angle measures. Notice (see DRAWINGS 2.18(a) at the end of this chapter) that the measure of an exterior angle at a vertex represents the *change of direction* of a person traveling around the boundary of the polygon.

Definition 2.17. A much older parameter for measuring angles is **degrees.** The goal is to have a complete circumference be 360 degrees. In general, for any real θ, θ radians is $\frac{180\theta}{\pi}$ degrees. $\frac{\pi}{2}$ radians, for example, is 90 degrees, π radians is 180 degrees, etc.

The only reason for preferring degrees to radians is custom, which is the worst possible reason for anything. Degrees were introduced by the Babylonians, who also had base 60 numeration, and introduced the custom of 60 minutes to an hour and 60 seconds to a minute.

Definitions 2.18. Our definition of angle (Definitions 2.10) leads naturally to the following completions of Definitions 2.6.

Let's put a circle and disc of radius R centered at (the complex number) C in complex form: the circle is

$$\{z \,|\, |z - C| = R\} \quad \text{or} \quad \{C + Re^{i\theta} \,|\, 0 \leq \theta \leq 2\pi\};$$

the closed disc is

$$\{z \,|\, |z - C| \leq R\} \quad \text{or} \quad \{C + re^{i\theta} \,|\, 0 \leq \theta \leq 2\pi, 0 \leq r \leq R\}.$$

An **arc** of this circle is, for $0 \leq \theta_1 \leq \theta_2 < \theta_1 + 2\pi$ as in the definition of counterclockwise angle,

$$\{C + Re^{i\theta} \,|\, \theta_1 \leq \theta \leq \theta_2\};$$

the corresponding **closed sector** of the disc, determined by the arc, is

$$\{C + re^{i\theta} \,|\, \theta_1 \leq \theta \leq \theta_2, 0 \leq r \leq R\}.$$

See DRAWINGS 2.19 at the end of this chapter. The **boundary** of the closed sector is the arc combined with the two lines from the center to the circle, that enclose the sector. The **perimeter** of the sector is the length of the boundary. The **arclength** is the length of the arc; we apologize for terminology that neither shocks nor surprises.

Proposition 2.19. Consider an arc and sector as in Definitions 2.18.

(a) The measure of the angle between the two lines in the boundary of the sector is $(\theta_2 - \theta_1)$.

(b) The length of the arc is $R(\theta_2 - \theta_1)$.

(c) The perimeter of the sector is $2R + R(\theta_2 - \theta_1)$.

Proof: (a) is clear from Definitions 2.10, after we translate C to the origin (see comments after Definitions 2.10). (b) follows from Lemma 2.8, then (c) follows from (b). $\square$

We will address the *area* of the sector in Theorem 12.4; it turns out to be $\frac{1}{2}R^2(\theta_2 - \theta_1)$.

See DRAWINGS 2.19 at the end of this chapter.

Note that radian measures of angles are arclength divided by radius. Radians are *unitless*, e.g., meters divided by meters.

Examples 2.20. All curves that are not line segments are arcs of a circle.

(a) Find the length of the arc in DRAWING 2.20 at the end of this chapter.

(b) Find the perimeter of the shaded sector and the unshaded sector in DRAWING 2.21 at the end of this chapter.

(c) Find the length of the curve in DRAWING 2.22 at the end of this chapter.

Solutions. (a) $150(\frac{\pi}{180}) = \frac{5\pi}{6}$ radians, so the length of the arc is $5(\frac{5\pi}{6}) = \frac{25\pi}{6}$ meters.

(b) $240(\frac{\pi}{180}) = \frac{4\pi}{3}$ radians, so the perimeter of the shaded sector is $20 + 10(\frac{4\pi}{3}) = 20 + \frac{40\pi}{3}$.

The unshaded sector has angle $2\pi - \frac{4\pi}{3} = \frac{2\pi}{3}$, so the perimeter of the unshaded sector is $20 + 10(\frac{2\pi}{3}) = 20 + \frac{20\pi}{3}$.

(c) $16 + (\frac{3\pi}{8})16 = (16 + 6\pi)$ feet.

Terminology 2.21. We will indicate equality of length or angle measure as in DRAWINGS 2.23 at the end of this chapter.

HOMEWORK

HWII.1. Suppose m is a real number. Show that a line has slope m (see Definitions 0.4) if and only if it has direction vector $< s, sm >$, for all real $s \neq 0$.

HWII.2. Show that a line has slope ∞ (see Definitions 0.4) if and only if it has direction vector $< 0, t >$, for all real $t \neq 0$.

HWII.3. Prove that two lines or rays or line segments are parallel if and only if the set of direction vectors for one line equals the set of direction vectors for the other line.

HWII.4. Give four points P_1, P_2, P_3, P_4 such that the line segments in Definitions 2.3, P_1P_2 (from P_1 to P_2), P_2P_3 (from P_2 to P_3), P_3P_4, and P_4P_1, do not produce a polygon as described in Definitions 2.3.

HWII.5. Write a parametric form $P = P_0 + t\vec{v}$ for the line $y = -2x + 5$.

HWII.6. For arbitrary real numbers a, b, c, get a direction vector for $ax + by = c$ in terms of a, b, c.

HWII.7. Find the perimeter of the triangle formed by $< 1, -2 >$ and $< 3, 4 >$.

HWII.8. For what real number α is $< 1, \alpha >$ parallel to $< 2, -3 >$?

HWII.9. Let $P \equiv (-2, 1)$ and $Q \equiv (3, 5)$. Find

(a) The line segment from P to Q;

(b) The half line through Q with initial point P; and

(c) The line through P and Q.

HWII.10. Write the line $2x + 5y = 7$ in a parametric form.

HWII.11. Write each of the following in the form $(a + bi)$, for real numbers a, b.

(a) $e^{4\pi i}$;

(b) $e^{5\pi i}$;

(c) $e^{\frac{13\pi}{4}i}$ HINT: $\frac{13}{4} = 3 + \frac{1}{4}$;

(d) $e^{\frac{17\pi}{4}i}$;

(e) i^3;

(f) i^{18};

(g) $e^{\frac{\pi}{3}i}$ HINT: for any complex z and w, $(z + w)^3 = z^3 + 3z^2w + 3zw^2 + w^3$. See Examples 6.5 and 7.11(e) for other techniques, after we've acquired more tools.

HWII.12. Find the clockwise and counterclockwise angle from $< -1, 1 >$ to $< 1, 1 >$, and find the measure of each angle.

HWII.13. All curves drawn are either line segments or arcs of a circle centered on a dot inside the corresponding disc.

(a) Find the length of the arc in DRAWING 2.24 at the end of this chapter.

(b) Find the perimeter of the shaded sector and the unshaded sector in DRAWING 2.25 at the end of this chapter.

(c) Find the length of the curve in DRAWING 2.26 at the end of this chapter.

HWII.14. Find a unit vector that points in the same direction as $< 2, -3 >$.

HWII.15. Write the line $(-3, 2) + t < 2, 5 >$ in the form $ax + by = c$, for some real numbers a, b, c.

HOMEWORK ANSWERS

HWII.1. If a line has direction vector $< s, sm >$, for some nonzero s, then it has, for some real a, b, with (a, b) on the line, the form (see Definitions 2.1)

$$P = (a, b) + t < s, sm > \quad (t \text{ real}),$$

thus (a, b) and $(a + s, b + sm) = (a, b)+ < s, sm >$ are two points on the line, so that the slope is

$$\frac{(b + sm) - b}{(a + s) - a} = m.$$

Conversely, if a line has slope m, then $y = mx + b$ is the equation of the line, for some real b; thus, for any $s \neq 0$,

$$\{\text{points on the line}\} = \{(x, mx+b) \,|\, x \text{ is real}\} = \{(0, b)+x < 1, m > \,|\, x \text{ is real}\} = \{(0, b)+\frac{x}{s} < s, sm > \,|\, x \text{ is real}\}$$
$$= \{(0, b) + t < s, sm > \,|\, t \text{ is real}\},$$

thus the line has direction vector $< s, sm >$, for any $s \neq 0$.

HWII.2. A line has slope $\infty \iff$ the line has the form $x = c \iff$

$$\text{the line equals } \{(c, y) \,|\, y \text{ is real}\} \iff$$
$$\text{the line equals } \{(c, 0) + y < 0, 1 > \,|\, y \text{ is real}\} \iff$$
$$\text{the line equals, for any } t \neq 0, \{(c, 0) + \frac{y}{t} < 0, t > \,|\, y \text{ is real}\} \iff$$
$$\text{the line equals, for any } t \neq 0, \{(c, 0) + s < 0, t > \,|\, s \text{ is real}\} \iff$$

the line has direction vector $< 0, t >$, for any $t \neq 0$.

HWII.3. For a fixed line, we need to characterize the set of all possible direction vectors for the line.

First suppose the line has finite slope m. We have already seen, in HWII.1, that anything of the form $< s, sm >$, for s nonzero, is a direction vector for the line. We will now show that all direction vectors for said line have this form: suppose $\vec{v} =< v_1, v_2 >$ is a direction vector for a line of finite slope m. Since m is finite, $v_1 \neq 0$. The parametric form for this line is then, for some real x_0, y_0,

$$(x, y) = P = P_0 + t\vec{v} = (x_0, y_0) + t < v_1, v_2 >= (x_0 + tv_1, y_0 + tv_2),$$

so that

$$m = \frac{(y_0 + tv_2) - y_0}{(x_0 + tv_1) - x_0} = \frac{v_2}{v_1},$$

thus $v_2 = mv_1$, so that $\vec{v} =< v_1, mv_1 >$, exactly the form of the direction vectors in HWII.1.

A similar argument, as in the proof of HWII.2, shows that the set of all direction vectors for a line of infinite slope equals $\{< 0, s > \,|\, s \neq 0\}$.

We will find it convenient to state what we've accomplished so far as follows.

HWII.3 Lemma. (a) If ℓ is a nonvertical line, then the set of all direction vectors for ℓ is

$$S_m \equiv \{\vec{v} =< v_1, v_2 > \,|\, \frac{v_2}{v_1} = m\},$$

where m is the slope of ℓ.

(b) If ℓ is a vertical line (of the form $x = c$, for some fixed real c), then the set of all direction vectors for ℓ is

$$S_\infty \equiv \{< 0, s > \,|\, s \neq 0\}.$$

Any two vertical lines $x = c_1$ and $x = c_2$ are parallel because they have parallel direction vectors of the form $< 0, s_1 >$ and $< 0, s_2 >$, for some nonzero s_1, s_2. It is also the case that any two vertical lines have the same set of direction vectors S_∞.

If one line is vertical and the other line is nonvertical, then it is clear from HWII.3 Lemma that they are not parallel, since nothing in S_∞ is parallel to anything in S_m, for any real m. It is also the case that the set of direction vectors of a vertical line does not equal the set of direction vectors for a nonvertical line, again by HWII.3 Lemma.

We have shown the equivalence of being parallel and having the same set of direction vectors, for a pair of lines, at least one of which is vertical. Thus we may assume, for the rest of HWII.3, that we have two lines ℓ_1 and ℓ_2 neither of which is vertical.

Suppose ℓ_1 and ℓ_2 are parallel. Then there are parallel vectors $\vec{v} =< v_1, v_2 >$ and $\vec{w} =< w_1, w_2 >$ such that $\vec{v}$ is a direction vector for ℓ_1 and $\vec{w}$ is a direction vector for ℓ_2. Being parallel and nontrivial implies that there is nonzero r so that

$$< v_1, v_2 >= \vec{v} = r\vec{w} =< rw_1, rw_2 >,$$

so that $\frac{v_2}{v_1} = \frac{rw_2}{rw_1} = \frac{w_2}{w_1}$. Thus, by HWII.3 Lemma(a),

$$\{\text{direction vectors for } \ell_1\} = S_{\frac{v_2}{\vec{v}_1}} = S_{\frac{w_2}{\vec{w}_1}} = \{\text{direction vectors for } \ell_2\}.$$

Conversely, if

$$\{\text{direction vectors for } \ell_1\} = \{\text{direction vectors for } \ell_2\},$$

then there is a vector $\vec{v}$ that is a direction vector for both ℓ_1 and ℓ_2, thus ℓ_1 and ℓ_2 are parallel.

HWII.4. See DRAWING 2.27 at the end of this chapter, where we have drawn a parallelogram with edges $\overrightarrow{P_1P_3}, \overrightarrow{P_3P_2}\overrightarrow{P_2P_4}$ and $\overrightarrow{P_4P_1}$. Here the sequence of points P_1, P_2, P_3, P_4 do not produce a quadrilateral, as in Examples 2.3.

HWII.5. $P = (0, 5) + t < 1, -2 >$ (t real)

HWII.6. For $b \neq 0$, a direction vector is $< b, -a >$ (or $< sb, -sa >$, for any nonzero real s).
For $b = 0$, a direction vector is $< 0, 1 >$ (or $< 0, s >$, for any nonzero real s).

HWII.7. $[\| < 1, -2 > \| + \| < 2, 6 > \| + \| < 3, 4 > \|] = 2(\sqrt{5} + \sqrt{10})$.

HWII.8. Setting $< 1, \alpha >= c < 2, -3 >$, for some real c, implies that $c = \frac{1}{2}$, so that $\alpha = -\frac{3}{2}$.

HWII.9. For all parts, we need $\overrightarrow{PQ} =< 5, 4 >$.

(a) All points of the form $P = (-2, 1) + t < 5, 4 >$ $(0 \leq t \leq 1)$.

(b) All points of the form $P = (-2, 1) + t < 5, 4 >$ $(t \geq 0)$.

(c) All points of the form $P = (-2, 1) + t < 5, 4 >$ (t real).

HWII.10. The slope $m = -\frac{2}{5}$ and the y intercept $b = \frac{7}{5}$, so we may write

$$P = (0, \frac{7}{5}) + t < 1, -\frac{2}{5} > \quad (t \text{ real}),$$

or, to avoid fractions in the direction vector,

$$P = (0, \frac{7}{5}) + t < 5, -2 > \quad (t \text{ real}).$$

HWII.11. (a) 1 (b) -1 (c) $-\frac{1}{\sqrt{2}}(1 + i)$ (d) $\frac{1}{\sqrt{2}}(1 + i)$ (e) $-i$ (f) -1

(g) Setting $(-1) = e^{i\pi} = (e^{i\frac{\pi}{3}})^3 = (a + bi)^3$ leads to $a = \frac{1}{2}, b = \frac{\sqrt{3}}{2}$; that is, our desired exponential is $\frac{1}{2}(1 + \sqrt{3})$.

HWII.12. The clockwise angle is $\{e^{i\theta} \mid \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}\}$. The counterclockwise angle is $\{e^{i\theta} \mid \frac{3\pi}{4} \leq \theta \leq 2\pi + \frac{\pi}{4}\}$.

The measure of the clockwise angle is $(\frac{3\pi}{4} - \frac{\pi}{4}) = \frac{\pi}{2}$.

The measure of the counterclockwise angle is $((2\pi + \frac{\pi}{4}) - \frac{3\pi}{4}) = (2\pi - \frac{\pi}{2}) = \frac{3\pi}{2}$.

HWII.13. (a)

$$(10 \text{ feet}) \left(120 \text{ degrees}(\frac{\pi \text{ radians}}{180 \text{ degrees}}) \right) = \frac{20\pi}{3} \text{ feet}.$$

(b)

$$\text{shaded sector perimeter is } (16 \text{ feet}) + (8 \text{ feet})(\frac{11\pi}{12}) = (16 + \frac{22\pi}{3}) \text{ feet}.$$

$$\text{unshaded sector perimeter is } (16 \text{ feet}) + (8 \text{ feet})(\frac{13\pi}{12}) = (16 + \frac{26\pi}{3}) \text{ feet}.$$

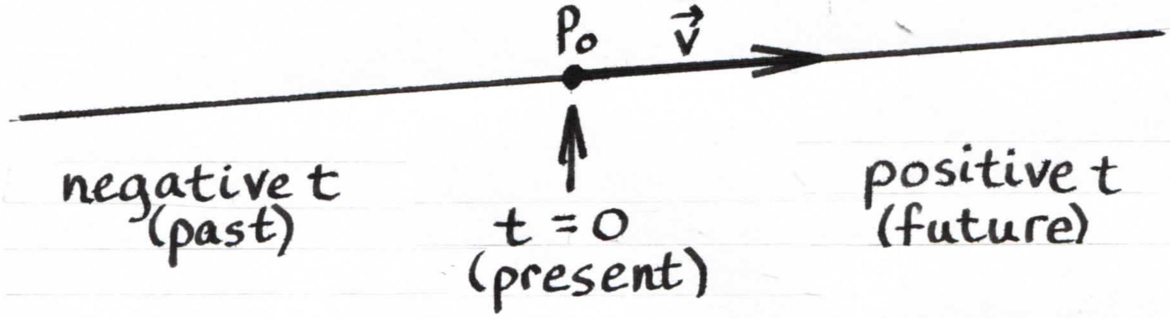
(c)

$$(24 \text{ meters}) + (12 \text{ meters})(\frac{9\pi}{8}) = (24 + \frac{27\pi}{2}) \text{ meters}.$$

HWII.14. $\frac{1}{\|<2,-3>\|} < 2, -3 >= \frac{1}{\sqrt{13}} < 2, -3 >$.

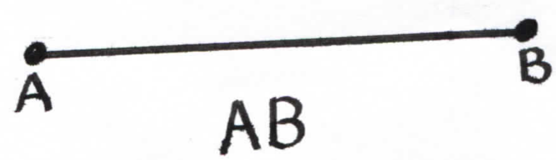
HWII.15. Slope $\frac{5}{2}$, with $(-3, 2)$ on line; $\frac{y-2}{x+3} = \frac{5}{2} \to -5x + 2y = 19$.

DRAWING 2.1

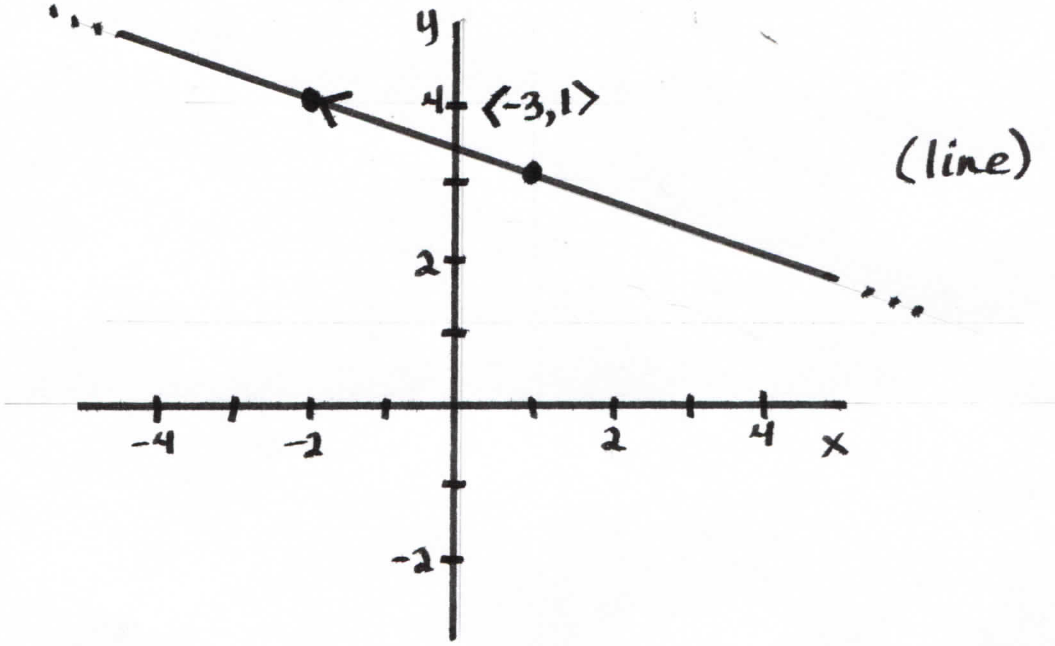


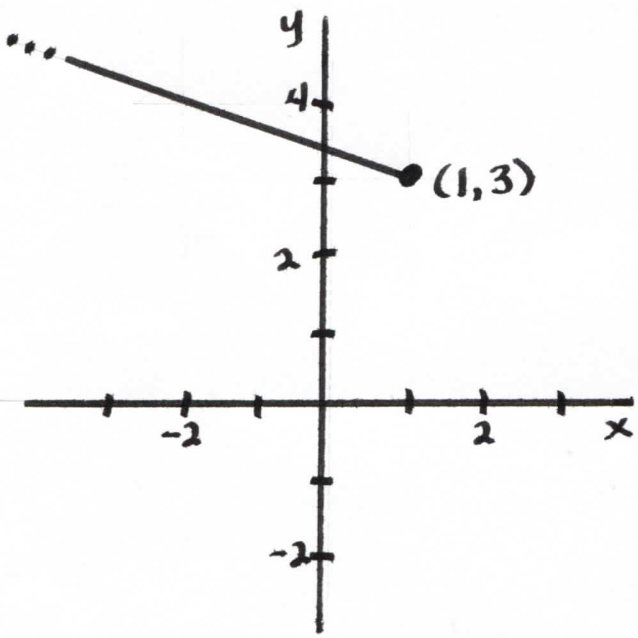
DRAWING 2.2

DRAWING 2.3

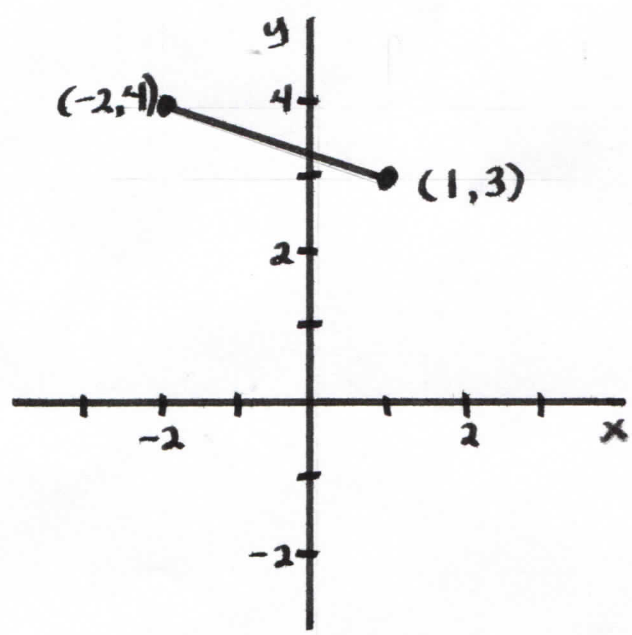


DRAWINGS 2.4



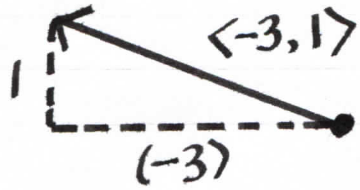


(half Line)

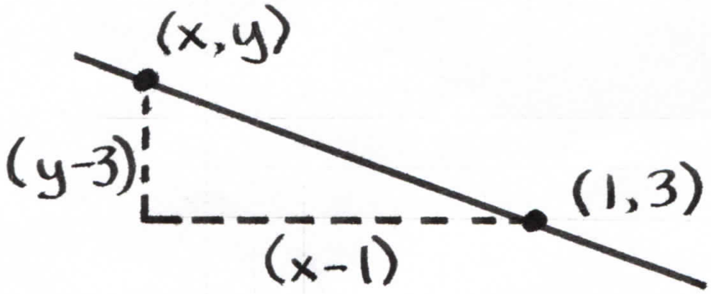


(line segment)

DRAWINGS 2.4 (continued)



$$\text{slope } m = \frac{1}{-3} = -\frac{1}{3}$$



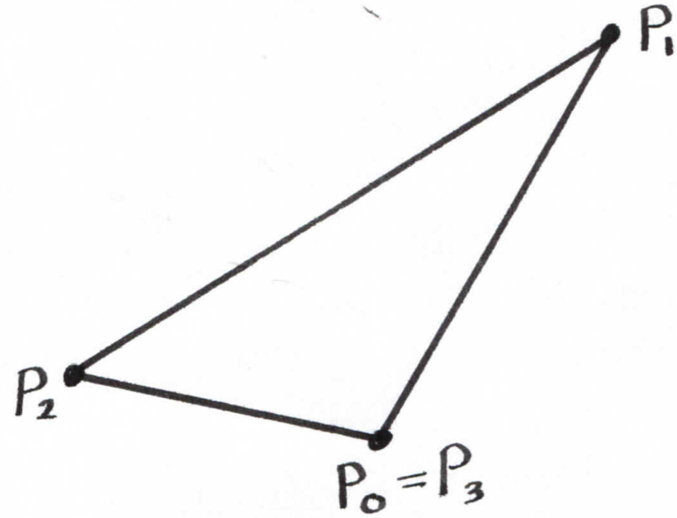
$$\frac{(y-3)}{(x-1)} = -\frac{1}{3} \longrightarrow y = -\frac{1}{3}x + \frac{10}{3}$$

or

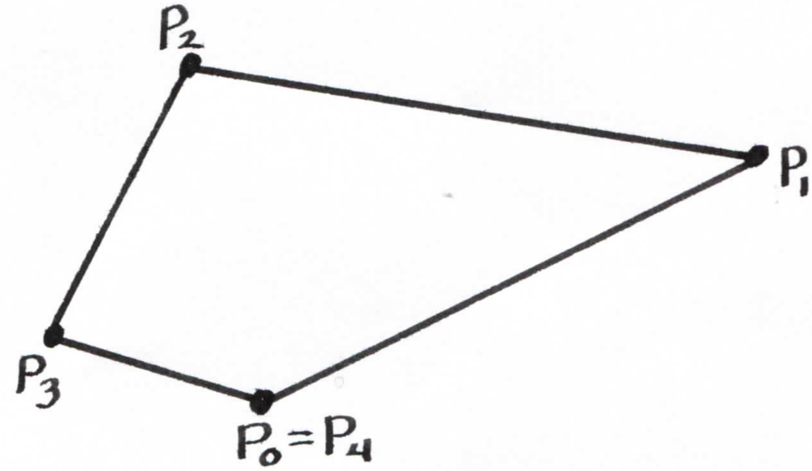
$$x + 3y = 10$$

DRAWING 2.5

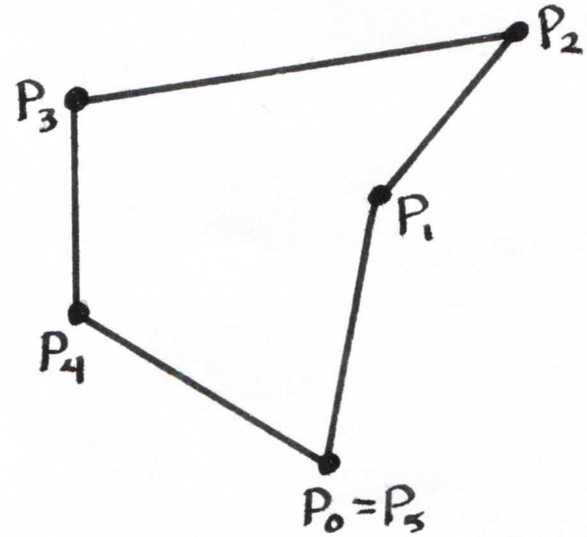
$n=3$



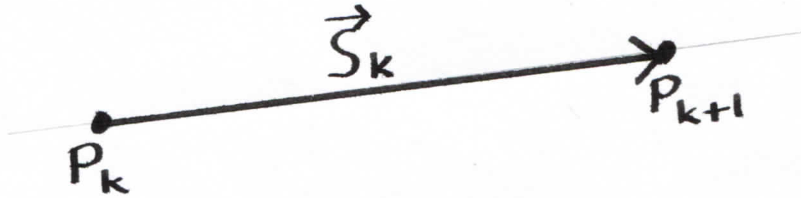
$n=4$



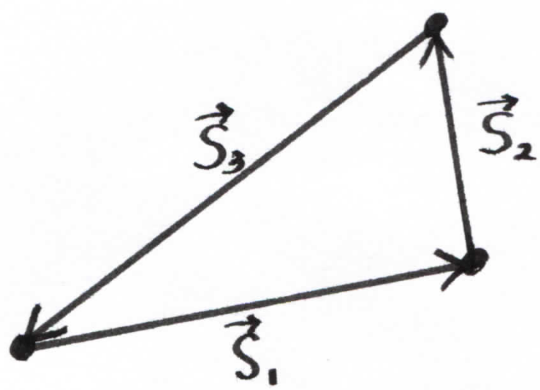
$n=5$



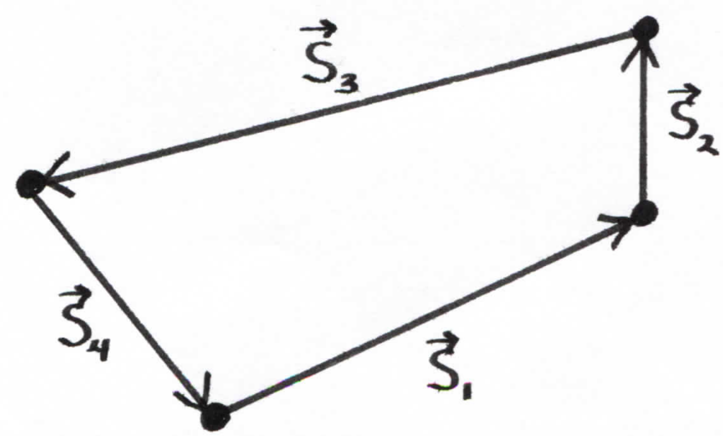
DRAWING 2.6



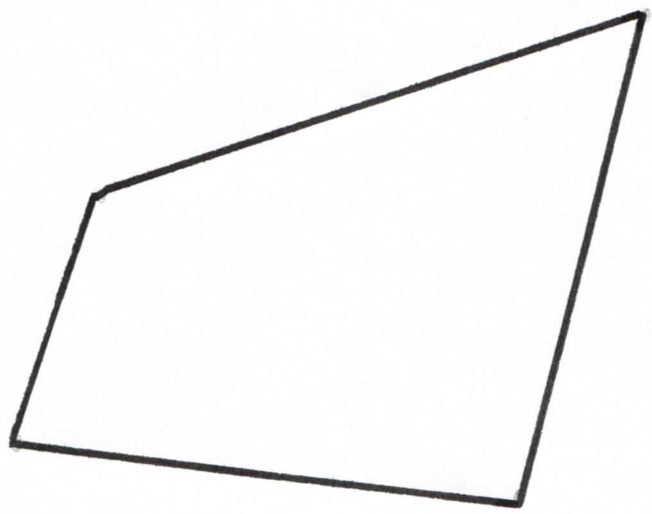
$n=3$



$n=4$



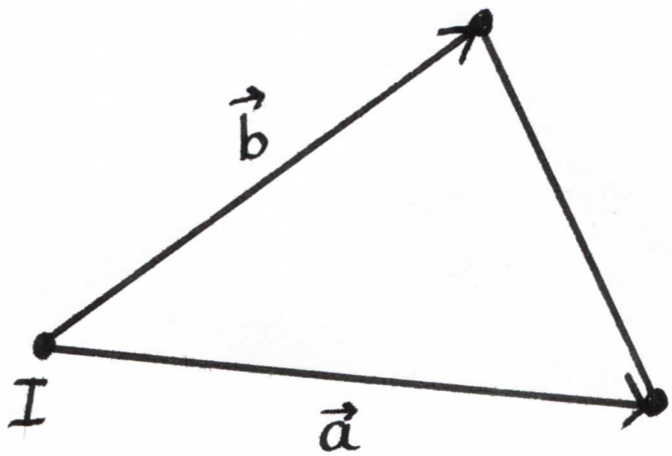
DRAWING 2.7



convex

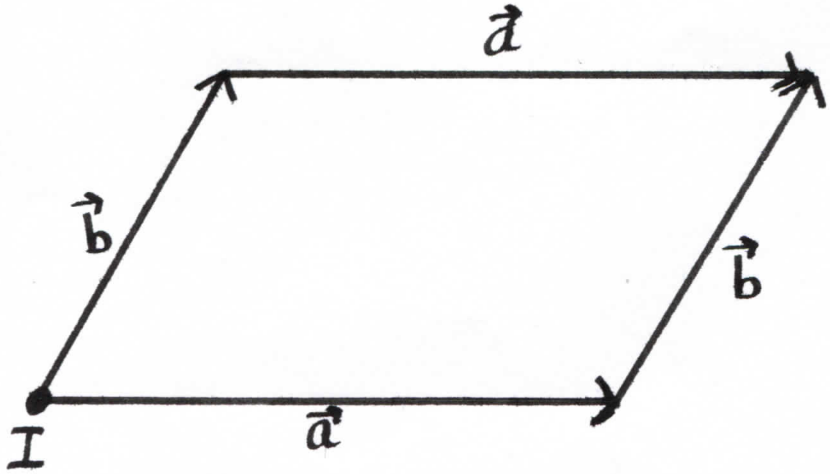
not convex

DRAWING 2.8 (a)



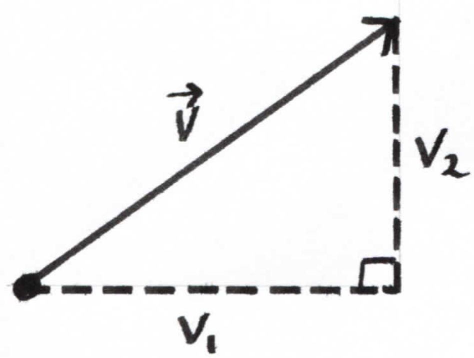
triangle

DRAWING 2.8 (b)

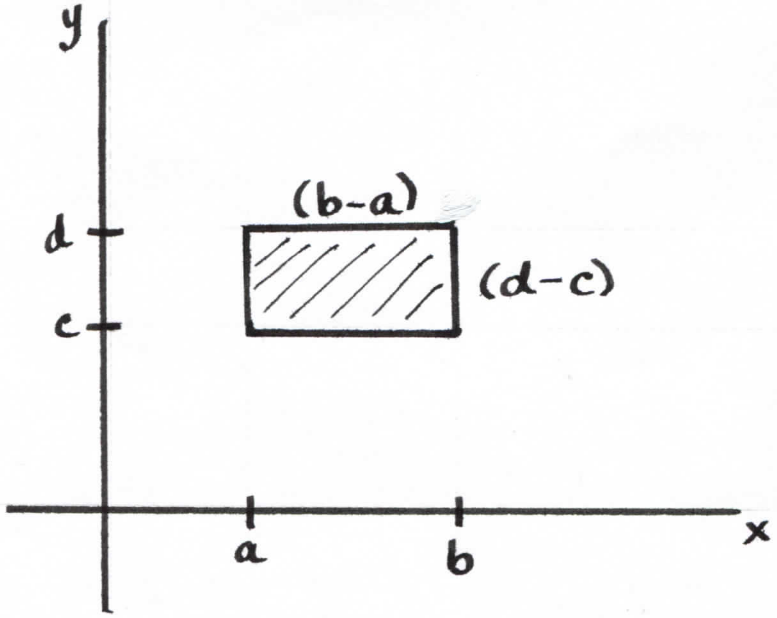


parallelogram

DRAWINGS 2.9

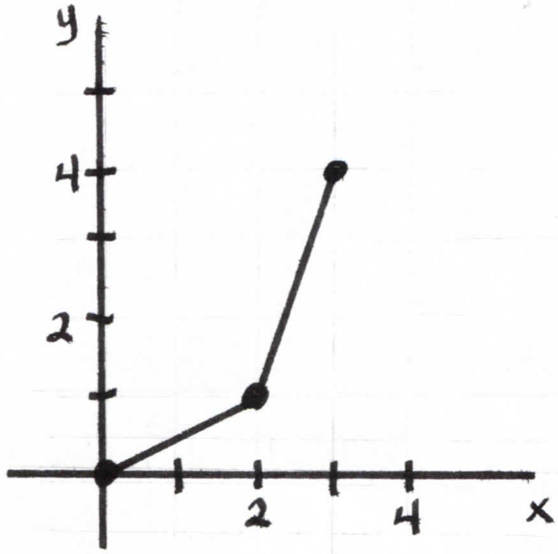


$$\|\vec{v}\| \equiv \sqrt{v_1^2 + v_2^2}$$



$$\text{area} \equiv (b-a)(d-c)$$

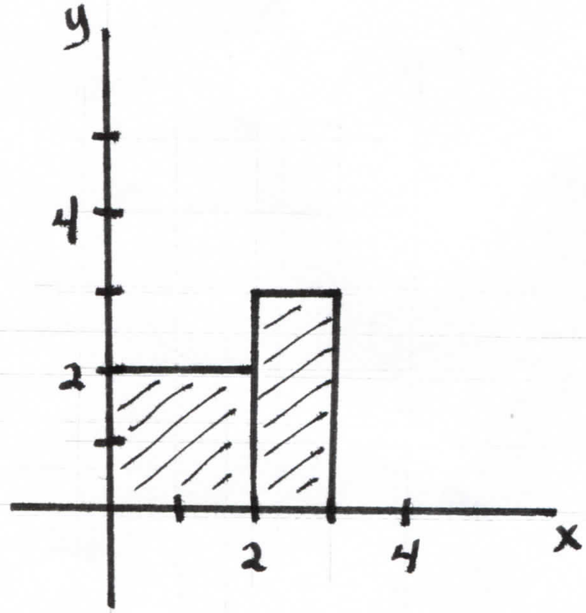
DRAWINGS 2.9 (continued)

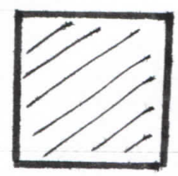
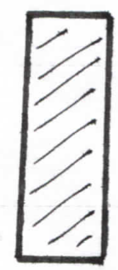


length =  +  =

$$\sqrt{2^2+1^2} + \sqrt{1^2+3^2} =$$

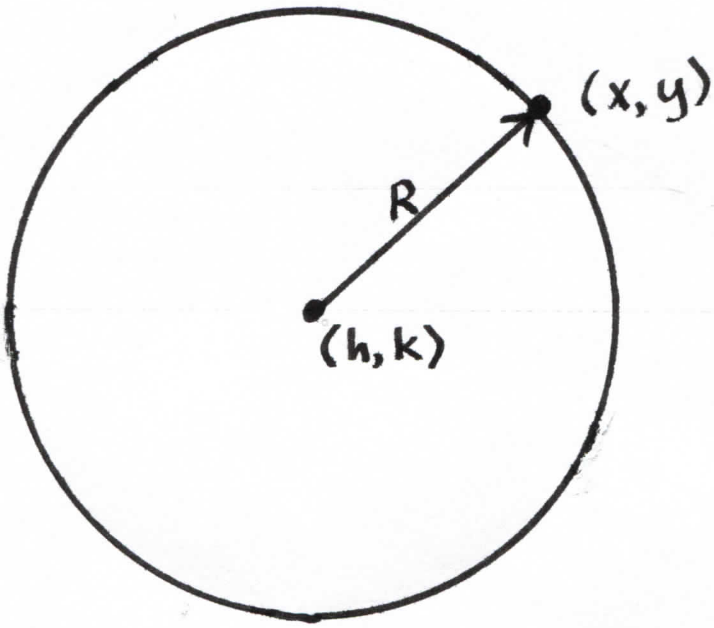
$$\sqrt{5} + \sqrt{10}$$

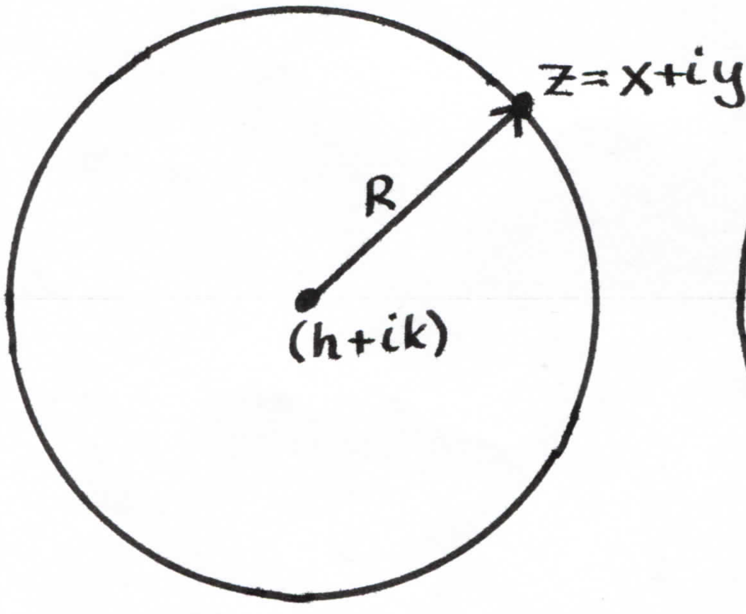


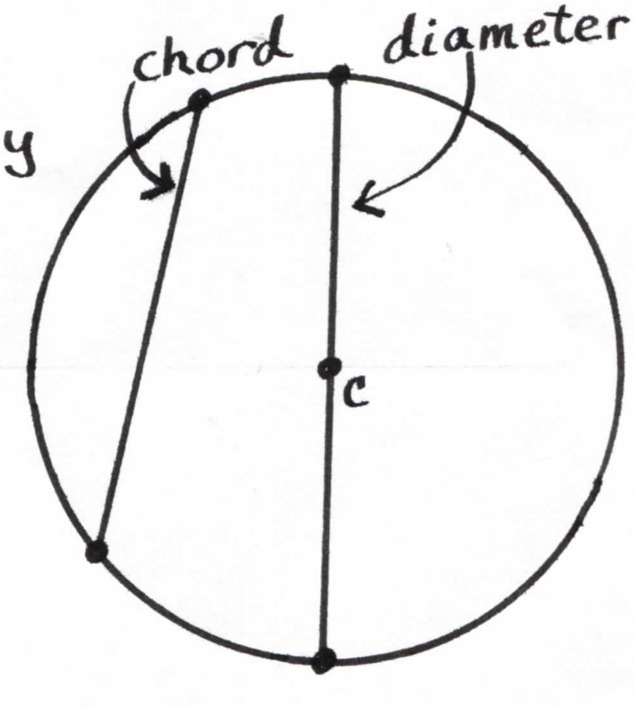
area =  + 

$$= (2\cdot 2) + (1\cdot 3) = 4+3 = 7$$

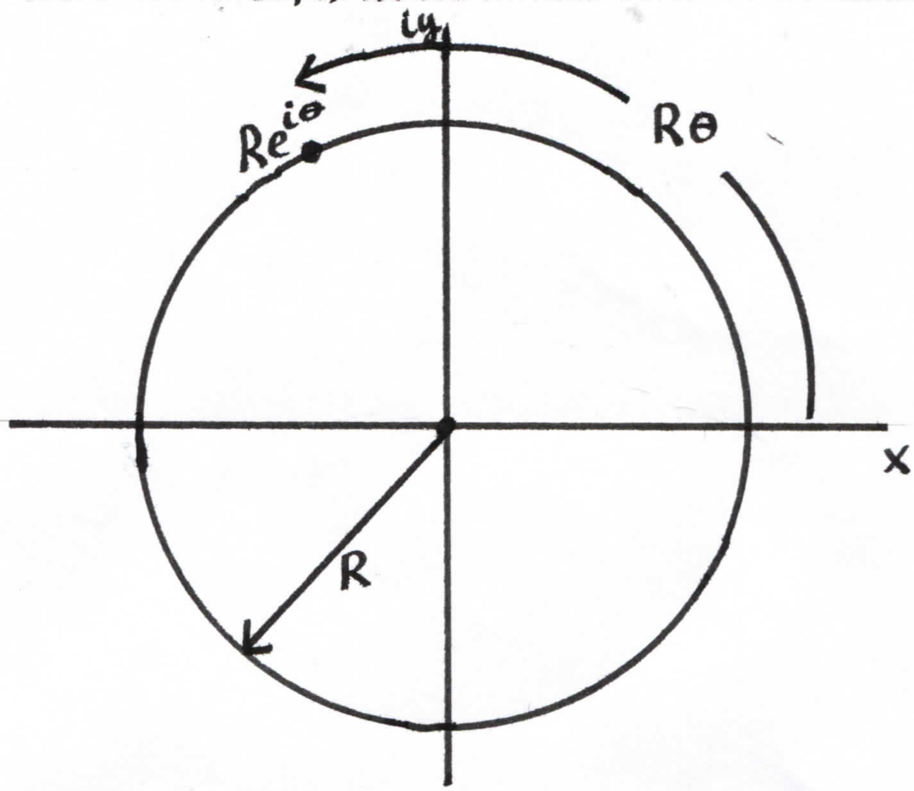
DRAWINGS 2.10



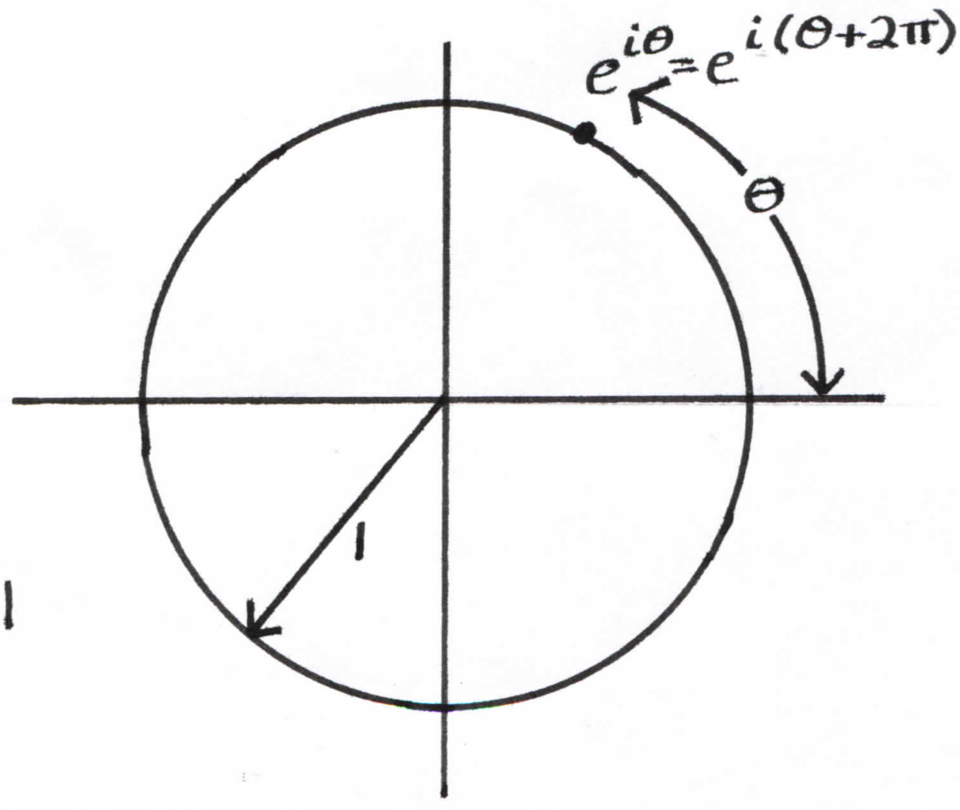




DRAWING 2.11

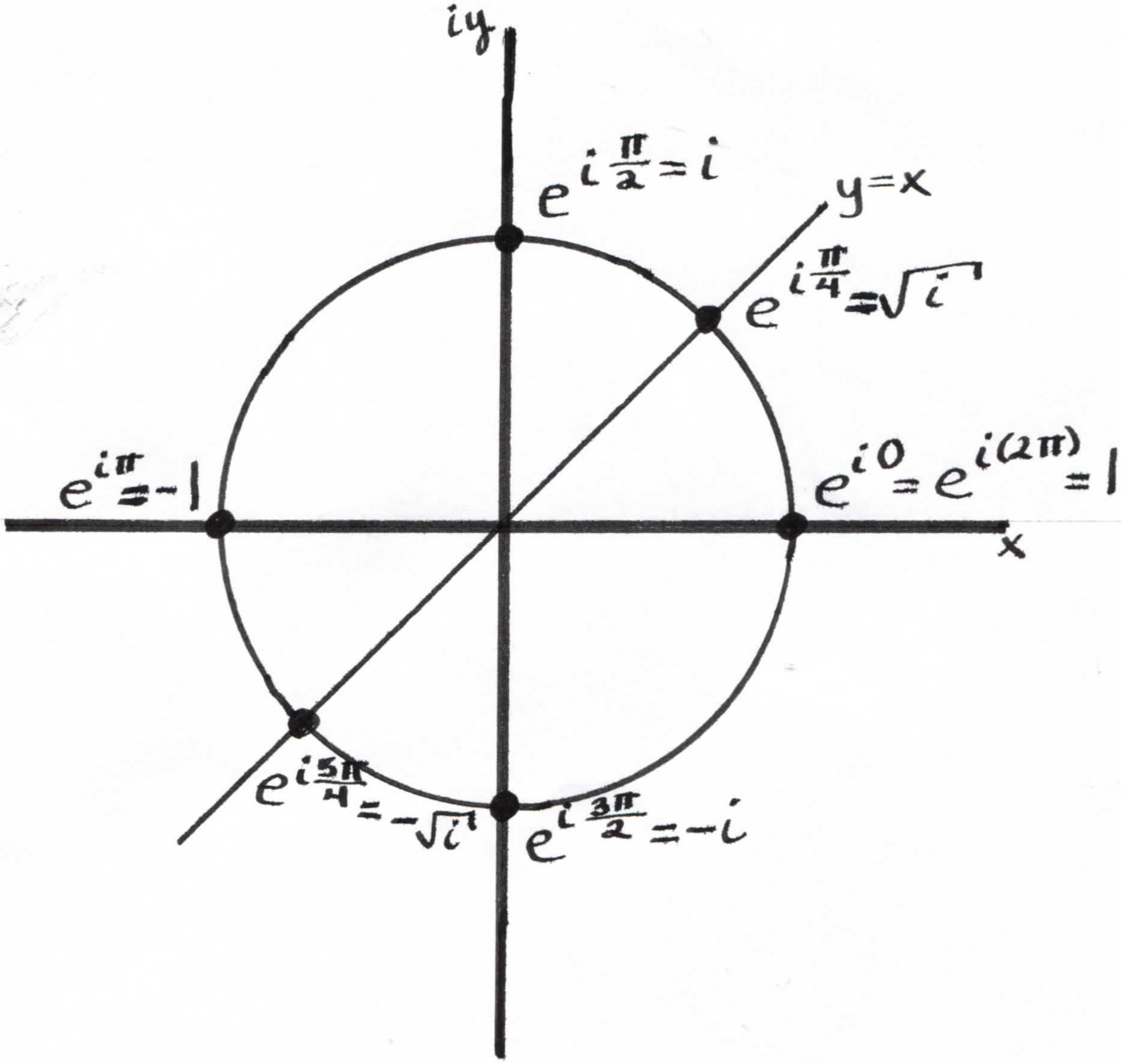


DRAWING 2.12



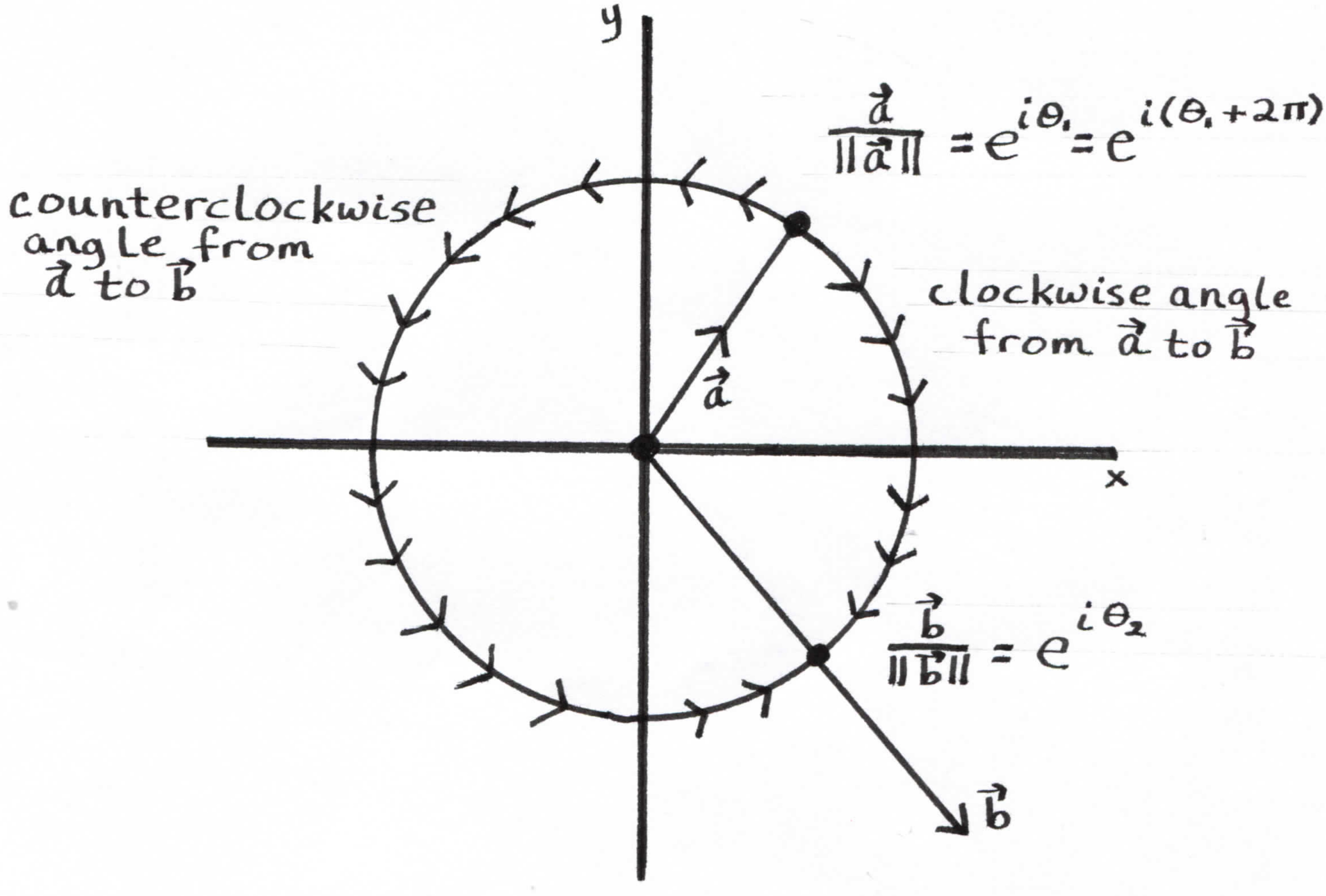
$x^2+y^2=1$

DRAWING 2.13



iy
$e^{i\frac{\pi}{2}} = i$
y=x
$e^{i\frac{\pi}{4}} = \sqrt{i}$
$e^{i\pi} = -1$
$e^{i0} = e^{i(2\pi)} = 1$
x
$e^{i\frac{5\pi}{4}} = -\sqrt{i}$
$e^{i\frac{3\pi}{2}} = -i$

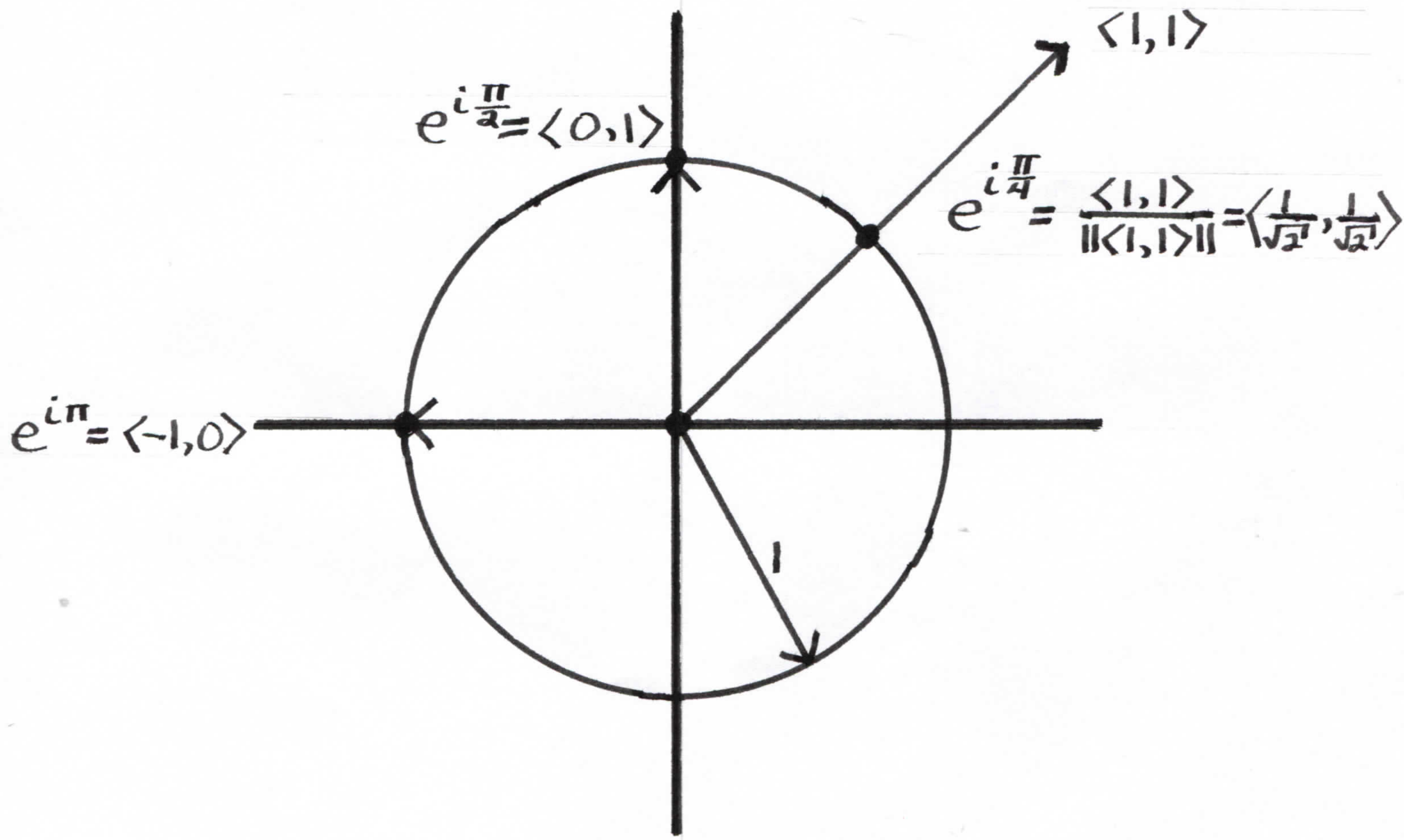
DRAWING 2.14



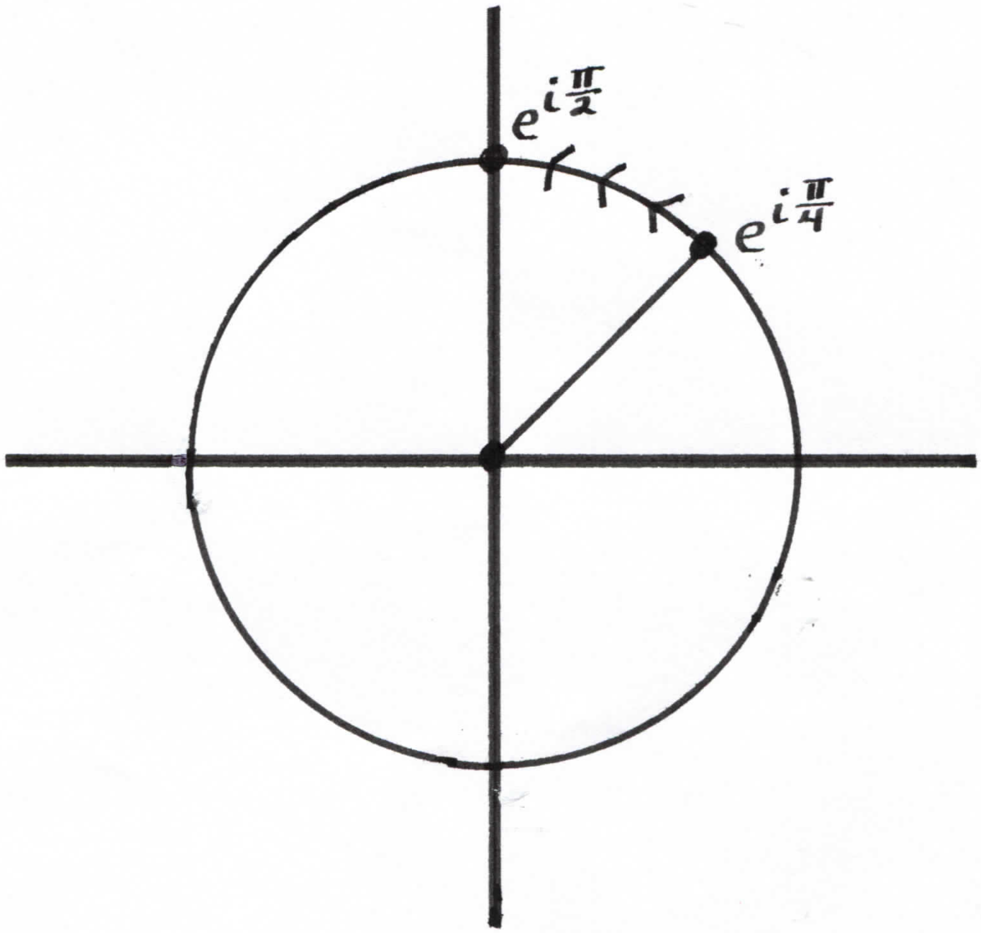
<u>unit</u> <u>circle</u>

$\{(x,y) \mid x^2 + y^2 = 1\}$

DRAWINGS 2.15 (1)

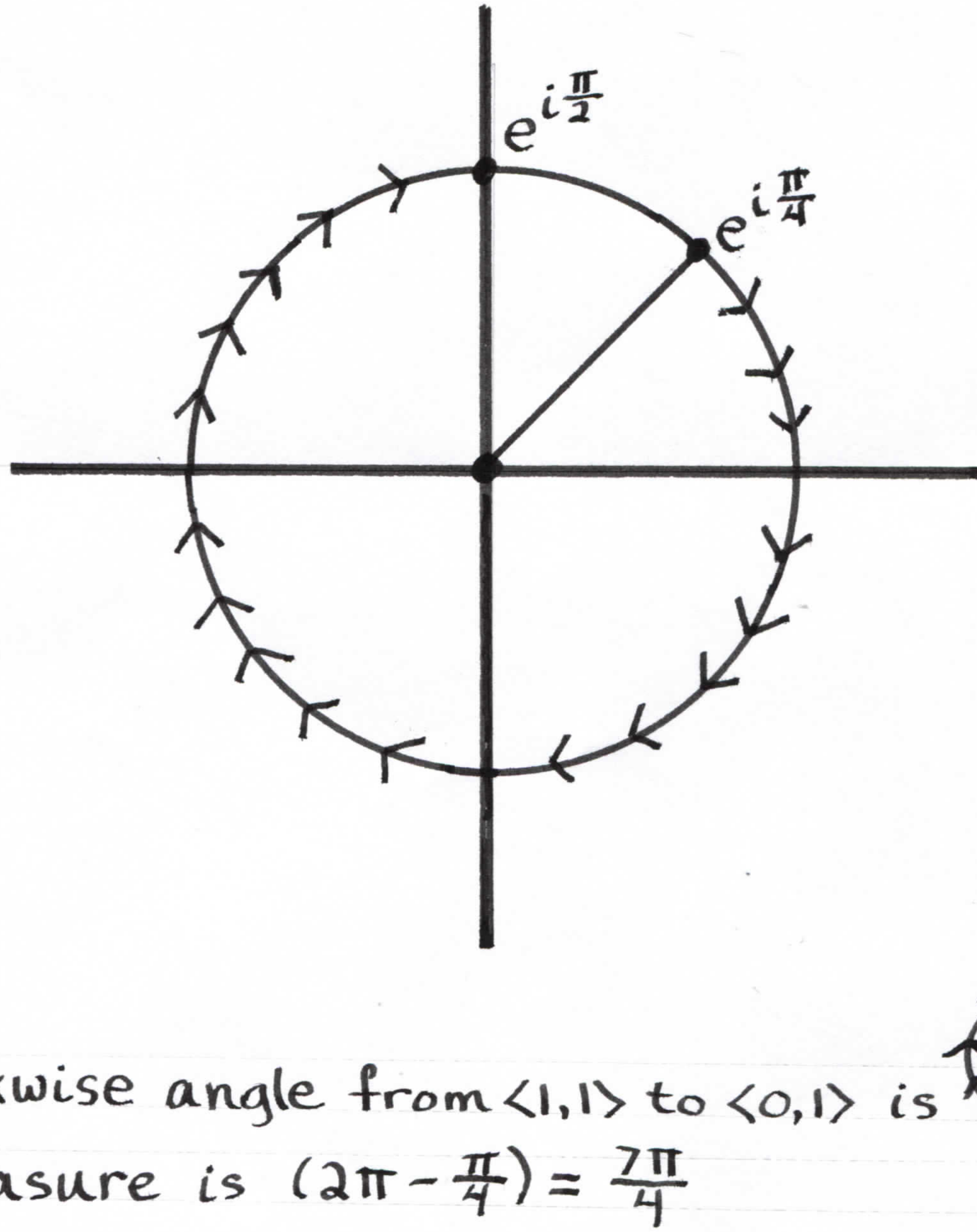


DRAWINGS2.15 (2)



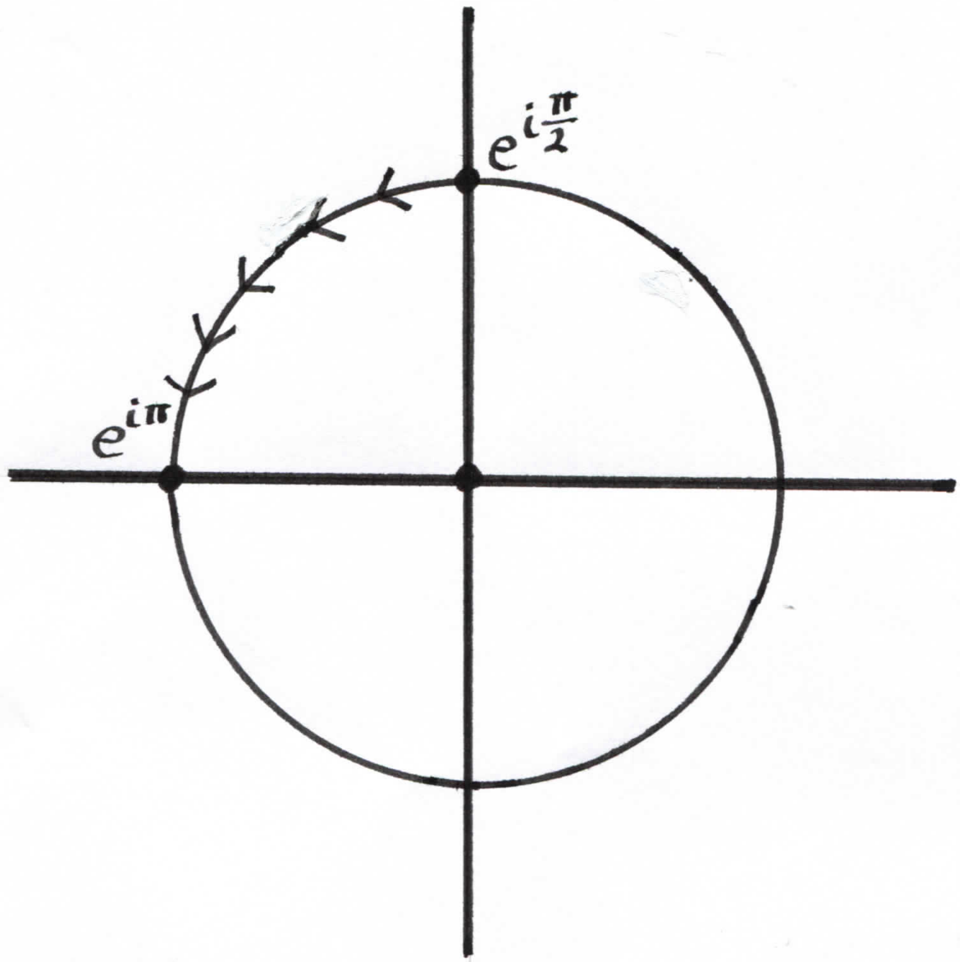
counterclockwise angle from $\langle 1,1\rangle$ to $\langle 0,1\rangle$ is  ;
measure is $\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi}{4}$ radians

DRAWINGS 2.15 (3)



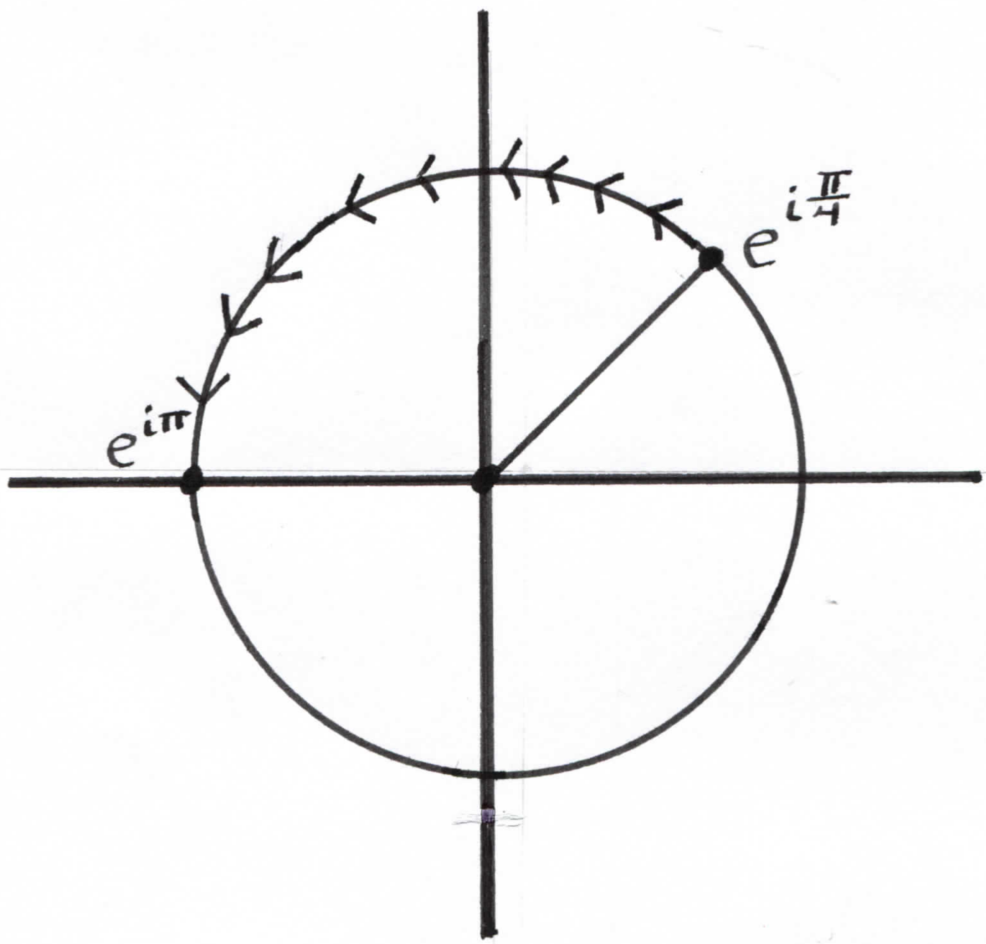
clockwise angle from $\langle 1,1\rangle$ to $\langle 0,1\rangle$ is  ;
measure is $(2\pi - \frac{\pi}{4}) = \frac{7\pi}{4}$

DRAWINGS 2.15 (4)



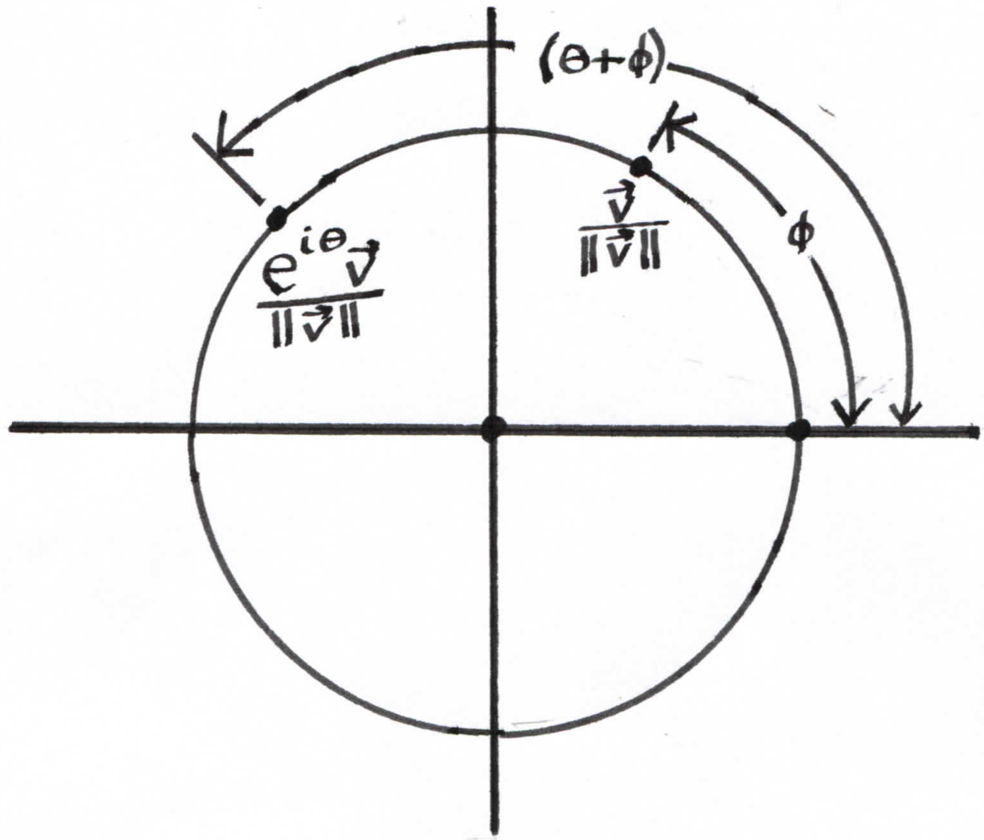
counterclockwise angle from $\langle 0,1 \rangle$ to $\langle -1,0 \rangle$ is ;
measure is $(\pi - \frac{\pi}{2}) = \frac{\pi}{2}$ radians

DRAWINGS 2.15 (5)

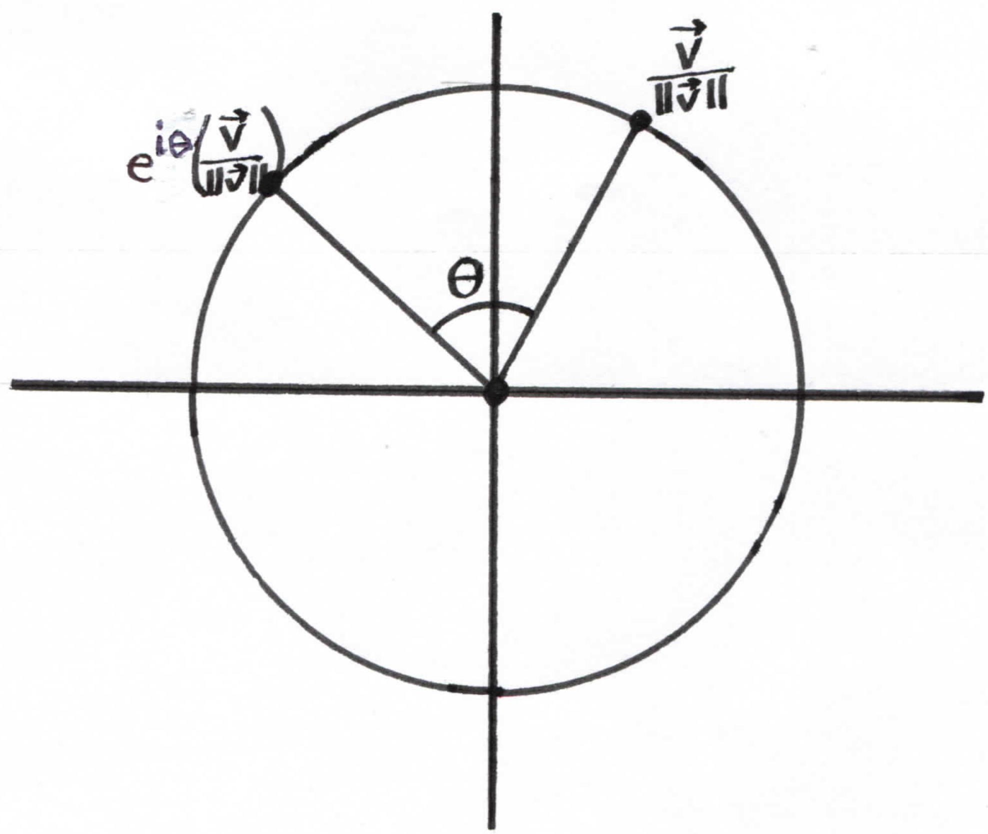


counterclockwise angle from $\langle 1,1\rangle$ to $\langle -1,0\rangle$ is  ;
measure is $\left(\pi - \frac{\pi}{4}\right) = \frac{3\pi}{4} = \left(\frac{\pi}{4} + \frac{\pi}{2}\right)$ radians
(see DRAWINGS 2.15 (2) and (4))

DRAWING 2.16



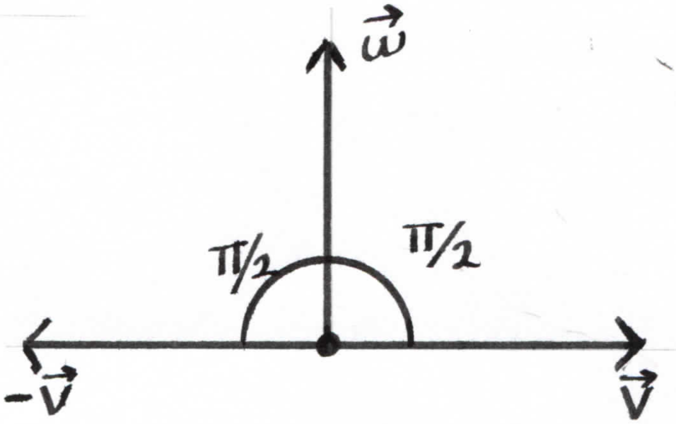
$(\theta+\phi)$
ϕ
$\frac{\vec{v}}{\|\vec{v}\|}$
$e^{i\theta}\frac{\vec{v}}{\|\vec{v}\|}$



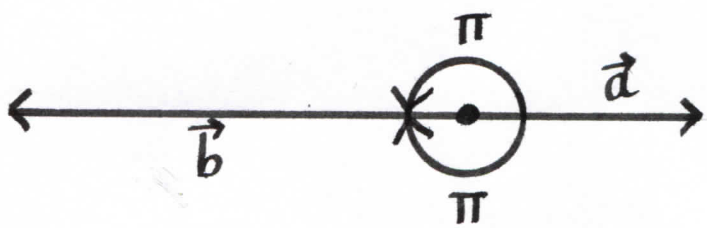
$\frac{\vec{v}}{\|\vec{v}\|}$
$e^{i\theta}\left(\frac{\vec{v}}{\|\vec{v}\|}\right)$
θ

DRAWINGS 2.17

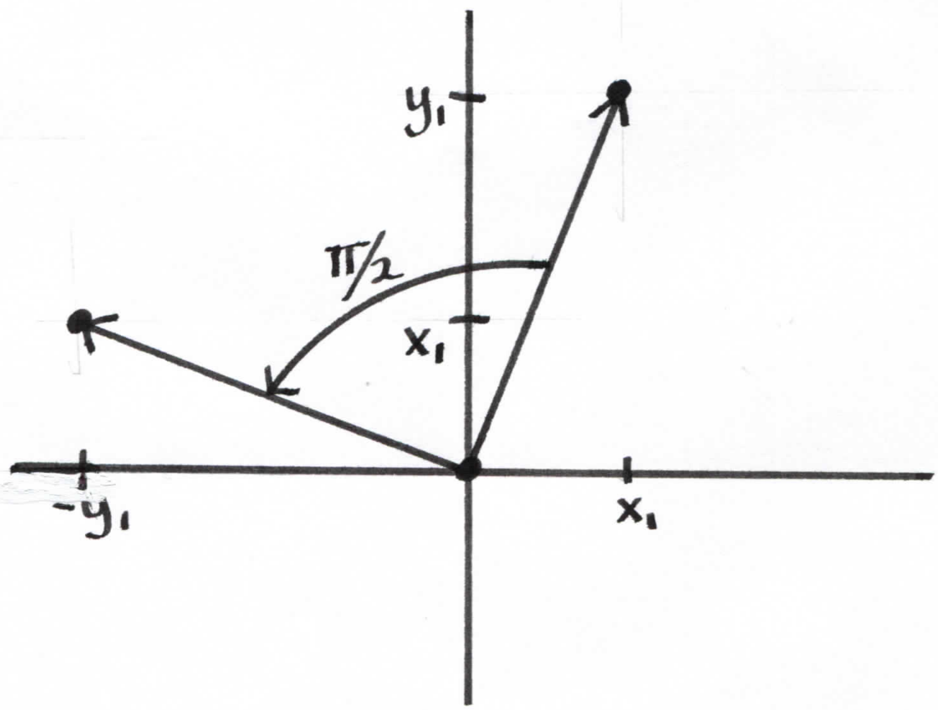
(a)



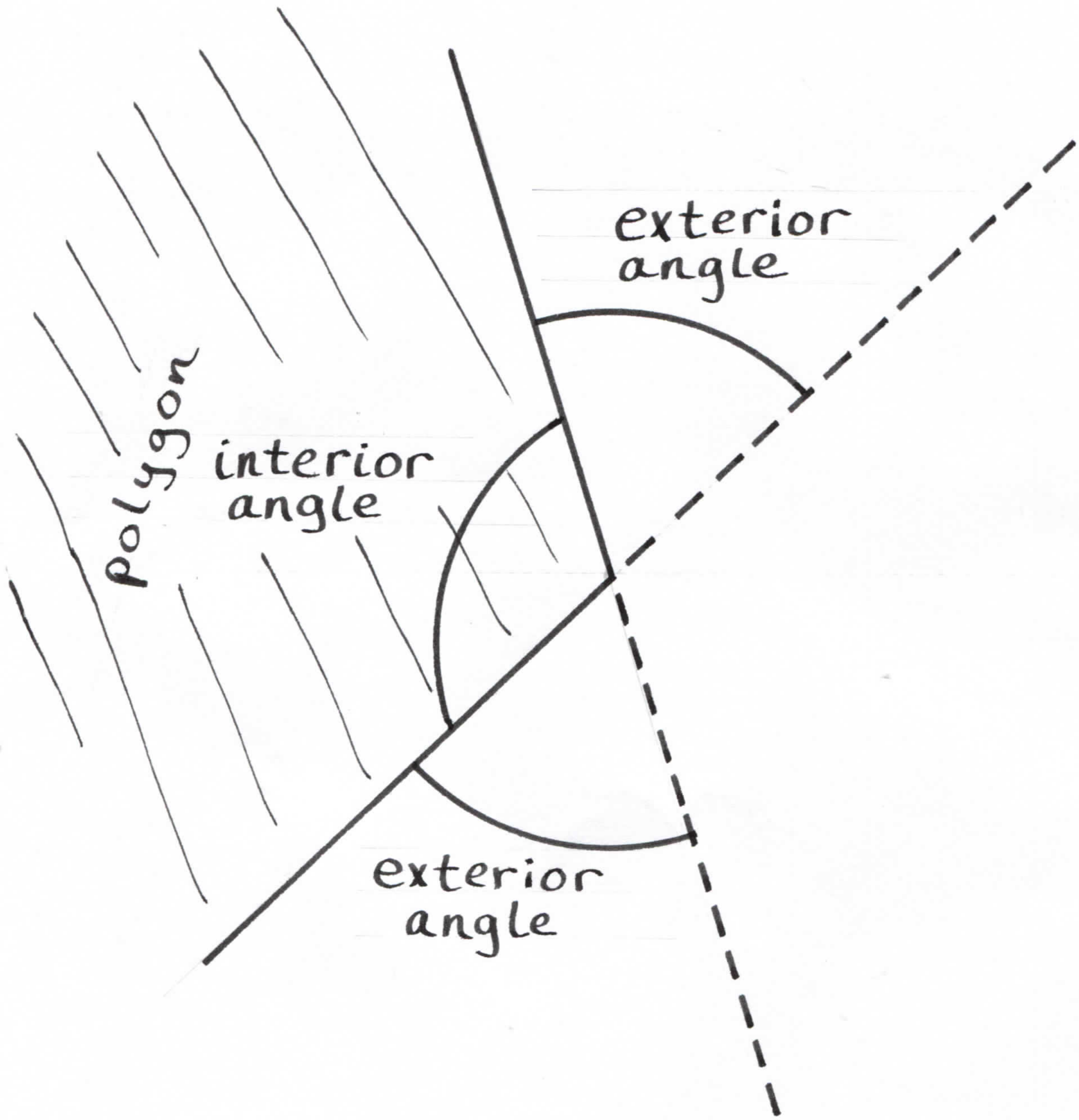
(b)



(c)

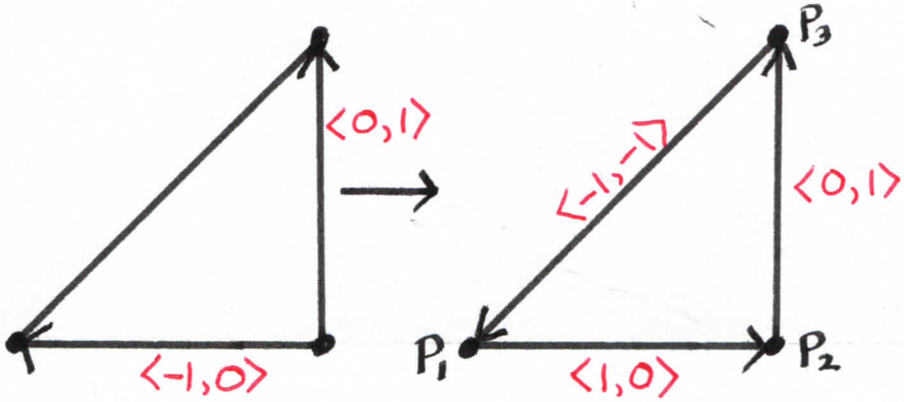


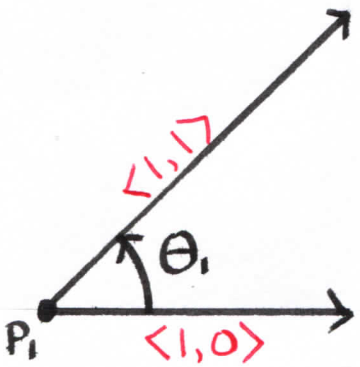
DRAWINGS 2.18 (a)



DRAWINGS 2.18 (b)

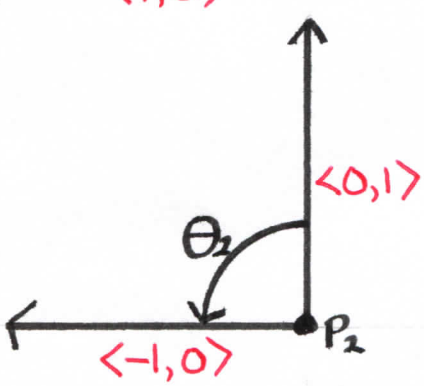
Triangle formed by $\langle -1,0\rangle$ and $\langle 0,1\rangle$:





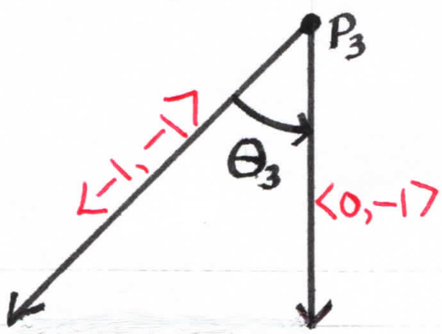
from e^0 counterclockwise to $e^{i\frac{\pi}{4}}$:

$\boxed{\frac{\pi}{4} \text{ radians}}$



from $e^{i\frac{\pi}{2}}$ counterclockwise to $e^{i\pi}$:

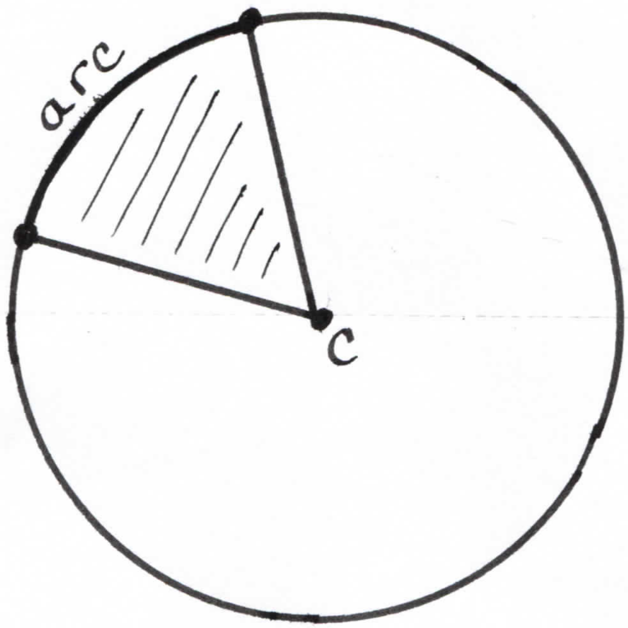
$\boxed{\frac{\pi}{2} \text{ radians}}$



from $e^{i\left(\frac{5\pi}{4}\right)}$ counterclockwise to $e^{i\left(\frac{3\pi}{2}\right)}$:

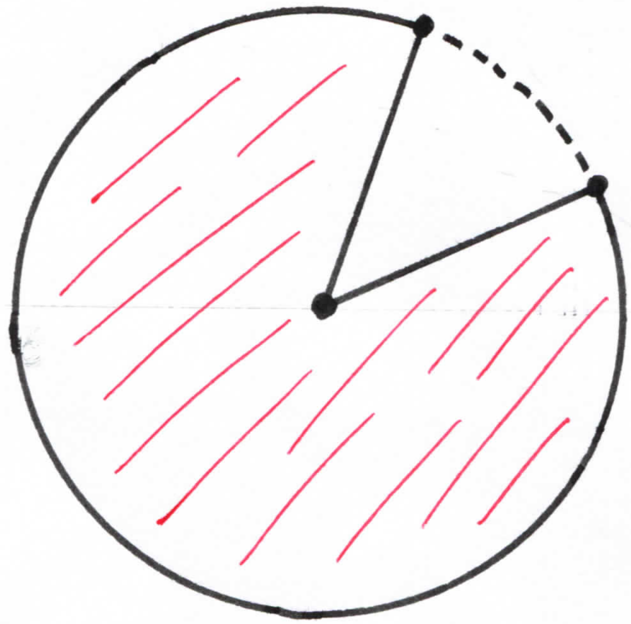
$\boxed{\frac{\pi}{4} \text{ radians}}$

DRAWINGS 2.19



shaded area is closed sector ///
determined by arc

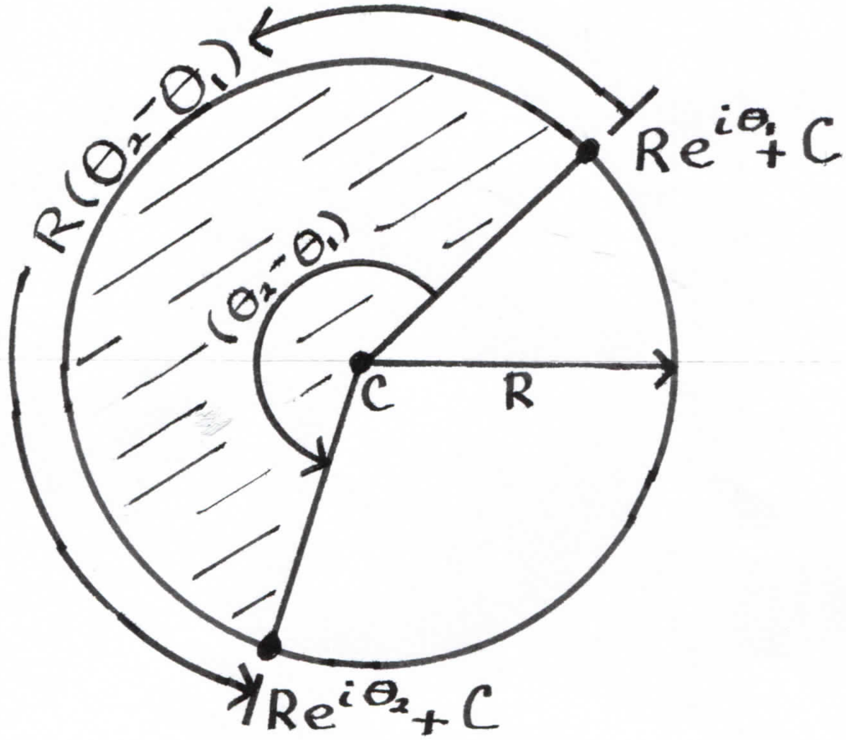
DRAWINGS 2.19 (continued)



boundary of closed sector in solid black

closed sector in red

DRAWINGS 2.19 (continued)



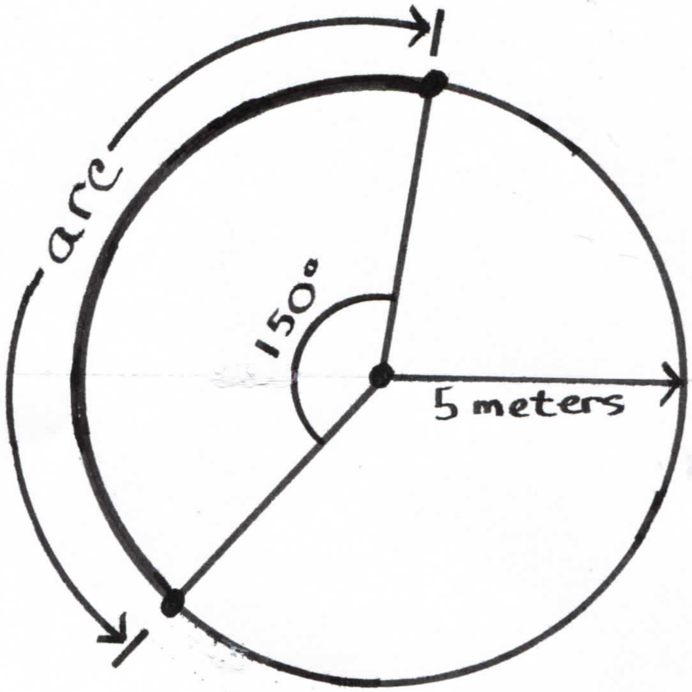
Measure of angle of sector: $(\Theta_2 - \Theta_1)$ radians

Length of arc: $R(\Theta_2 - \Theta_1)$

Perimeter of sector: $2R + R(\Theta_2 - \Theta_1)$

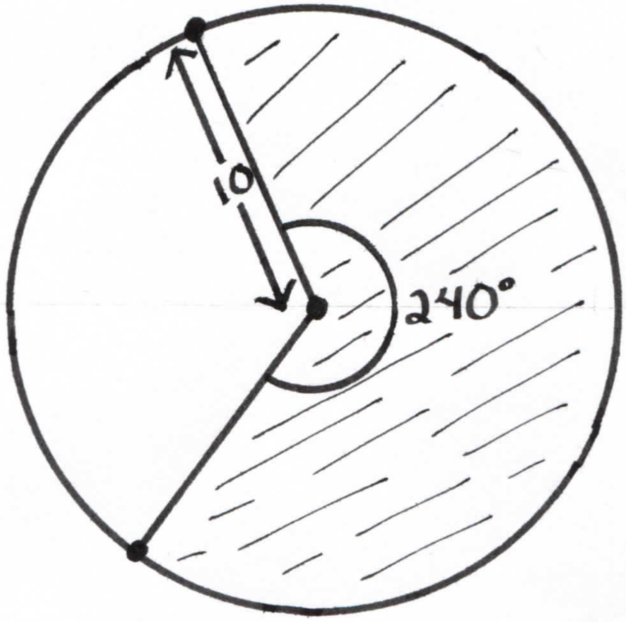
Area of sector: $\frac{1}{2}R^2(\Theta_2 - \Theta_1)$

DRAWING 2.20



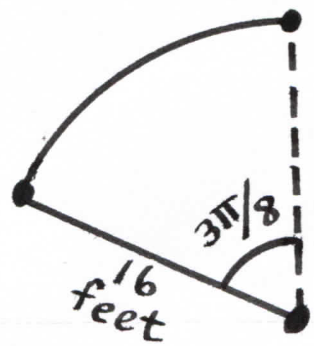
(radius equals 5 meters)

DRAWING 2.21



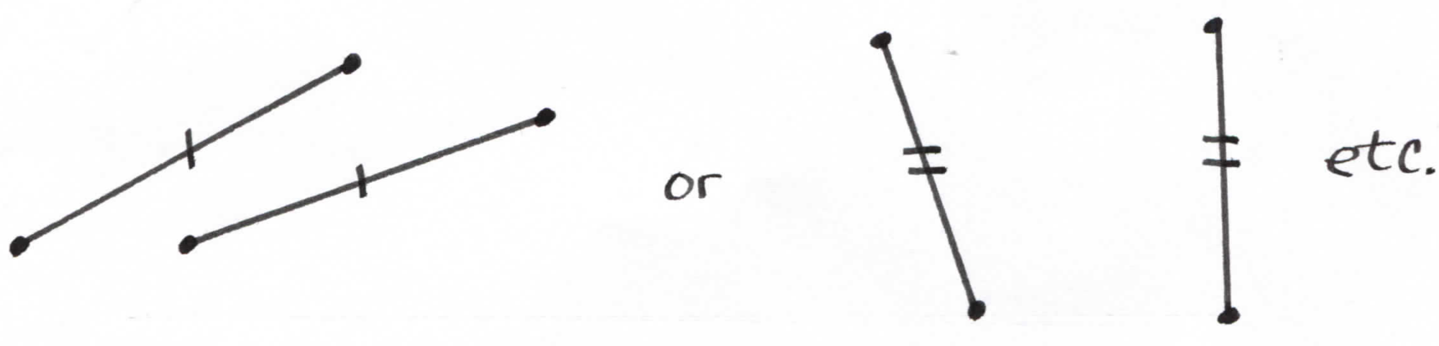
(radius equals 10)

DRAWING 2.22

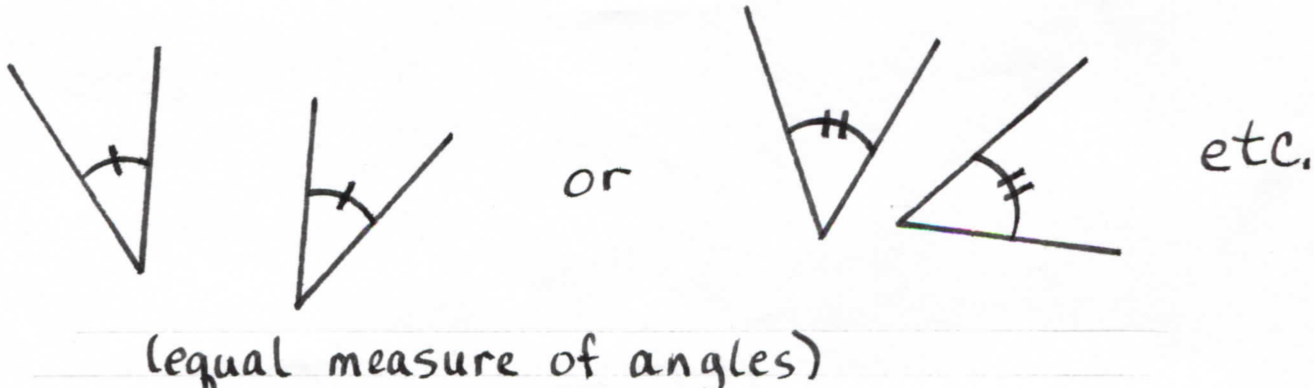


(radius equals 16 feet)

DRAWINGS 2.23

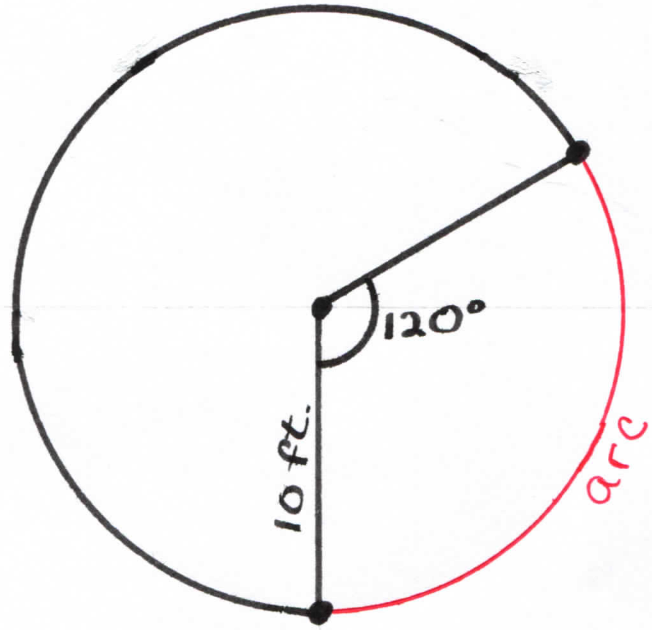


(equal length of line segments)



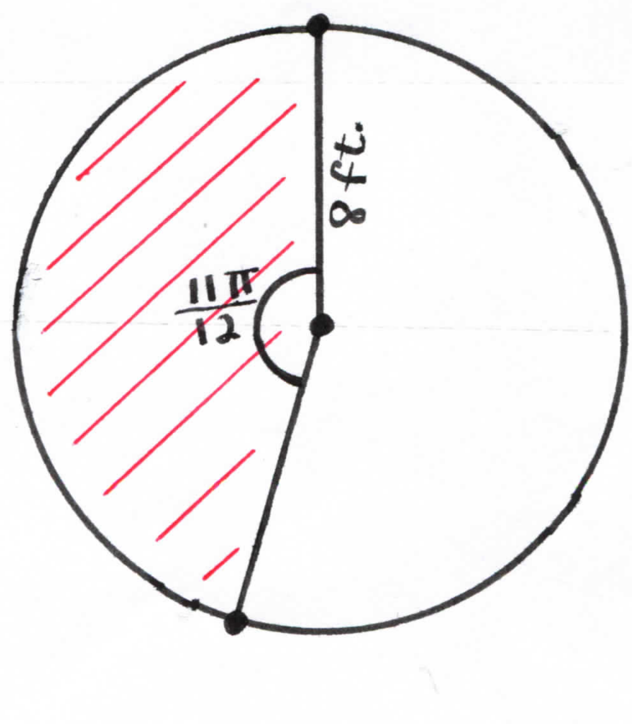
(equal measure of angles)

DRAWING 2.24



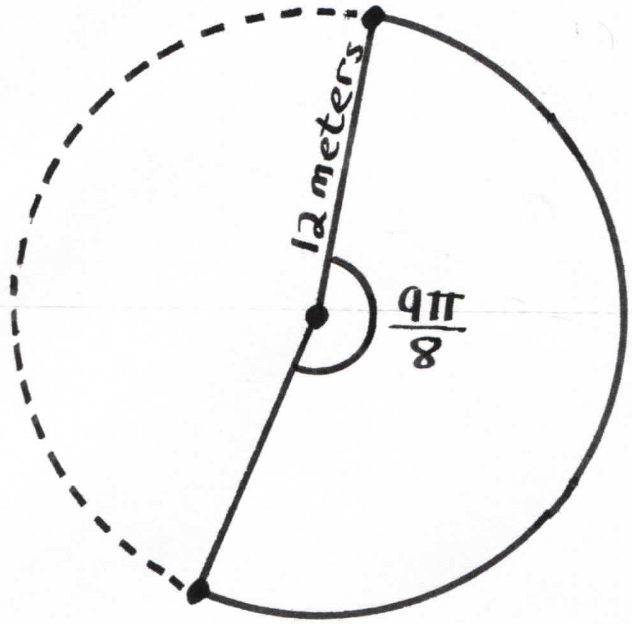
(radius is 10 ft.)

DRAWING 2.25



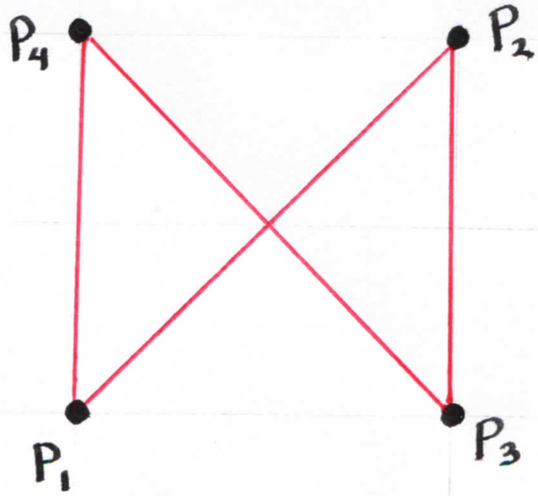
(radius is 8 ft.)

DRAWING 2.26



(radius is 12 meters)

DRAWING 2.27



CHAPTER III: Some Parallel Geometry.

This section will present some geometric consequences of the concept of being *parallel* (see Definitions 1.5 and 2.1). Parallelograms (Definitions 2.3) will receive particular attention; see 3.3–3.5, 3.8, and 3.13. Triangles also will begin to be of interest in Propositions 3.9, 3.14, and 3.15; they will make a much stronger appearance in later chapters.

Although we begin with matters related to parallel lines, one of the consequences will be information about interior angles of triangles and parallelograms; see Propositions 3.6–3.9.

Let's begin by giving some more familiar characterizations of being parallel; another intuitive equivalence will be given in Chapter V (Theorem 5.2), after we've used orthogonality to discuss distance from a point to a line.

Theorem 3.1. The following are equivalent, for two different lines l_1 and l_2.

(a) l_1 and l_2 never intersect.

(b) l_1 and l_2 are parallel.

(c) l_1 and l_2 have the same slope.

Proof: Let l_1 be described by

$$P = P_1 + t\vec{v}_1 \quad (t \text{ real}),$$

l_2 by

$$P = P_2 + t\vec{v}_2 \quad (t \text{ real}).$$

(b) $\to$ (a). Suppose l_1 and l_2 are parallel. Then we may assume $\vec{v}_1 = \vec{v}_2$ (see Definitions 2.1). If l_1 and l_2 intersect, let P_0 be a point of intersection. There then exist real t_1, t_2 so that

$$P_0 = P_1 + t_1\vec{v}_1 = P_2 + t_2\vec{v}_2;$$

thus l_1 may be rewritten as the set of points

$$P = P_1 + t\vec{v}_1 = P_1 + t_1\vec{v}_1 + (t - t_1)\vec{v}_1 = P_0 + (t - t_1)\vec{v}_1 \quad (t \text{ real});$$

identically, l_2 may be rewritten as the set of points

$$P = P_0 + (t - t_2)\vec{v}_1 \quad (t \text{ real}).$$

These are the same lines; we have shown that, if l_1 and l_2 intersect, then they are the same lines. Since we are assuming l_1 and l_2 are different lines, we have shown that l_1 and l_2 cannot intersect, when l_1 and l_2 are parallel.

(a) $\to$ (b). Suppose $\vec{v}_1$ and $\vec{v}_2$ are not parallel. We wish to find a point of intersection of the lines l_1 and l_2; this means we need real s, t so that

$$P_1 + s\vec{v}_1 = P_2 + t\vec{v}_2;$$

in purely vector language,

$$-s\vec{v}_1 + t\vec{v}_2 = \overrightarrow{P_2P_1}.$$

Writing components $\vec{v}_1 =< x_1, y_1 >, \vec{v}_2 =< x_2, y_2 >, \overrightarrow{P_2P_1} =< b_1, b_2 >$, this vector equation is equivalent to (setting components equal) two real equations

$$\begin{array}{rcrcl} -x_1 s & + & x_2 t & = & b_1 \\ -y_1 s & + & y_2 t & = & b_2 \end{array}$$

We leave it to the student to solve for s and t, by whatever method he/she is familiar with. All we care about is that there *is* a solution: the exact expression turns out to be

$$(s, t) = \frac{1}{(y_1x_2 - x_1y_2)}(y_2b_1 - x_2b_2, y_1b_1 - x_1b_2).$$

NOTE that the denominator $(y_1x_2 - x_1y_2)$ is zero precisely when $\vec{v}_1$ and $\vec{v}_2$ are parallel. Thus, when the direction vectors $\vec{v}_1$ and $\vec{v}_2$ are not parallel, we are guaranteed a point of intersection for the lines l_1 and l_2.

(b) $\iff$ (c). First suppose neither line has infinite slope. Let m_1 be the slope of l_1, m_2 the slope of l_2. Then l_1 has direction vector $< 1, m_1 >$, while l_2 has direction vector $< 1, m_2 >$ (HWII.1).

ℓ_1 and ℓ_2 are parallel if and only if $< 1, m_1 >= \alpha < 1, m_2 >$ for some real α if and only if $m_1 = m_2$.

l_1 has infinite slope if and only if $< 0, 1 >$ is a direction vector for it (HWII.2). This is parallel to l_2 if and only if l_2 has a direction vector of $< 0, t >$, for some nonzero t, which is equivalent to l_2 having infinite slope (HWII.2). □

The following assertion is famously known as the "parallel postulate" because it turned out to be independent of Euclid's first four postulates. For us, it is not a postulate, but follows from our postulates stated in the Introduction; see Chapter XIII for constructing the desired parallel line.

Proposition 3.2. (parallel "postulate") Given a line ℓ and a point P, there is a unique line thru P parallel to ℓ. See DRAWING 3.1 at the end of this chapter.

Proof: Let $\vec{v}_0$ be a direction vector for ℓ. Then $\ell_0 \equiv \{P + t\vec{v}_0 \mid t \text{ is real}\}$ is a line thru P parallel to ℓ (see Definitions 1.5 and 2.1).

For uniqueness, suppose ℓ_1 is a line thru P parallel to ℓ. Since ℓ_1 is parallel to ℓ, we may use $\vec{v}_0$ as a direction vector for ℓ_1; that is, for some point P_1 on ℓ_1,

$$\ell_1 = \{P_1 + s\vec{v}_0 \mid s \text{ is real}\}$$

Since P is on ℓ_1, there's real s_1 so that

$$P = P_1 + s_1\vec{v}_0$$

so that ℓ_1 is the set of all points of the form

$$P + (s - s_1)\vec{v}_0 \quad (s \text{ real}),$$

which is the same as the line ℓ_0 (see Definitions 2.1). □

Proposition 3.3. In a parallelogram, opposite sides have equal length.

Proof: By definition of a parallelogram (Definitions 2.3), there are consecutive vectors $\vec{a}, \vec{b}, s\vec{a}, t\vec{b}$, for some nonzero s, t, such that we may denote the sides by $\vec{a}, \vec{b}, s\vec{a}, t\vec{b}$, with

$$\vec{a} + \vec{b} + s\vec{a} + t\vec{b} = \vec{0},$$

as in DRAWING 3.2 at the end of this chapter.

Then

$$(1 + s)\vec{a} = -(1 + t)\vec{b};$$

Since $\vec{a}$ and $\vec{b}$ are not parallel (see Definitions 2.3), this implies that

$$(1 + s) = 0 = -(1 + t),$$

so that $s = t = -1$, quickly implying that $s\vec{a}$, the side opposite $\vec{a}$, has the same length as $\vec{a}$, likewise for $\vec{b}$. □

Corollary 3.4. Any parallelogram may be characterized by two vectors and one point: vertices $I, I + \vec{a}, I + \vec{b}, I + \vec{a} + \vec{b}$, as in DRAWING 2.8(b) at the end of Chapter II.

Proposition 3.5. A quadrilateral with a pair of nonconsecutive sides that are parallel and of equal length is a parallelogram.

Proof: If $\vec{a}$ and $\vec{d}$ are parallel and of equal length, then $\vec{d}$ equals $\vec{a}$ or $(-\vec{a})$. Thus we may denote the sides of the quadrilateral by $\vec{a}, \vec{b}, -\vec{a}, \vec{c}$, with

$$\vec{a} + \vec{b} + (-\vec{a}) + \vec{c} = \vec{0},$$

as in DRAWING 3.3 at the end of this chapter.

Simplifying, we get

$$\vec{b} = -\vec{c},$$

so that both pairs of opposite sides are parallel, showing that the quadrilateral is a parallelogram. □

Proposition 3.6. Suppose ℓ_1, ℓ_2, and ℓ_3 are lines, with ℓ_1 and ℓ_2 parallel, ℓ_3 intersecting both ℓ_1 and ℓ_2 and angles of measure $\theta_j, j = 1, 2, \ldots, 8$ as drawn in DRAWING 3.4 at the end of this chapter.

Then $\theta_1 + \theta_2 = \pi, \theta_1 = \theta_3 = \theta_5 = \theta_7$, and $\theta_2 = \theta_4 = \theta_6 = \theta_8$.

Proof: For $j = 1, 2, 3$, let $\vec{v}_j$ be a direction vector for ℓ_j. Since ℓ_1 and ℓ_2 are parallel, we may assume $\vec{v}_1 = \vec{v}_2$ (see Definitions 2.1).

θ_1 is the measure of the counterclockwise angle from $\vec{v}_1$ to $\vec{v}_3$ and θ_5 is the measure of the counterclockwise angle from $\vec{v}_2$ to $\vec{v}_3$; since $\vec{v}_1 = \vec{v}_2$, it follows that $\theta_1 = \theta_5$. See DRAWING 3.5 at the end of this chapter.

By definition of π and the additivity of angles (see Definitions 2.10 and Proposition 2.12), $\theta_1 + \theta_2 = \pi = \theta_3 + \theta_2$; it follows that $\theta_1 = \theta_3$. The other equalities follow identically. □

In DRAWING 3.4 at the end of this chapter, ℓ_3 is a **transversal**, the angles of measure θ_1 and θ_3 are called **vertical angles** and the angles of measure θ_1 and θ_4 are **supplementary**.

Example 3.7. In DRAWING 3.6 at the end of this chapter, find the measures of the angles $\theta_1, \theta_2, \ldots, \theta_7$. Assume ℓ_1 and ℓ_2 are parallel.

Solutions. By Proposition 3.6, $\theta_7 = \theta_5 = \theta_4 = \theta_2 = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ and $\theta_6 = \theta_3 = \theta_1 = \frac{\pi}{3}$.

Proposition 3.8. In a parallelogram, opposite interior angles are of equal measure and the measures of adjacent interior angles add up to π.

Proof: This is essentially a corollary of Proposition 3.6. But we would like to exploit the picture of a parallelogram guaranteed by Corollary 3.4 to give a proof entirely in terms of the vectors $\vec{a}, \vec{b}$ from Corollary 3.4 and DRAWING 2.8(b) at the end of Chapter II that characterize a parallelogram.

Take the picture of a parallelogram guaranteed by Corollary 3.4 (see DRAWING 2.8(b) at the end of Chapter II), and add on interior angles, of measures $\theta_1, \theta_2, \theta_3, \theta_4$, as in DRAWING 3.7 at the end of this chapter.

In DRAWING 3.7, we wish to show that $\theta_1 = \theta_3, \theta_2 = \theta_4$, and $\theta_1 + \theta_2 = \theta_2 + \theta_3 = \theta_3 + \theta_4 = \theta_4 + \theta_1 = \pi$.

In DRAWING 3.7, extend the sides vertically and to the right, as in DRAWING 3.8 at the end of this chapter; note that the angle of measure θ_1 appears in numerous places as the measure of the counterclockwise angle from $\vec{a}$ to $\vec{b}$, while the angle of measure θ_4 appears twice as the clockwise angle from $\vec{a}$ to $-\vec{b}$.

In DRAWING 3.8, exploiting vertical angles of measure θ_3 and θ_1, as in DRAWING 3.4 at the end of this chapter and Proposition 3.6, we see that $\theta_1 = \theta_3, \theta_1 + \theta_2 = \pi = \theta_1 + \theta_4$, from which all the desired equalities follow. $\square$

Proposition 3.9. The sum of the measures of the interior angles in a triangle is π.

Proof: Represent a triangle with vectors $\vec{a}, \vec{b}$, and $\vec{c}$, as in DRAWING 3.9 at the end of this chapter, where we have labeled the measures of the interior angles as $\theta_1, \theta_2, \theta_3$.

Extend each of the sides, add on a copy of $\vec{a}$ with initial point at the vertex with interior angle of measure θ_3, and observe that θ_1 appears as the measure of the counterclockwise angle from $\vec{a}$ to $\vec{b}$, while θ_2, after invoking Proposition 3.6 for a pair of vertical angles, appears as the measure of the counterclockwise angle from $\vec{c}$ to $\vec{a}$. See DRAWING 3.10 at the end of this chapter.

At the top of DRAWING 3.10 you will see (reading clockwise) $\theta_1 + \theta_2 + \theta_3 = \pi$, as desired. $\square$

Corollary 3.10. For $n = 3, 4, 5, \ldots$, the sum of the interior angles in an n-gon is $(n-2)\pi$.

$\sim$ **Proof:** This follows from Proposition 3.9 and the fact, that we choose (as mentioned near the end of Definitions 2.3) to not prove, that any n-gon may be written as the union of $(n-2)$ triangles whose interiors do not overlap, with each vertex of each triangle equal to a vertex of the n-gon. $\square$

Definitions 3.11. If ℓ_1 is the line segment from A to B, the **midpoint** of ℓ_1 is a point C on ℓ_1 such that $\|\overrightarrow{AC}\| = \|\overrightarrow{CB}\|$. Another line segment or line ℓ_2 **bisects** ℓ_1 if ℓ_2 intersects ℓ_1 at C. See DRAWINGS 3.11 at the end of this chapter.

The angle between $\vec{a}$ and $\vec{b}$ is **bisected** by the vector $\vec{c}$ if the measure of the angle between $\vec{a}$ and $\vec{c}$ equals the measure of the angle between $\vec{c}$ and $\vec{b}$. See DRAWINGS 3.11.

Definition 3.12. The **diagonals** of a parallelogram are line segments between nonconsecutive vertices.

If the parallelogram is formed by vectors $\vec{a}$ and $\vec{b}$, then the (directed) diagonals are $(\vec{a} + \vec{b})$ and $(\vec{b} - \vec{a})$. See DRAWINGS 3.12 at the end of this chapter.

Proposition 3.13. The diagonals of a parallelogram bisect each other.

Proof: In DRAWING 3.13 at the end of this chapter, let $\vec{a}$ and $\vec{b}$ be as in Corollary 3.4, with the diagonals $(\vec{a} + \vec{b})$ and $(\vec{b} - \vec{a})$ drawn in.

Let P be the intersection of the diagonals. Then (see DRAWING 3.13)

$$P = I + s(\vec{a} + \vec{b}) = I + \vec{a} + t(\vec{b} - \vec{a}),$$

for some real s, t, so that

$$(s + t - 1)\vec{a} = (t - s)\vec{b};$$

since $\vec{a}$ and $\vec{b}$ are not parallel (see Definitions 1.5 and 2.3),

$$(s + t - 1) = 0 = (t - s),$$

which implies that $s = t = \frac{1}{2}$, as desired. $\square$

Proposition 3.14. In any triangle, the vectors from vertices to midpoints of opposite sides intersect at a single point.

Discussion and Proof: Draw an arbitrary triangle with sides represented by vectors $\vec{c}_1, \vec{c}_2, \vec{c}_3$, opposite vertices I_1, I_2, I_3, with

$$\vec{c}_1 + \vec{c}_2 + \vec{c}_3 = \vec{0},$$

as drawn in DRAWING 3.14 at the end of this chapter.

For $j = 1, 2, 3$, let $\vec{a}_j$ be the vector from I_j to the midpoint of the side opposite I_j, as in DRAWING 3.15 at the end of this chapter.

Focus now on the intersection (if it exists), call it P, of $\vec{a}_1$ and $\vec{a}_2$, as in DRAWING 3.16 at the end of this chapter.

If P exists, we can calculate it, analogous to the proof of Proposition 3.13.

By definition of P, there are real s and t so that

$$P = I_2 + s\vec{a}_2 \quad \text{and} \quad P = I_1 + t\vec{a}_1 = I_2 + \vec{c}_1 + \vec{c}_2 + t\vec{a}_1. \qquad (*)$$

Let's see if we can figure out what s and t are.

We haven't used the midpoint hypotheses; our hope is to express $\vec{a}_1$ and $\vec{a}_2$, hence (*), in terms of $\vec{c}_1$ and $\vec{c}_2$.

Because of the midpoint assertions,

$$\vec{a}_2 = \vec{c}_1 + \frac{1}{2}\vec{c}_2 \quad \text{and} \quad \vec{a}_1 = -\frac{1}{2}\vec{c}_1 - \vec{c}_2.$$

Plug these into (*), to get

$$s(\vec{c}_1 + \frac{1}{2}\vec{c}_2) = s\vec{a}_2 = \vec{c}_1 + \vec{c}_2 + t\vec{a}_1 = \vec{c}_1 + \vec{c}_2 + t(-\frac{1}{2}\vec{c}_1 - \vec{c}_2)$$

or

$$(s - 1 + \frac{t}{2})\vec{c}_1 = (-\frac{1}{2}s + 1 - t)\vec{c}_2;$$

since $\vec{c}_1$ and $\vec{c}_2$ are not parallel,

$$(s - 1 + \frac{t}{2}) = 0 = (-\frac{1}{2}s + 1 - t).$$

The solution of this pair of equations is $s = \frac{2}{3} = t$. In other words, by (*),

$$P = I_1 + \frac{2}{3}\vec{a}_1 = I_2 + \frac{2}{3}\vec{a}_2$$

equals the intersection of the vector from I_1 to the midpoint of its opposite side and the vector from I_2 to the midpoint of its opposite side.

All of the above could be done as scratch work, then concealed. For a proof that gives you the aura of mystical certainty, seemingly pulling surprises out of a hat, *begin* a formal proof with the formulas we just calculated:

$$P_1 \equiv I_1 + \frac{2}{3}\vec{a}_1, \quad \text{on the line from } I_1 \text{ to the midpoint of } \vec{c}_1,$$

$$P_2 \equiv I_2 + \frac{2}{3}\vec{a}_2, \quad \text{on the line from } I_2 \text{ to the midpoint of } \vec{c}_2.$$

We will show that $P_1 = P_2$. Begin with the midpoint assertions implying that

$$\vec{a}_2 = \vec{c}_1 + \frac{1}{2}\vec{c}_2 \quad \text{and} \quad \vec{a}_1 = -\frac{1}{2}\vec{c}_1 - \vec{c}_2.$$

Thus we can rewrite

$$P_1 = I_1 + \frac{2}{3}\left(-\frac{1}{2}\vec{c}_1 - \vec{c}_2\right) = I_1 - \frac{1}{3}\vec{c}_1 - \frac{2}{3}\vec{c}_2 = I_2 + (\vec{c}_1 + \vec{c}_2) - \frac{1}{3}\vec{c}_1 - \frac{2}{3}\vec{c}_2 = I_2 + \frac{2}{3}\vec{c}_1 + \frac{1}{3}\vec{c}_2 = I_2 + \frac{2}{3}\left(\vec{c}_1 + \frac{1}{2}\vec{c}_2\right) = P_2.$$

This shows that $P_2 = P_1$ is the intersection of the vector from I_1 to the midpoint of its opposite side and the vector from I_2 to the midpoint of its opposite side.

By a similar argument,

$$P_3 \equiv I_3 + \frac{2}{3}\vec{a}_3 = I_2 + \frac{2}{3}\vec{a}_2 \equiv P_2;$$

that is, the intersection of the vector from I_3 to the midpoint of its opposite side and the vector from I_2 to the midpoint of its opposite side also equals P.

Thus $P \equiv P_1 = P_2 = P_3$ is a common intersection for all three vectors from vertices to midpoints of opposite sides. □

Notice that we have actually proved more, namely the following.

Proposition 3.15. The common intersection point of Proposition 3.14 is, for each vertex, two-thirds of the way from the vertex to the midpoint of the opposite side.

Examples 3.16. In each of the drawings in DRAWINGS 3.17 at the end of this chapter, use the results of this chapter to fill in any lengths of sides or measures of angles, whenever possible. Do not assume anything is drawn to scale. All quadrilaterals are parallelograms.

Solutions. See DRAWINGS 3.18 at the end of this chapter. (a) Propositions 3.3 and 3.8. (b) Proposition 3.15. (c) Proposition 3.13. (d) Proposition 3.3. (e) Proposition 3.9. (f) Proposition 3.13. (g) Proposition 3.8.
(i) Proposition 3.6.

Examples 3.17. In each of the drawings in DRAWINGS 3.19 at the end of this chapter, find x. All quadrilaterals are parallelograms.

Solutions. (a) By Proposition 3.3, $(2x + 5) = (5x - 10)$, so $x = 5$.

(b) By Proposition 3.6, $180 = (3x + 20) + (2x - 90)$, so $x = 50$ (degrees).

(c) By Proposition 3.8, $180 = (6x + 10) + (x + 50)$, so $x = \frac{120}{7}$.

(d) By Proposition 3.8, $(x + 50) = (6x + 10)$, so $x = 8$.

(e) By Proposition 3.9, $180 = 60 + (2x + 10) + 3x$, so $x = 22$.

(f) By Proposition 3.13, $(x + 7) = (3x + 5)$, so $x = 1$.

HOMEWORK

HWIII.1. In each of the drawings in DRAWINGS 3.20 at the end of this chapter, use the results of this chapter to fill in lengths and angles, where possible. All quadrilaterals are parallelograms.

HWIII.2. Prove that, in the drawing of a diagonal of a parallelogram in DRAWING 3.21 at the end of this chapter, $\theta_1 = \theta_4$ and $\theta_2 = \theta_3$.

HWIII.3. In a quadrilateral, draw line segments between midpoints of consecutive sides. Show that the new quadrilateral that is formed is a parallelogram. Use vector methods. HINT: DRAWING 3.22 at the end of this chapter and Proposition 3.5.

HWIII.4. Show that, in any triangle, the line segment between the midpoints of two sides is parallel to the remaining side, with length half the length of the remaining side. Use vector methods. See DRAWING 3.23 at the end of this chapter for a hint; see also Example 14.19 for a generalization.

HOMEWORK ANSWERS

HWIII.1. See DRAWINGS 3.24 at the end of this chapter.

HWIII.2. Proposition 3.6.

HWIII.3. In DRAWING 3.22,

$$\vec{e} = \frac{1}{2}\vec{d} + \frac{1}{2}\vec{a} \quad \text{and} \quad \vec{f} = \frac{1}{2}\vec{b} + \frac{1}{2}\vec{c},$$

thus

$$\vec{0} = \frac{1}{2}\left(\vec{a} + \vec{b} + \vec{c} + \vec{d}\right) = \vec{e} + \vec{f},$$

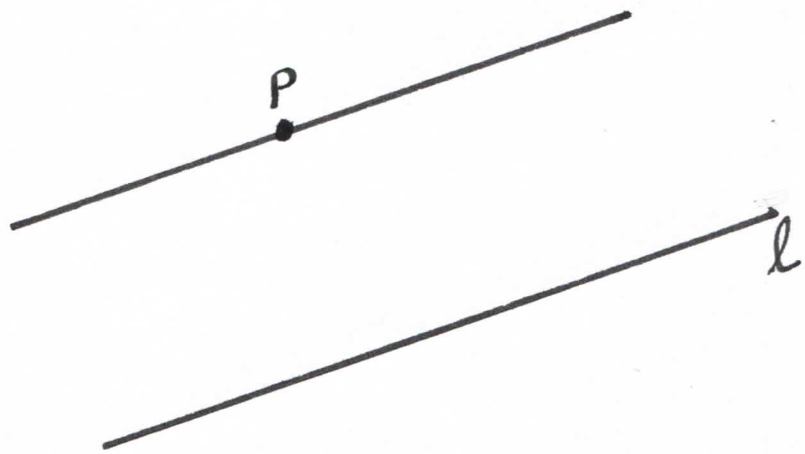
so that $\vec{e} = -\vec{f}$, so that Proposition 3.5 implies we have a parallelogram.

HWIII.4. In DRAWING 3.23,

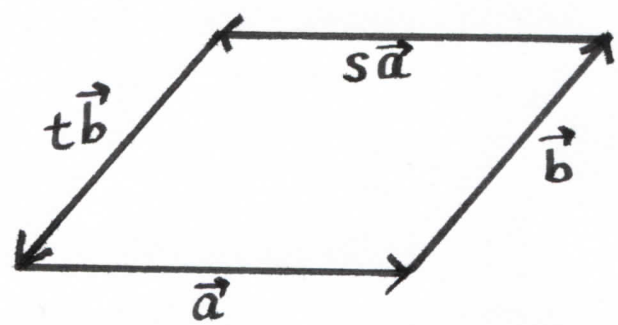
$$\vec{d} = \frac{1}{2}\vec{c} + \vec{a} + \frac{1}{2}\vec{b} = \frac{1}{2}\left(\vec{a} + \vec{b} + \vec{c}\right) + \frac{1}{2}\vec{a} = \frac{1}{2}\vec{a},$$

so that $\vec{a} = 2\vec{d}$.

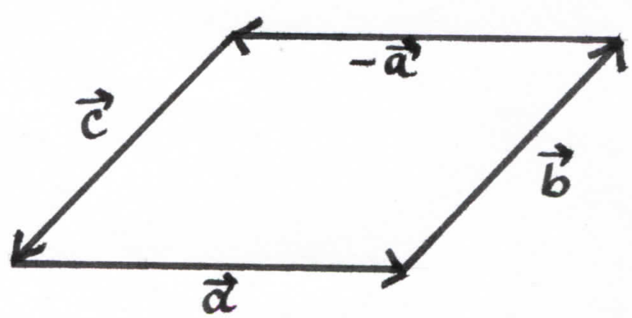
DRAWING 3.1



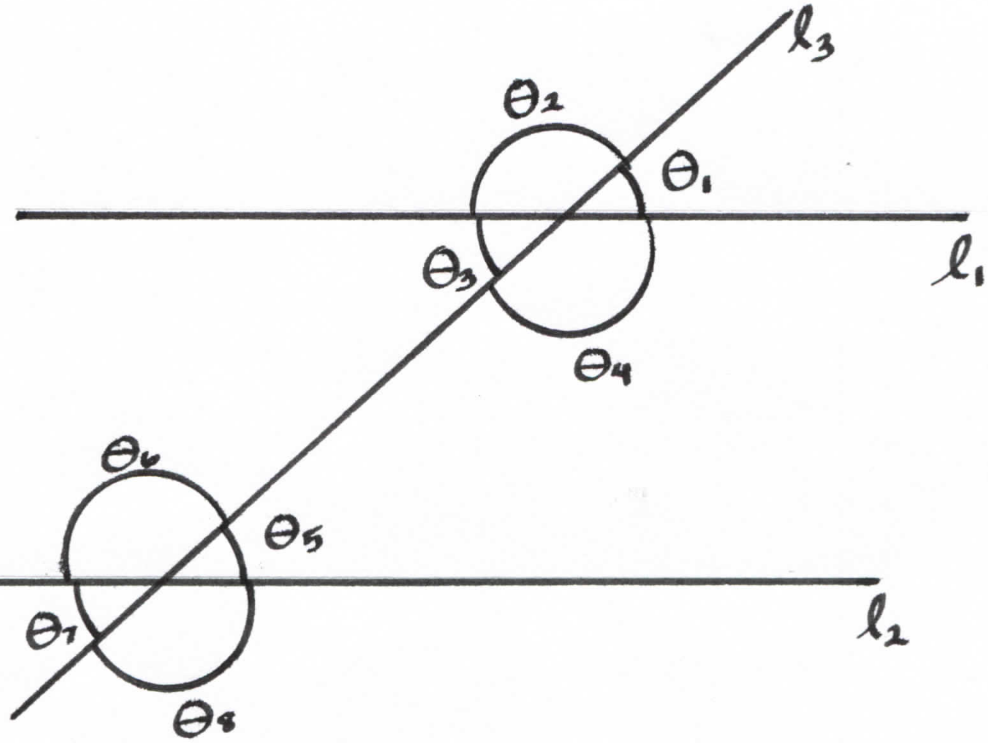
DRAWING 3.2



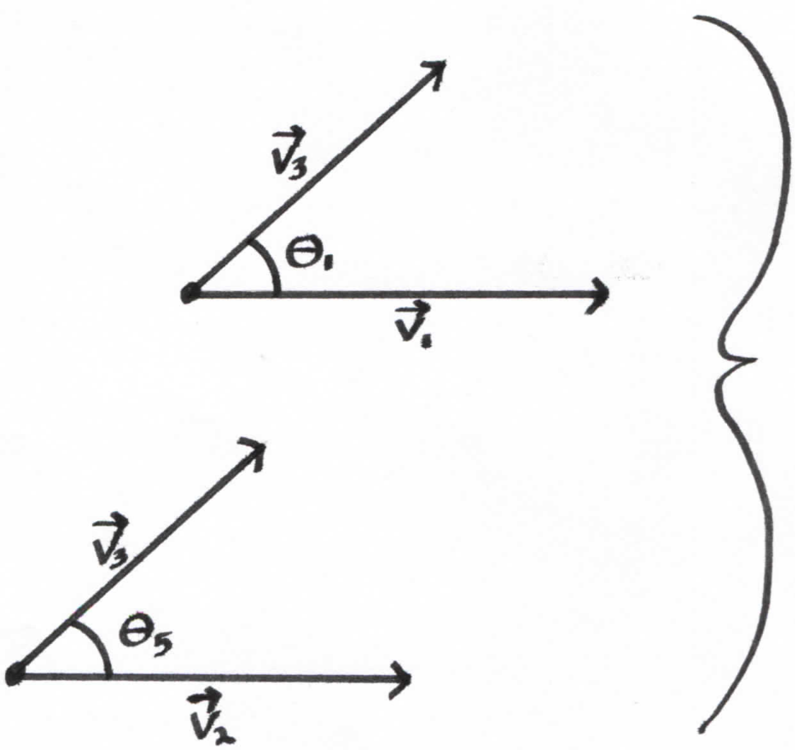
DRAWING 3.3



DRAWING 3.4

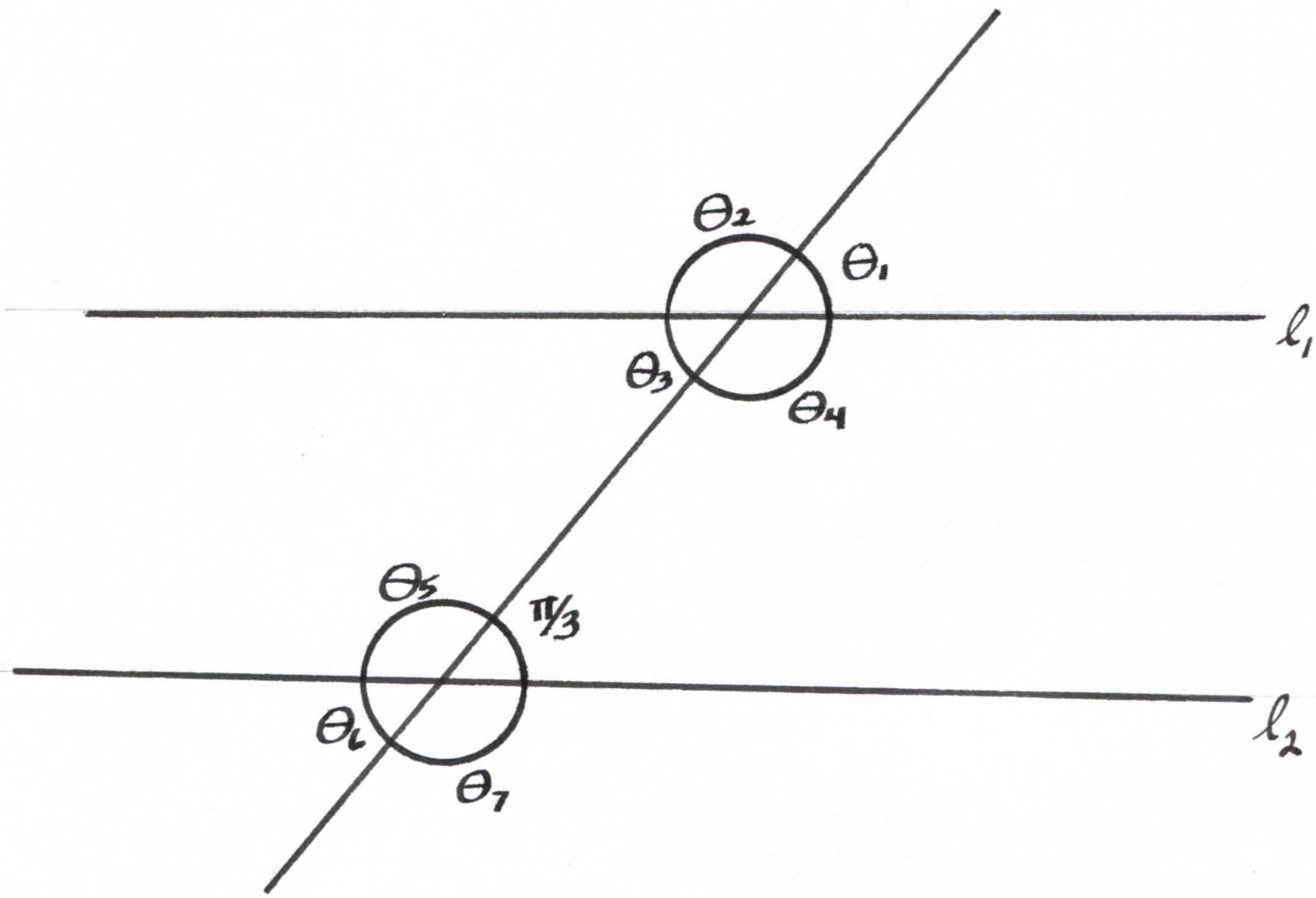


DRAWING 3.5



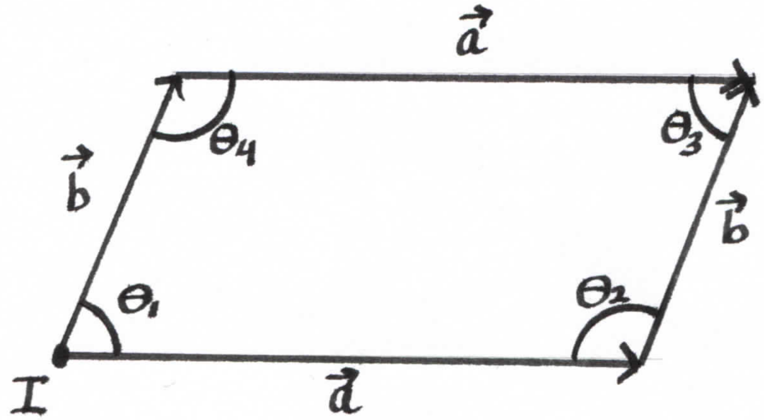
$$\vec{V}_1 = \vec{V}_2 \longrightarrow \theta_1 = \theta_5$$

DRAWING 3.6

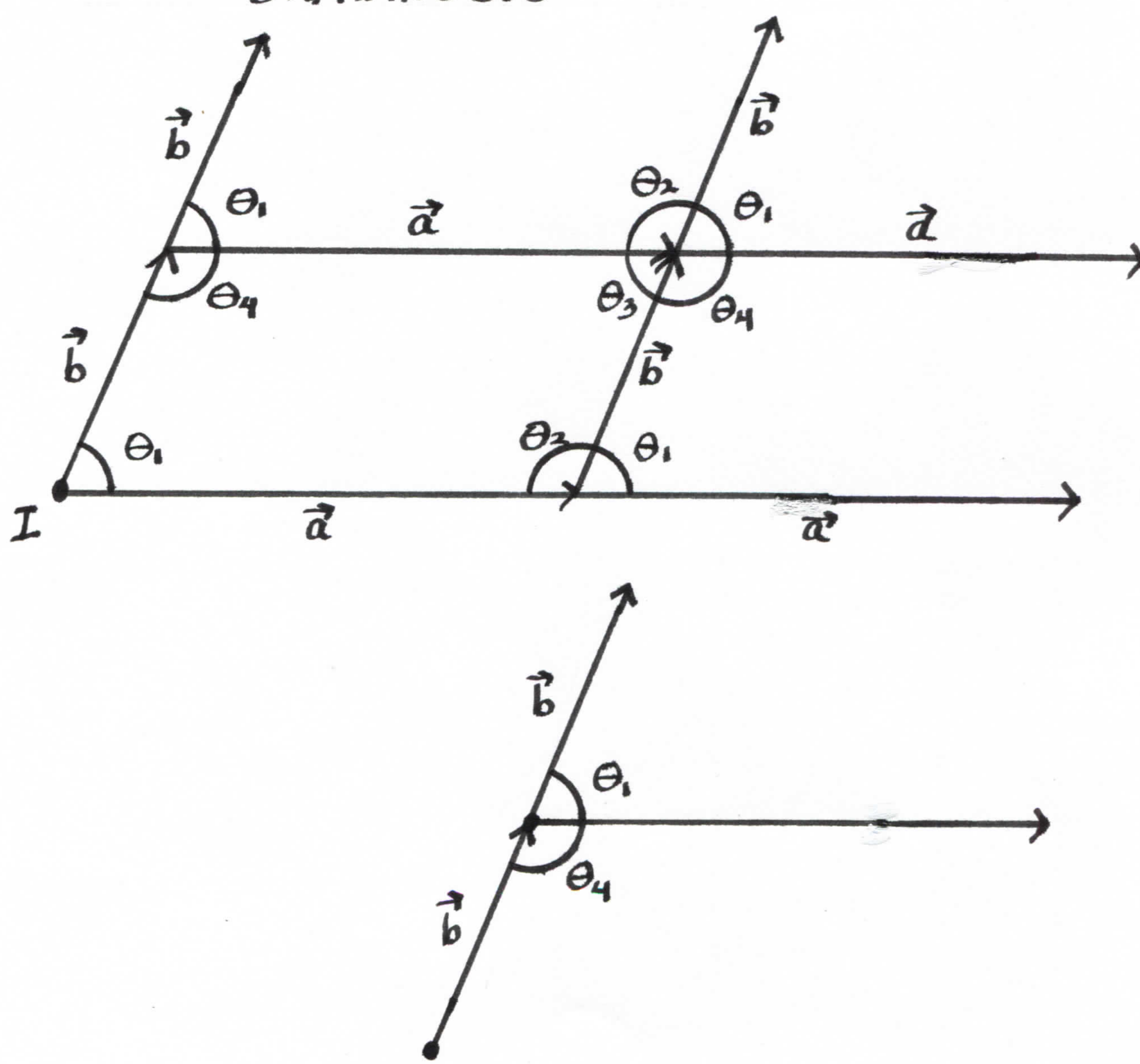


θ_2
θ_1
ℓ_1
θ_3
θ_4
θ_5
$\pi/3$
ℓ_2
θ_6
θ_7

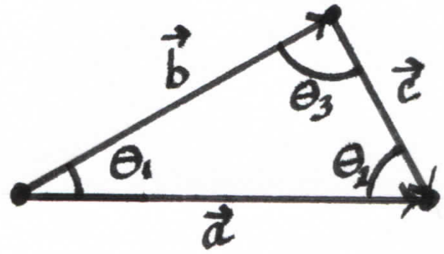
DRAWING 3.7



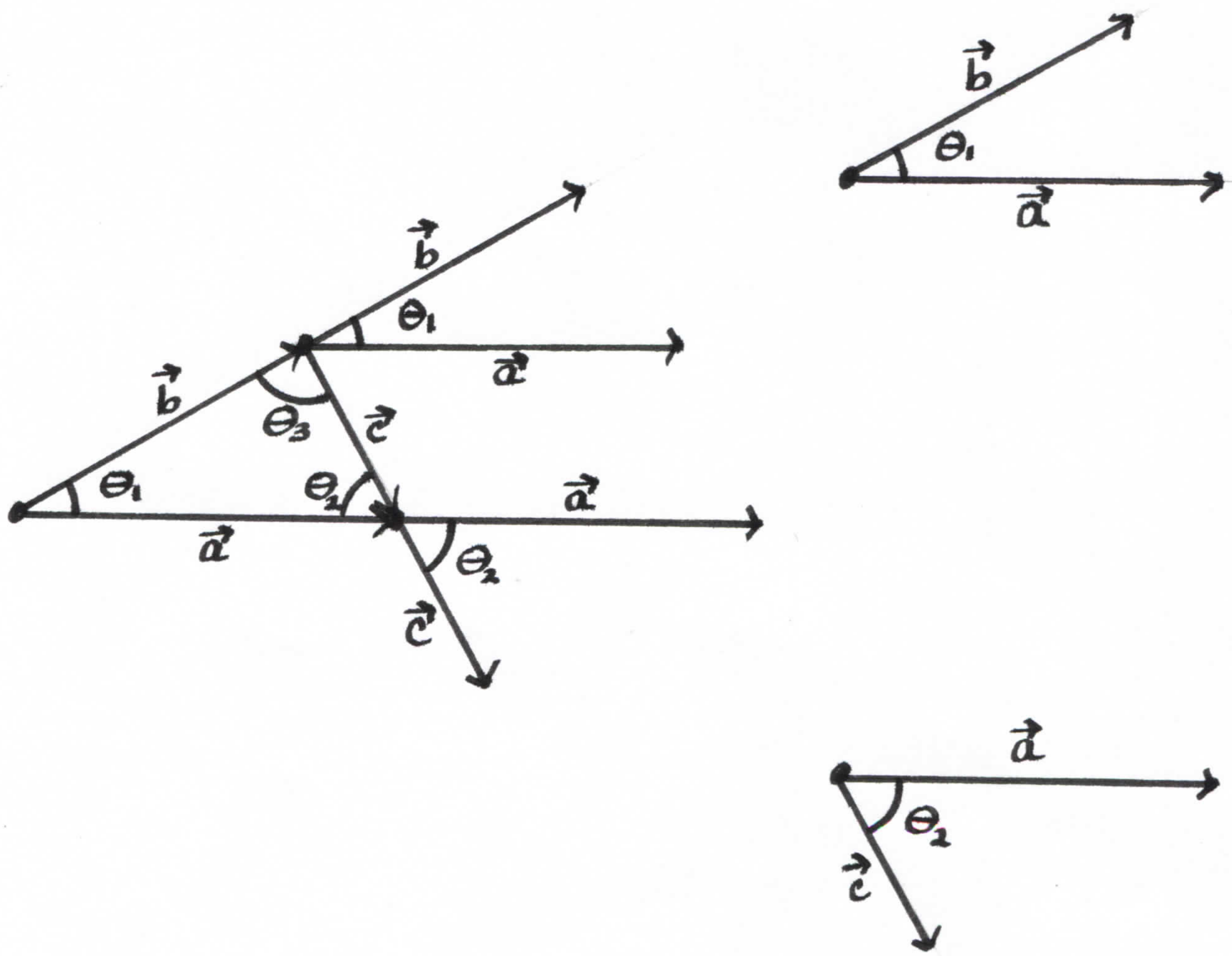
DRAWING 3.8



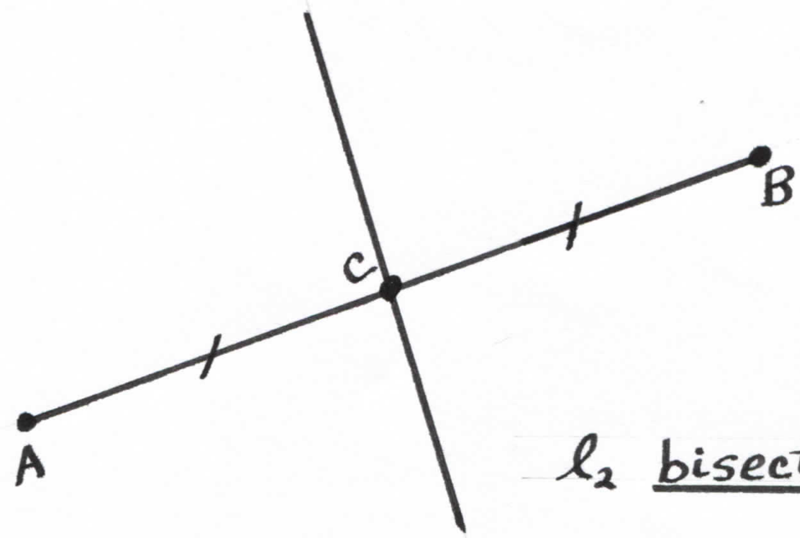
DRAWING 3.9



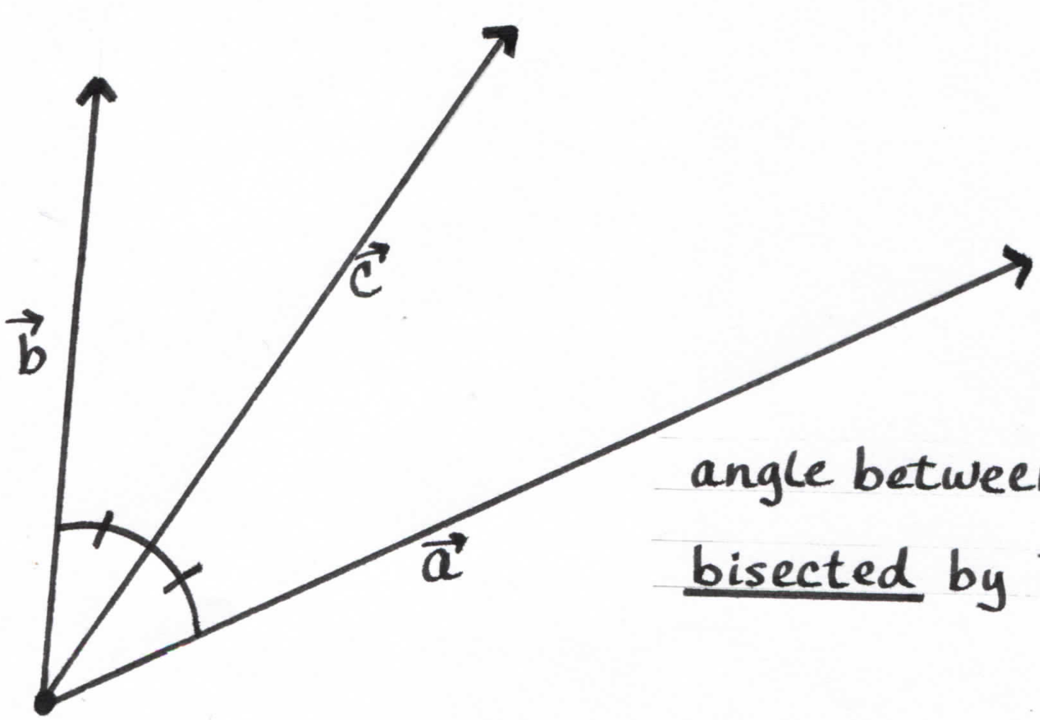
DRAWING 3.10



DRAWINGS 3.11

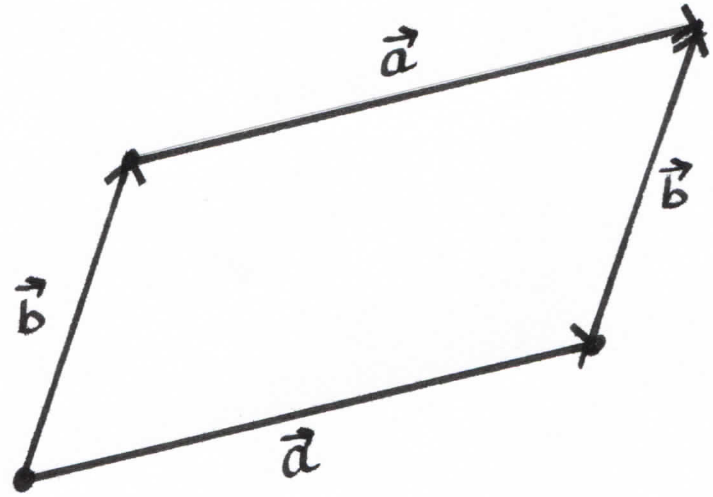


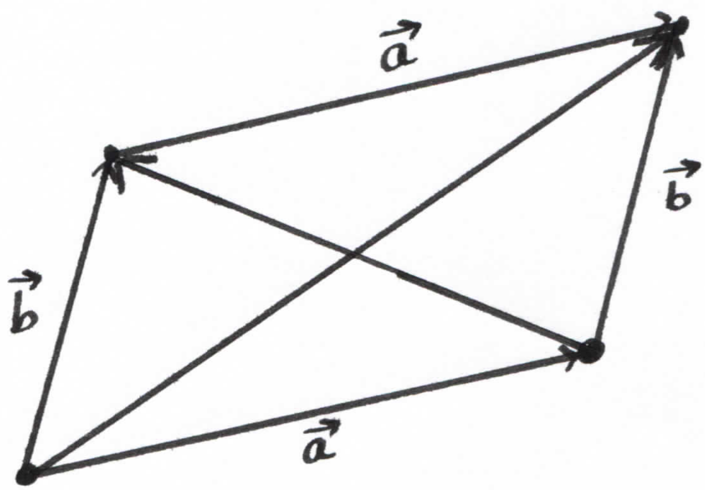
ℓ_2 **bisects** AB



angle between $\vec{a}$ and $\vec{b}$ **bisected** by $\vec{c}$

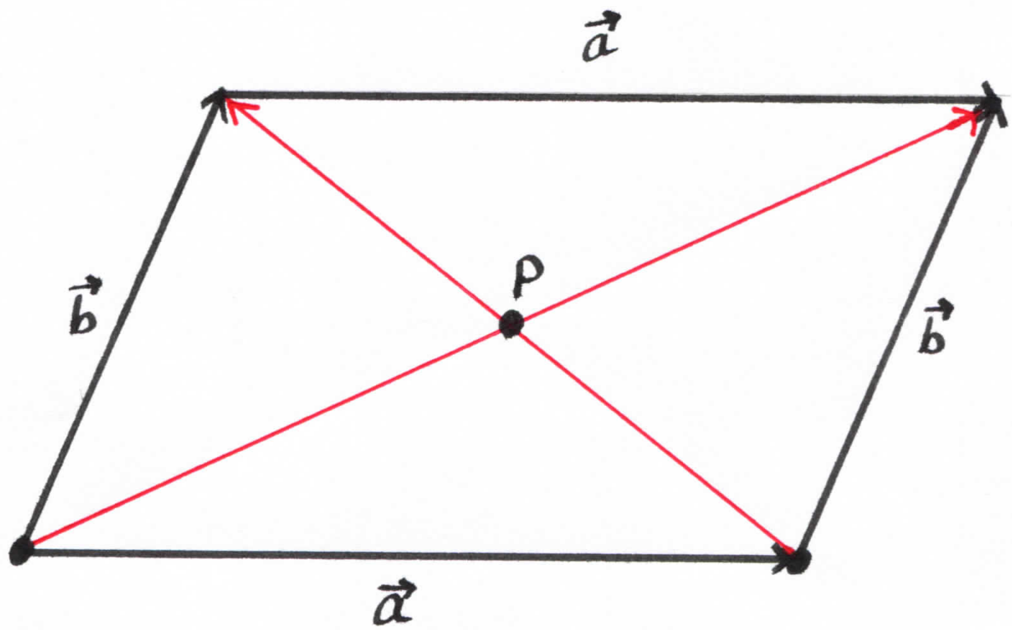
DRAWINGS 3.12





diagonals $(\vec{a}+\vec{b})$ and $(\vec{b}-\vec{a})$

DRAWING 3.13

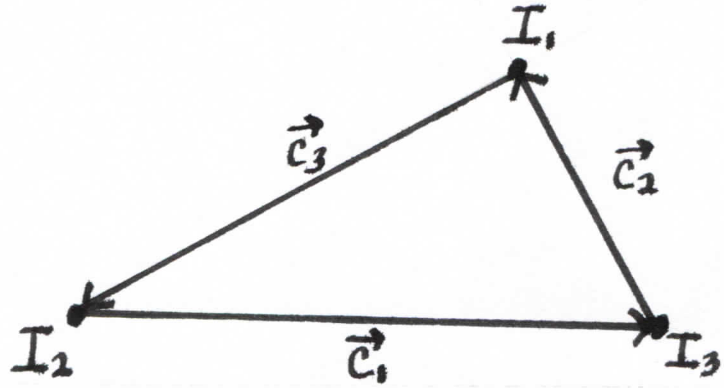


diagonals $(\vec{a}+\vec{b})$ and $(\vec{b}-\vec{a})$

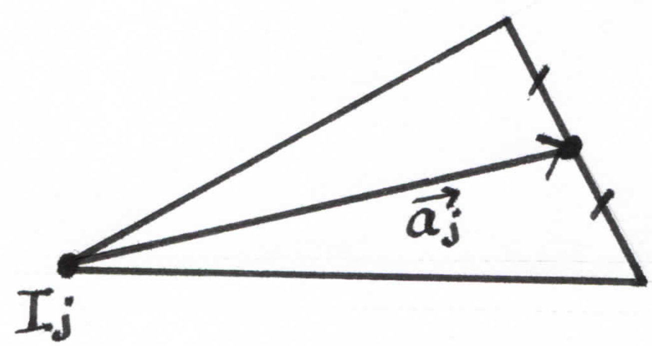
$P = I + s(\vec{a}+\vec{b})$ AND

$P = I + \vec{a} + t(\vec{b}-\vec{a})$

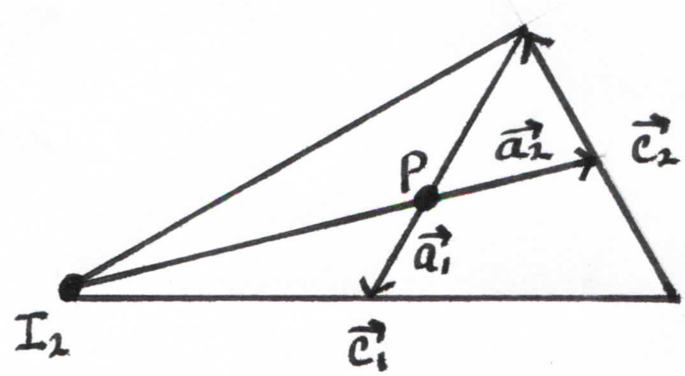
DRAWING 3.14



DRAWING 3.15

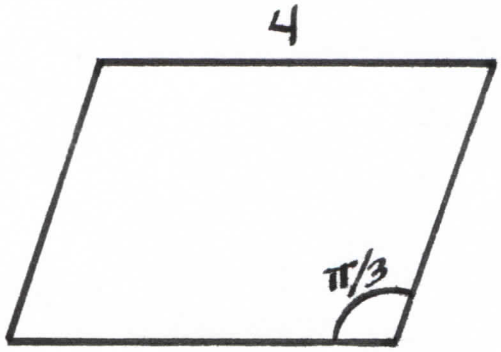


DRAWING 3.16

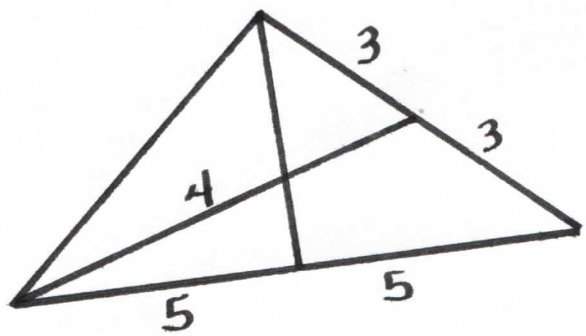


DRAWINGS 3.17

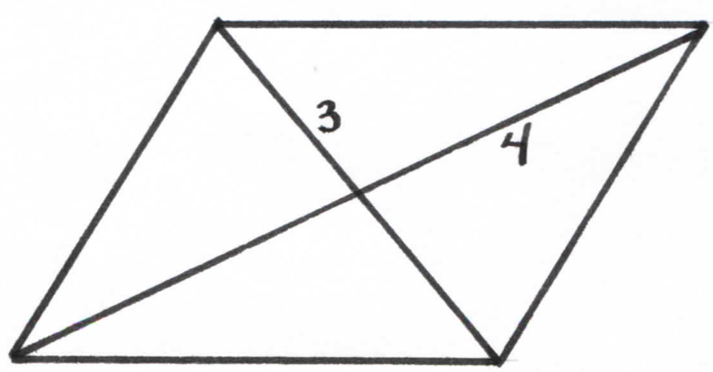
(a)



(b)

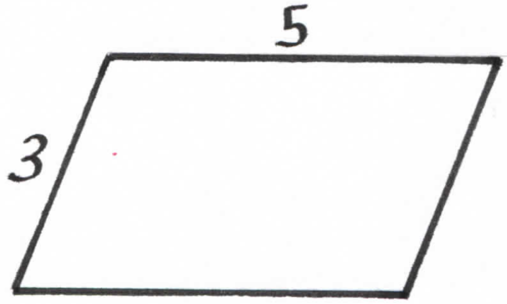


(c)

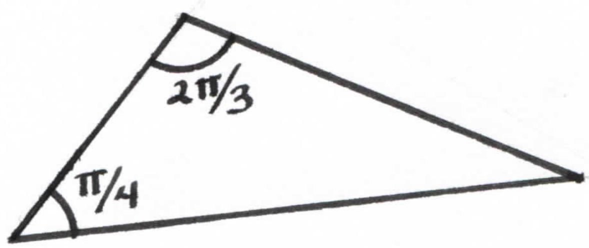


DRAWINGS 3.17 (continued)

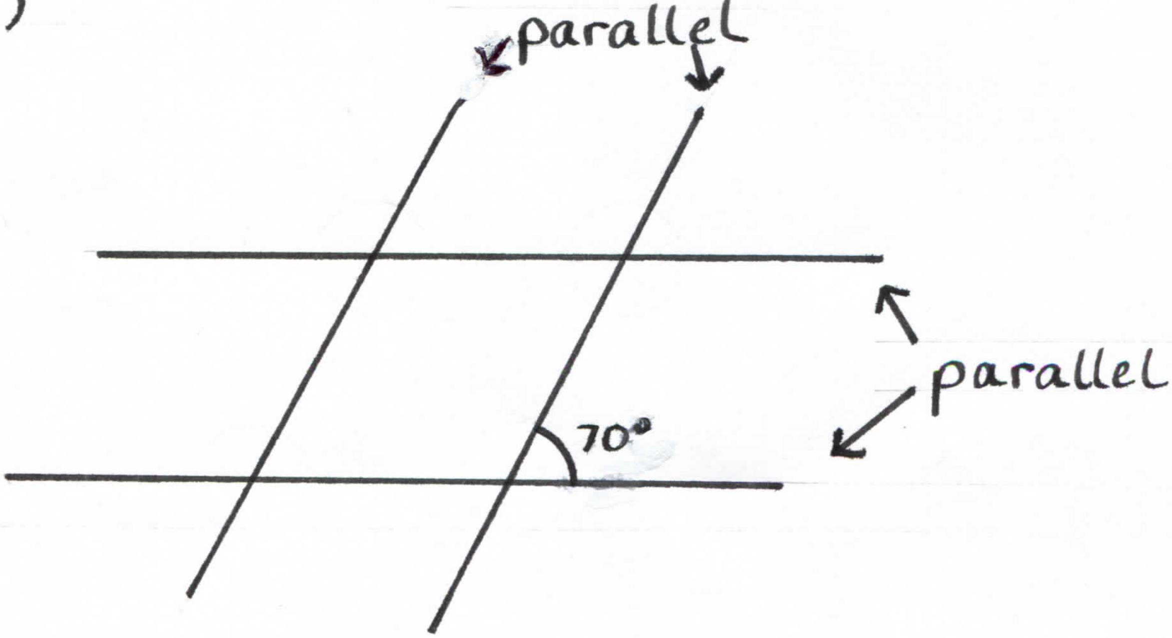
(d)



(e)

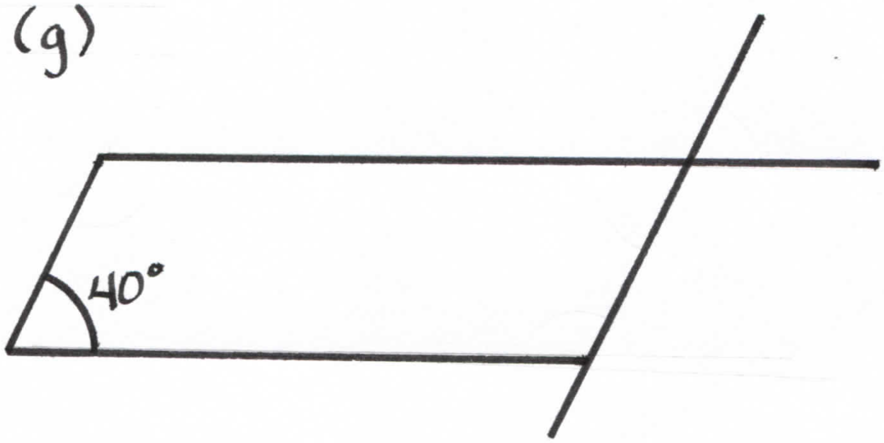


(f)

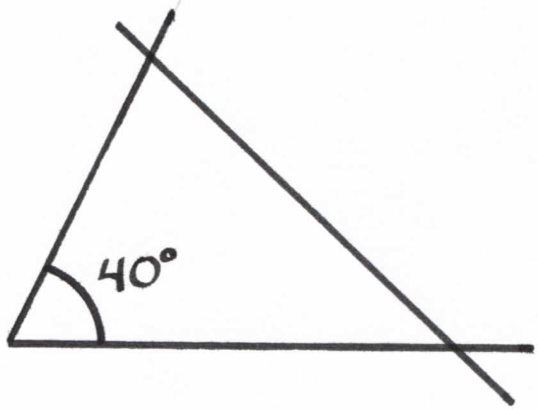


DRAWINGS 3.17 (continued)

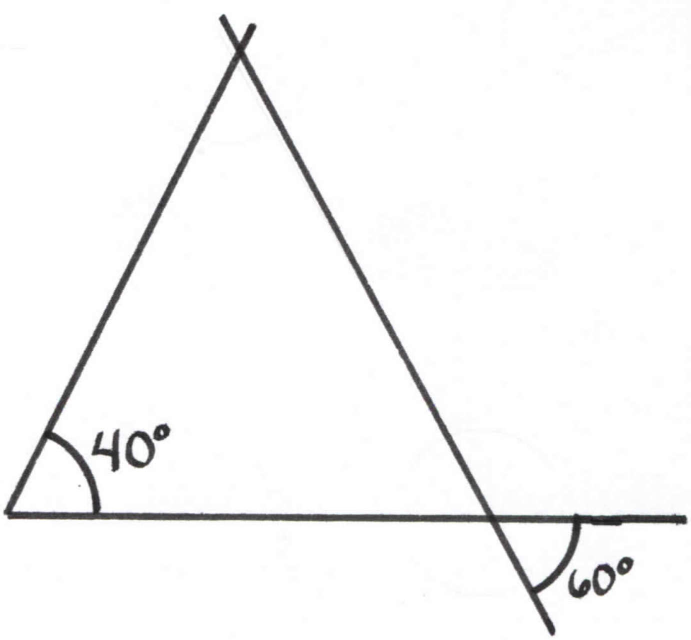
(g)



(h)

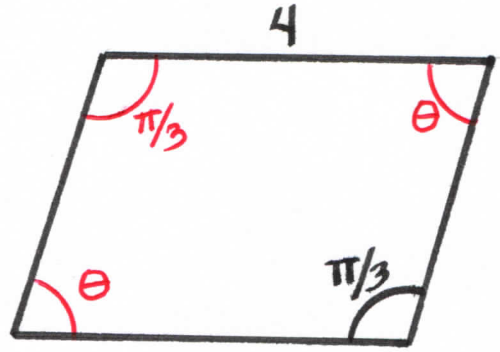


(i)



DRAWINGS 3.18

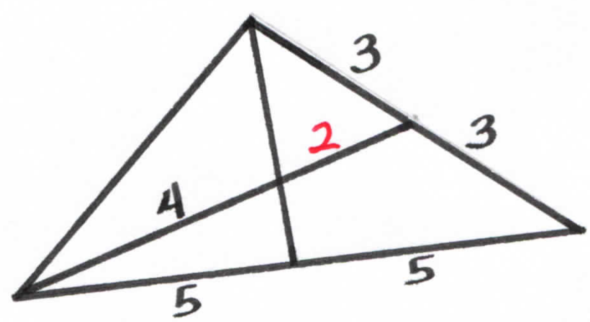
(a)



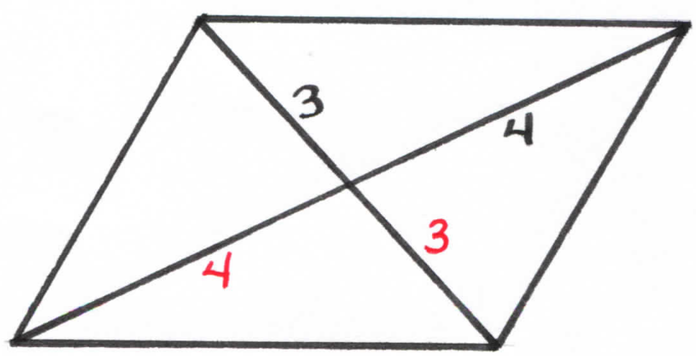
$2\pi = 2\theta + 2\left(\frac{\pi}{3}\right)$

$\rightarrow \theta = \frac{2\pi}{3}$

(b)

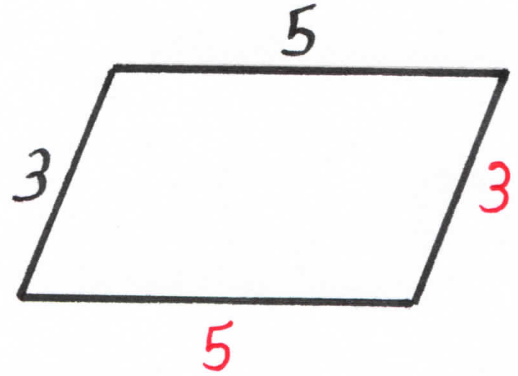


(c)



DRAWINGS 3.18 (continued)

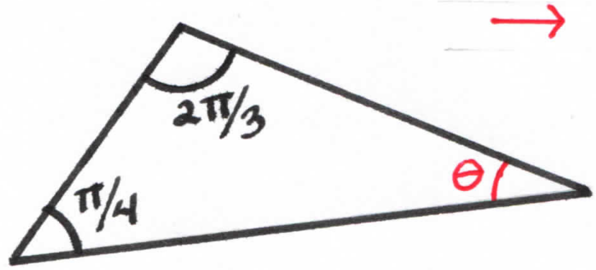
(d)



(e)

$\pi = \theta + \frac{\pi}{4} + \frac{2\pi}{3}$

$\longrightarrow \theta = \frac{\pi}{12}$



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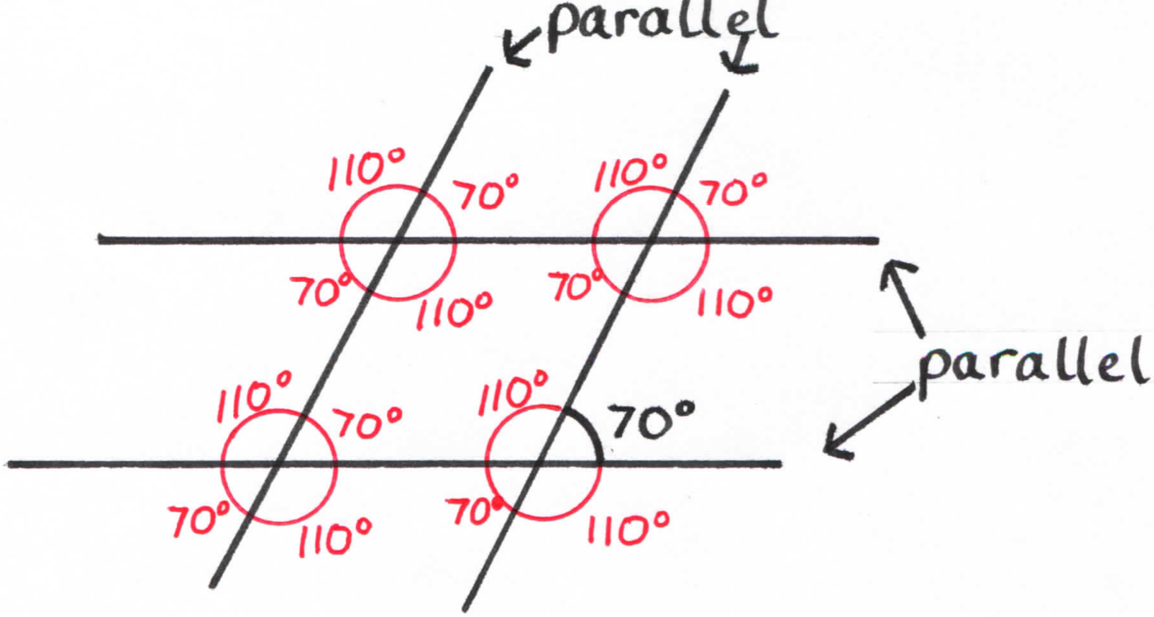
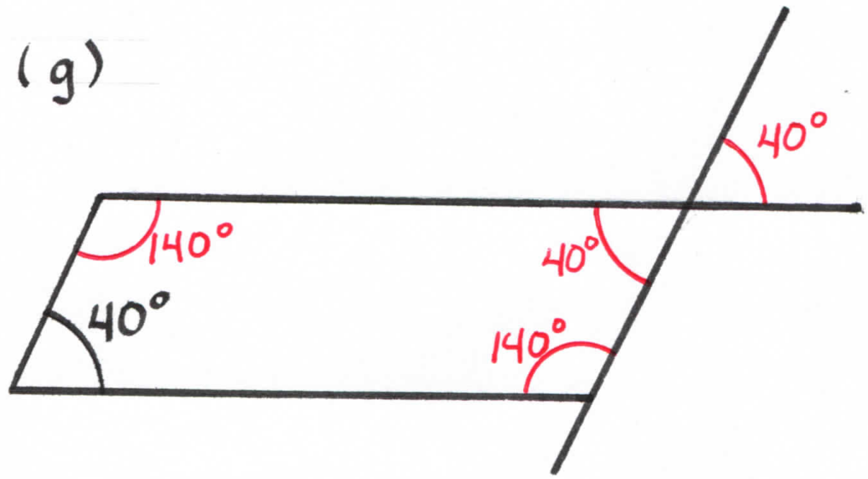
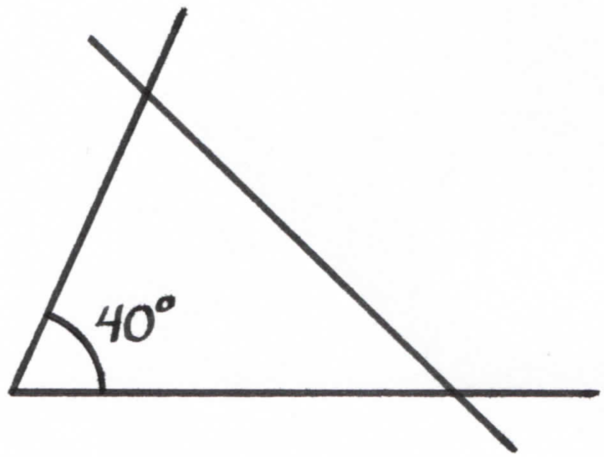


DRAWINGS 3.18 (continued)

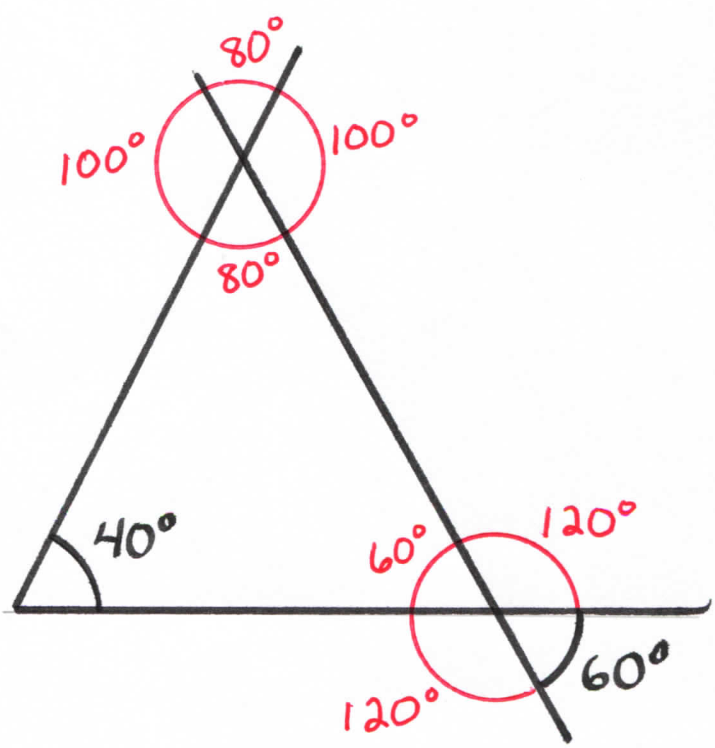
(g)



(h)

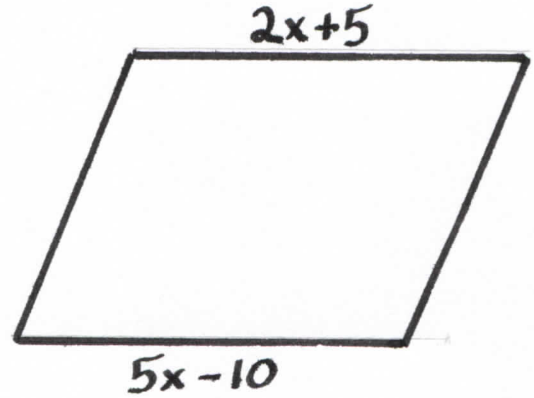


(i)

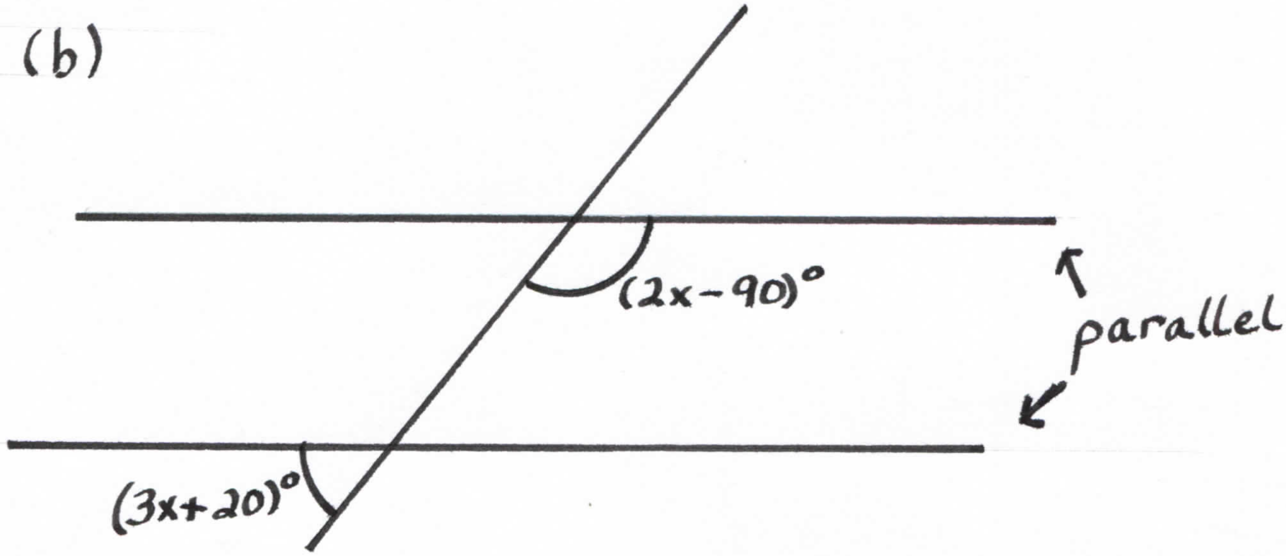


DRAWINGS 3.19

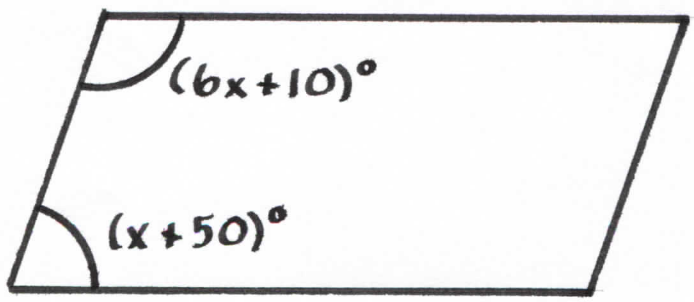
(a)



(b)

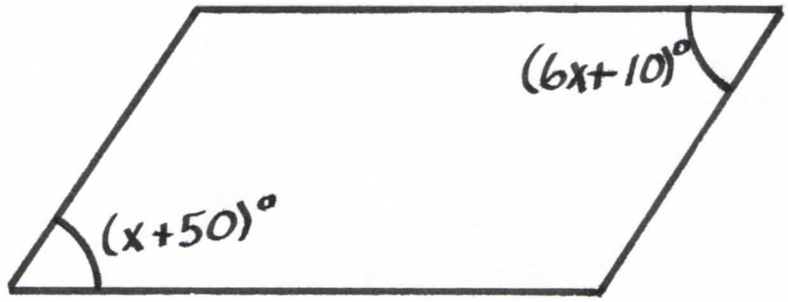


(c)

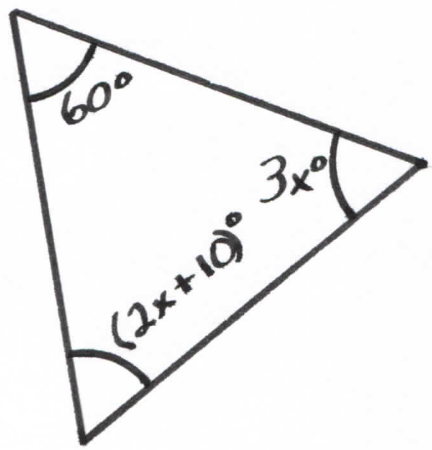


DRAWINGS 3.19 (continued)

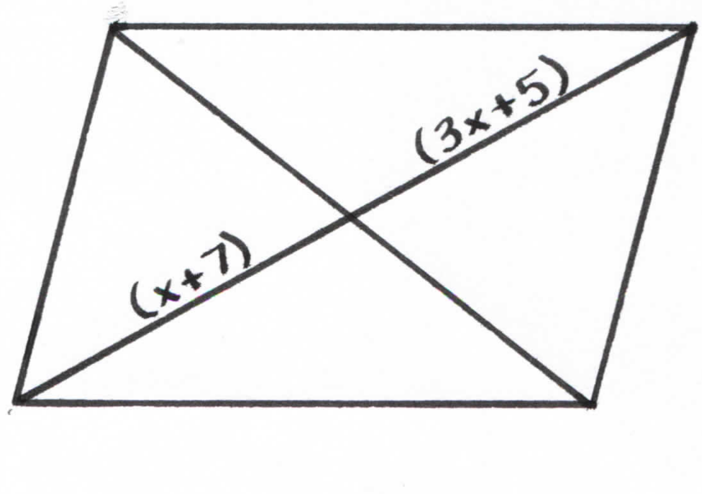
(d)



(e)

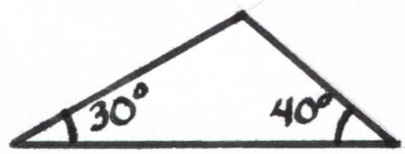


(f)

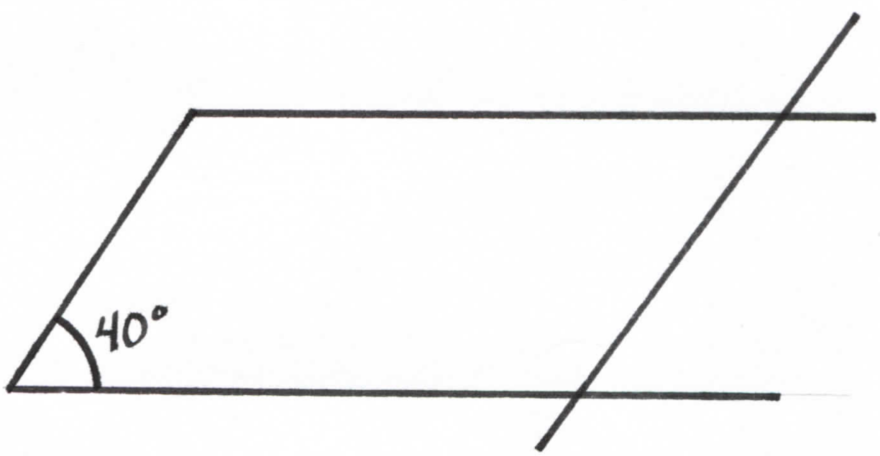


DRAWINGS 3.20

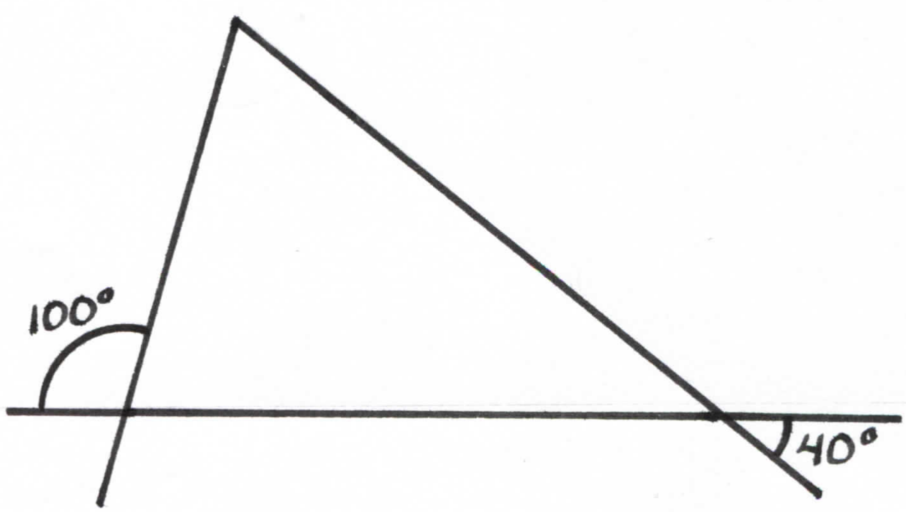
(a)



(b)

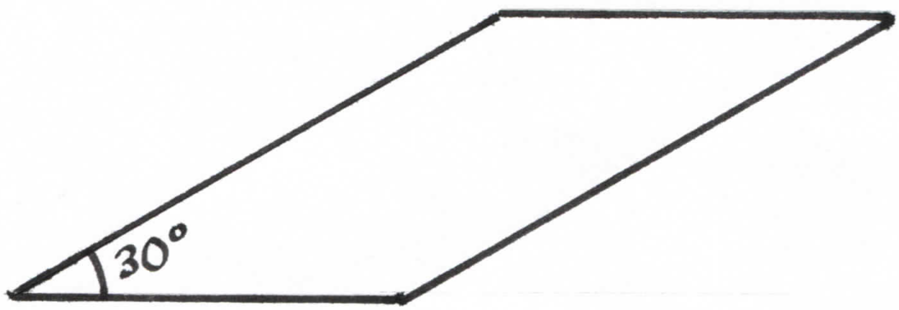


(c)

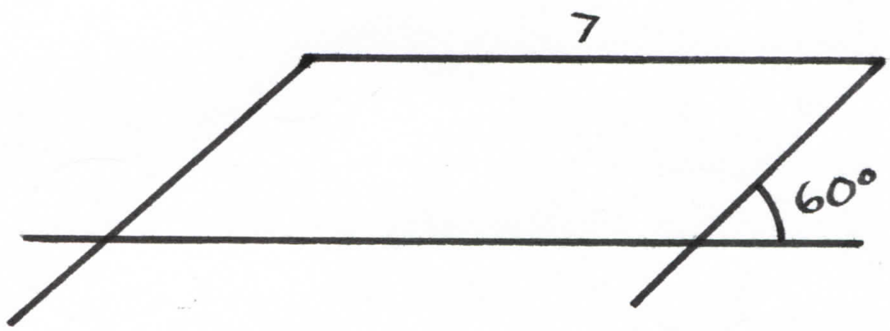


DRAWINGS 3.20 (continued)

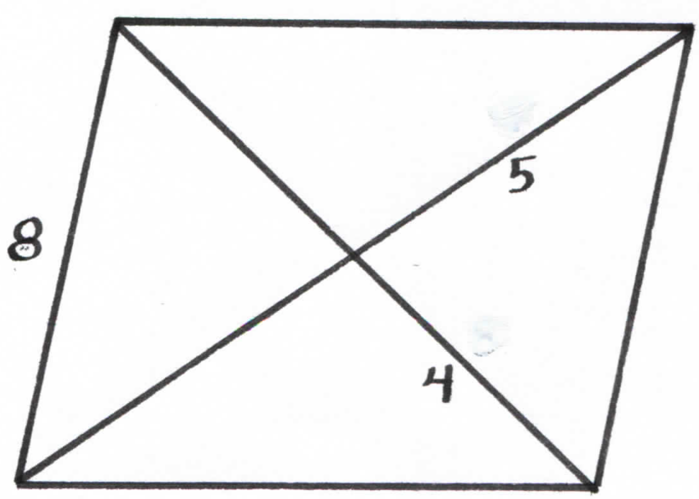
(d)



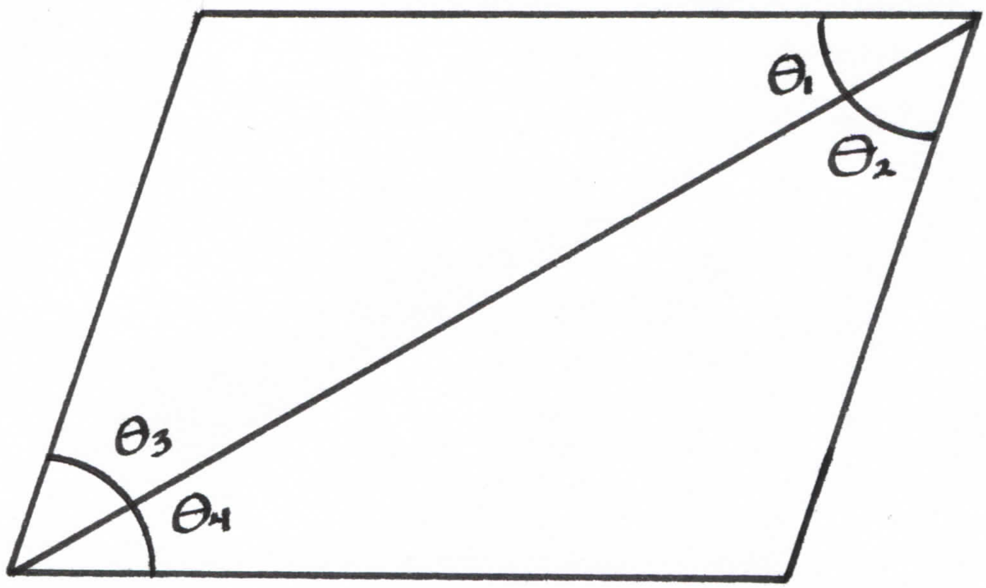
(e)



(f)

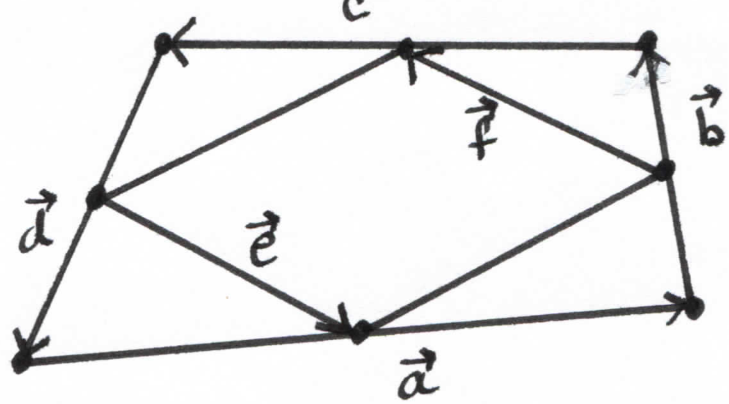


DRAWING 3.21



DRAWING 3.22

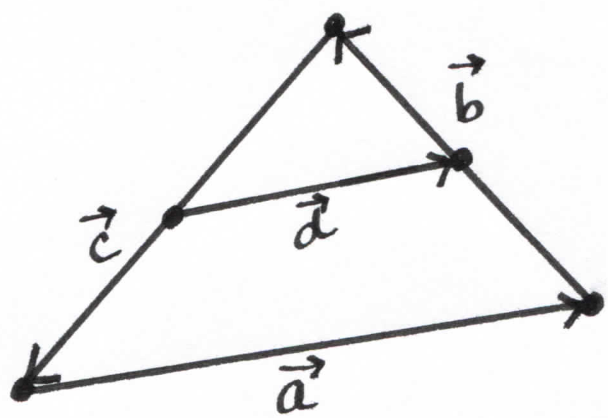
quadrilateral: $\vec{a} + \vec{b} + \vec{c} + \vec{d} = \vec{0}$



Show that $\vec{e} = -\vec{f}$

DRAWING 3.23

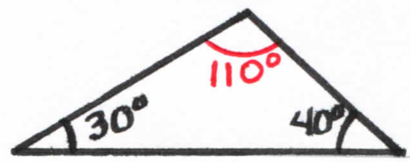
triangle: $\vec{a} + \vec{b} + \vec{c} = \vec{0}$



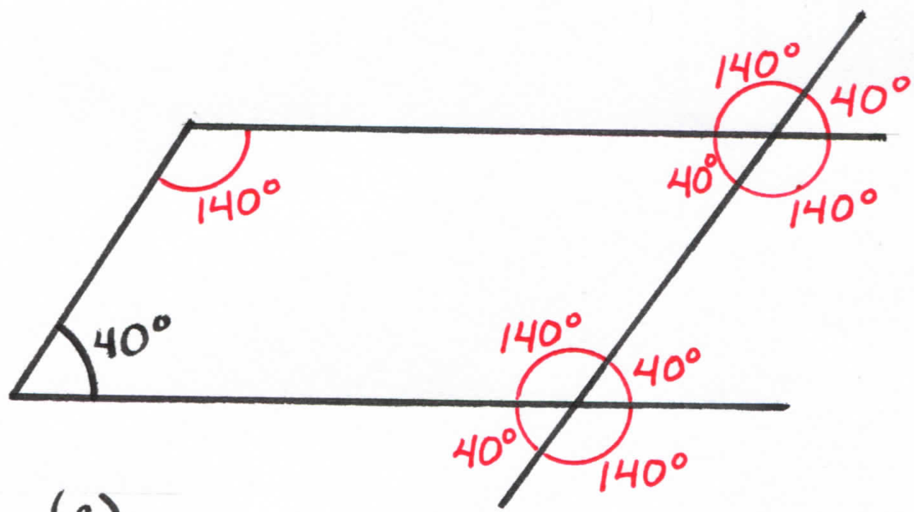
Show that $\vec{a} = 2\vec{d}$

DRAWINGS 3.24

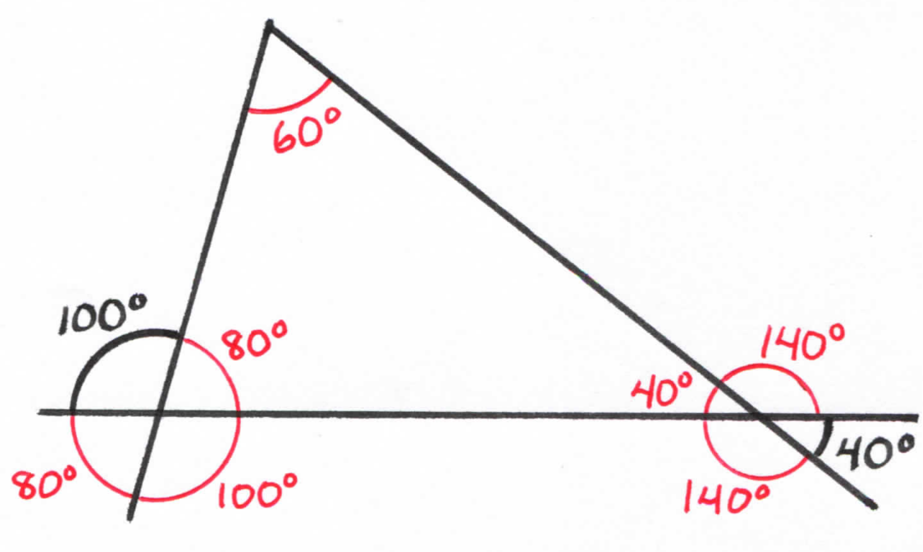
(a)



(b)

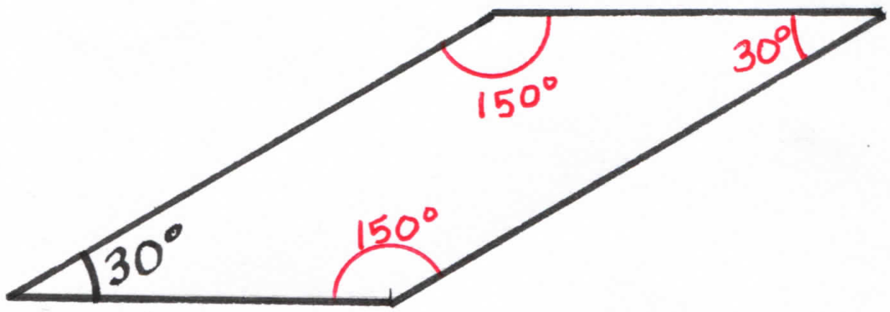


(c)

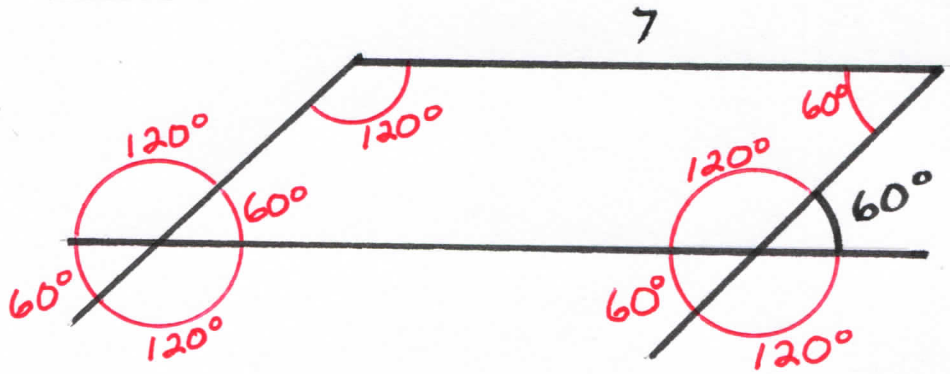


DRAWINGS 3.24 (continued)

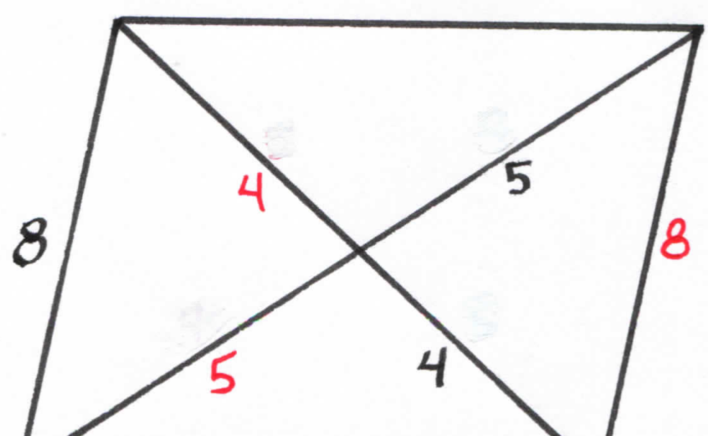
(d)



(e)



(f)



CHAPTER IV: Dot Product and Orthogonality.

The idea of being *perpendicular* or *orthogonal* arises in numerous places. It describes the most comfortable relationship of our body, when standing, to the ground. If our feet are being burned by sand on a beach, the fastest route to the cooling ocean is perpendicular to the shoreline. See DRAWING 4.1 at the end of this chapter.

We would like an algebraic characterization of being perpendicular. Motivation comes from considering the *Pythagorean theorem* (see Proposition 4.15 and DRAWING 4.2 at the end of this chapter).

For any vectors

$$\vec{a} \equiv < a_1, a_2 >, \quad \vec{b} \equiv < b_1, b_2 >,$$

make the following calculation:

$$\begin{aligned}\|\vec{a}+\vec{b}\|^2 = \| < a_1+b_1, a_2+b_2 > \|^2 = (a_1+b_1)^2 + (a_2+b_2)^2 = (a_1^2 + 2a_1b_1 + b_1^2) + (a_2^2 + 2a_2b_2 + b_2^2) \\ = (a_1^2 + a_2^2) + (b_1^2 + b_2^2) + 2(a_1b_1 + a_2b_2) \equiv \|\vec{a}\|^2 + \|\vec{b}\|^2 + 2(a_1b_1 + a_2b_2).\end{aligned}$$

Note that the Pythagorean theorem (see Proposition 4.15 and Definition 4.3) holds precisely when that last quantity $(a_1b_1 + a_2b_2)$ is zero. This motivates the following definition.

Definition 4.1. If $\vec{a}$ and $\vec{b}$ are two vectors, their **dot** or **inner** product is

$$\vec{a} \cdot \vec{b} \equiv (a_1b_1 + a_2b_2).$$

Example 4.2. $< 1, -2 > \cdot < 2, 3 > = 2 - 6 = -4.$

The Pythagorean theorem and our calculations above (see also Corollary 2.14(c) and Remarks 2.15) motivate the following definition.

Definitions 4.3. Two vectors $\vec{a}$ and $\vec{b}$ are said to be **orthogonal** or **perpendicular** if their dot product is zero. This will be denoted $\vec{a} \perp \vec{b}$.

Two lines are perpendicular if their direction vectors are perpendicular (Definitions 2.1).

Example 4.4. Are the vectors $< 1, 2 >$ and $< 3, -2 >$ orthogonal? This is a geometric question, so we feel that we should be able to answer by drawing a picture, as in DRAWING 4.3 at the end of this chapter.

The vectors *look* perpendicular, I think. I hate to rely on anything related to my art ability or sharpness of vision.

But we don't need a picture, because of Definition 4.3. Calculate the dot product of the two vectors

$$< 1, 2 > \cdot < 3, -2 > = 3 - 4 = -1;$$

all we care about is that their dot product is *not* zero, therefore the vectors are *not* perpendicular.

As in Chapter I, we have the interaction of algebra and geometry. We get the best of both worlds: the precision of algebra and the intuition of geometry.

We leave the calculations for the following properties to the reader.

Some Properties 4.5. Suppose $\vec{a}, \vec{b}$, and $\vec{c}$ are vectors and α is a real number.

(a) $\vec{a} \cdot (\vec{b} + \vec{c}) = (\vec{a} \cdot \vec{b}) + (\vec{a} \cdot \vec{c})$.

(b) $\vec{a} \cdot (\alpha \vec{b}) = \alpha(\vec{a} \cdot \vec{b})$.

(c) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$.

(d) $\|\vec{a} + \vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 + 2(\vec{a} \cdot \vec{b})$.

(e) $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$.

For the remainder of this section, let $\vec{b}$ be a nontrivial (that is, not equal to $\vec{0}$) vector, let $\vec{a}$ be a vector or point (see 1.16), and let ℓ be a line with direction vector $\vec{b}$ (see Definitions 2.1).

We would like a best approximation of $\vec{a}$ from ℓ; that is, a point on ℓ that is closest to $\vec{a}$. Intuitively, this means dropping a perpendicular from $\vec{a}$ onto ℓ (see DRAWING 4.4 at the end of this chapter; as at the beginning of this section, think of ℓ as the shoreline, $\vec{a}$ as where you stand on the beach).

Definition 4.6. The **orthogonal projection** of $\vec{a}$ onto ℓ, denoted $\mathrm{proj}_\ell(\vec{a})$, is a point on ℓ such that

$$(\vec{a} - \mathrm{proj}_\ell(\vec{a})) \perp \vec{b}.$$

See DRAWING 4.4 at the end of this chapter.

Note that, if $\vec{a}$ is on the line ℓ, then the projection of $\vec{a}$ onto ℓ is $\vec{a}$ itself: if you're already laid out on the ground, dropping to the ground, as in "drop and give me twenty," means you stay where you are.

The **orthogonal projection of $\vec{a}$ onto $\vec{b}$**, denoted $\mathrm{proj}_{\vec{b}}(\vec{a})$, is the projection of $\vec{a}$ onto the line through $\vec{0}$ and $\vec{b}$; that is, $\mathrm{proj}_{\vec{b}}(\vec{a})$ is a multiple of $\vec{b}$ such that

$$\left(\vec{a} - \mathrm{proj}_{\vec{b}}(\vec{a})\right) \perp \vec{b}.$$

See DRAWING 4.4 at the end of this chapter.

We are premature in our use of the article "the"; we must verify that, if the orthogonal projection exists, it is unique.

Proposition 4.7 (uniqueness of orthogonal projection) Suppose $\vec{x}_1$ and $\vec{x}_2$ both satisfy Definition 4.6; that is, they are both points on ℓ such that

$$(\vec{a} - \vec{x}_j) \perp \vec{b}, \;\; j = 1, 2.$$

Then $\vec{x}_1 = \vec{x}_2$.

Proof: Note that $(\vec{x}_1 - \vec{x}_2)$ is a multiple of $\vec{b}$, thus is orthogonal to $(\vec{a} - \vec{x}_1)$ and $(\vec{a} - \vec{x}_2)$, by Properties 4.5(b). This (see Definition 4.3 and Properties 4.5(d)) and the facts that

$$(\vec{a} - \vec{x}_1) = (\vec{a} - \vec{x}_2) + (\vec{x}_2 - \vec{x}_1), \;\; (\vec{a} - \vec{x}_2) = (\vec{a} - \vec{x}_1) + (\vec{x}_1 - \vec{x}_2),$$

imply

$$\|\vec{a} - \vec{x}_2\|^2 = \|\vec{a} - \vec{x}_1\|^2 + \|\vec{x}_1 - \vec{x}_2\|^2 = \|\vec{a} - \vec{x}_2\|^2 + \|\vec{x}_2 - \vec{x}_1\|^2 + \|\vec{x}_1 - \vec{x}_2\|^2 = \|\vec{a} - \vec{x}_2\|^2 + 2\|\vec{x}_1 - \vec{x}_2\|^2,$$

so that

$$\|\vec{x}_1 - \vec{x}_2\| = 0,$$

which implies that $\vec{x}_1 = \vec{x}_2$, as desired. □

At this point, we do not know that this projection exists (see Theorem 4.11). But if it does, it has a desirable property: it gives us the best approximation of $\vec{a}$ by points in ℓ (see Corollary 4.10 and DRAWING 4.4 at the end of this chapter).

Theorem 4.8. For any point $\vec{y}$ on ℓ,
$$\|\vec{a}-\vec{y}\|^2 = \|(\vec{a}-\text{proj}_\ell(\vec{a}))\|^2 + \|(\text{proj}_\ell(\vec{a})-\vec{y})\|^2.$$

Proof: Since $(\text{proj}_\ell(\vec{a})-\vec{y})$ is a multiple of $\vec{b}$,
$$\|\vec{a}-\vec{y}\|^2 = \|(\vec{a}-\text{proj}_\ell(\vec{a}))+(\text{proj}_\ell(\vec{a})-\vec{y})\|^2 = \|(\vec{a}-\text{proj}_\ell(\vec{a}))\|^2 + \|(\text{proj}_\ell(\vec{a})-\vec{y})\|^2,$$
by 4.5(d), since $(\vec{a}-\text{proj}_\ell(\vec{a}))$ is orthogonal to all multiples of $\vec{b}$, by Properties 4.5(b). □

Corollary 4.9. $\|\vec{a}\|^2 = \|a-\text{proj}_\ell(\vec{a})\|^2 + \|\text{proj}_\ell(\vec{a})\|^2$.

Corollary 4.10. $\text{proj}_\ell(\vec{a})$ minimizes the distance from $\vec{a}$ to points on the line ℓ; that is,
$$\|\vec{a}-\vec{y}\| \geq \|\vec{a}-\text{proj}_\ell(\vec{a})\|,$$
for any $\vec{y}$ in ℓ.

Proof: This follows from Theorem 4.8, since norm is always nonnegative. □

Using dot product, it is not hard to get an algebraic formula for $\text{proj}_\ell(\vec{a})$. Note first that, for any $\vec{x}_0$ in ℓ, $\text{proj}_\ell(\vec{a}) = \vec{x}_0 + t_0\vec{b}$, for some number t_0 (depending on $\vec{x}_0$). The desired orthogonality implies that
$$0 = (\vec{a}-(\vec{x}_0+t_0\vec{b}))\cdot\vec{b} = \vec{a}\cdot\vec{b}-\vec{x}_0\cdot\vec{b}-t_0\vec{b}\cdot\vec{b} = (\vec{a}-\vec{x}_0)\cdot\vec{b}-t_0(\vec{b}\cdot\vec{b});$$
we may solve for t_0:
$$t_0 = \frac{(\vec{a}-\vec{x}_0)\cdot\vec{b}}{\vec{b}\cdot\vec{b}} = \frac{(\vec{a}-\vec{x}_0)\cdot\vec{b}}{\|\vec{b}\|^2},$$
so that Definition 4.6 is brought to life.

Theorem 4.11. The orthogonal projection $\text{proj}_\ell(\vec{a})$ exists and, for any $\vec{x}_0$ in ℓ, direction vector $\vec{b}$ for ℓ, equals
$$\left[\vec{x}_0 + \left(\frac{(\vec{a}-\vec{x}_0)\cdot\vec{b}}{\|\vec{b}\|^2}\right)\vec{b}\right].$$

Proof: As in deriving the formula for the projection, look at
$$\left(\vec{a}-\left[\vec{x}_0+\left(\frac{(\vec{a}-\vec{x}_0)\cdot\vec{b}}{\|\vec{b}\|^2}\right)\vec{b}\right]\right)\cdot\vec{b} = (\vec{a}-\vec{x}_0)\cdot\vec{b}-\left(\frac{(\vec{a}-\vec{x}_0)\cdot\vec{b}}{\|\vec{b}\|^2}\right)(\vec{b}\cdot\vec{b}) = (\vec{a}-\vec{x}_0)\cdot\vec{b}-\left(\frac{(\vec{a}-\vec{x}_0)\cdot\vec{b}}{\|\vec{b}\|^2}\right)\|\vec{b}\|^2 = 0.$$
□

Corollary 4.12. The orthogonal projection $\text{proj}_{\vec{b}}(\vec{a})$ exists and equals $\left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\right)\vec{b}$.

Examples 4.13. (a) Get the orthogonal projection of $<-1,-2>$ onto $<2,-3>$.

(b) Get the orthogonal projection of $(-1,2)$ onto the line through $(2,1)$ and $(5,2)$.

Solutions. (a) This is Corollary 4.12, with $\vec{a}\equiv<-1,-2>$ and $\vec{b}\equiv<2,-3>$, thus we want
$$\text{proj}_{<2,-3>}(<-1,-2>) = \left(\frac{<-1,-2>\cdot<2,-3>}{\|<2,-3>\|^2}\right)<2,-3> = \left(\frac{-2+6}{4+9}\right)<2,-3> = \frac{4}{13}<2,-3>.$$
See DRAWING 4.5(a) at the end of this chapter.

(b) In the language of Definition 4.6, $\vec{a}=(-1,2)$ (or $<-1,2>$; see 1.16) and ℓ is the line through $(2,1)$ and $(5,2)$. To exploit Theorem 4.11, we need a direction vector $\vec{b}$; let's choose
$$\vec{b}\equiv\overrightarrow{(2,1)(5,2)} = <3,1>.$$

Finally, we need a point on the line; we could use

$$\vec{x}_0 \equiv (2,1).$$

Theorem 4.11 now gives us

$$\text{proj}_\ell(-1,2) = \left[(2,1) + \left(\frac{(<-1,2> - <2,1>)\cdot <3,1>}{\| <3,1> \|^2}\right) <3,1>\right] = \left[(2,1) + \left(\frac{(<-3,1>)\cdot <3,1>}{10}\right) <3,1>\right]$$

$$= \left[(2,1) + \left(\frac{(-8)}{10}\right) <3,1>\right] = \frac{1}{10}\left[(20,10) - <24,8>\right] = \frac{1}{10}(-4,2).$$

What if we had chosen $\vec{x}_0 = (5,2)$ instead of $(2,1)$? Let's run through the same calculations; it will be disturbing if we get a different answer:

$$\text{proj}_\ell(-1,2) = \left[(5,2) + \left(\frac{(<-1,2> - <5,2>)\cdot <3,1>}{\| <3,1> \|^2}\right) <3,1>\right] = \left[(5,2) + \left(\frac{(<-6,0>)\cdot <3,1>}{10}\right) <3,1>\right]$$

$$= \left[(5,2) + \left(\frac{(-18)}{10}\right) <3,1>\right] = \frac{1}{10}\left[(50,20) - <54,18>\right] = \frac{1}{10}(-4,2),$$

exactly what we got when we used $\vec{x}_0 = (2,1)$.

In general, Theorem 4.11 guarantees that we will get the same result (namely, the unique projection onto the line) regardless of which point $\vec{x}_0$ we choose on the line.

Let's check directly that our alleged projection $\frac{1}{10}(-4,2)$ satisfies Definition 4.6:

$$\left(<-1,2> - \frac{1}{10} <-4,2>\right)\cdot <3,1> = \frac{1}{10}\left(<-10,20> - <-4,2>\right)\cdot <3,1> = \frac{1}{10}\left(<-6,18>\right)\cdot <3,1> = 0,$$

so that

$$\left(<-1,2> - \frac{1}{10} <-4,2>\right) \perp <3,1>,$$

as demanded by Definition 4.6. See DRAWING 4.5(b) at the end of this chapter.

Let's reformulate the dot product in terms of complex numbers. See 1.15 and 2.8 for the *polar form* of a complex number, and 1.16 for the equating of vector with complex number. Explicitly, if $\vec{a} \equiv <x_1, y_1>$ is a vector, then "$\vec{a} = re^{i\theta}$" means that $re^{i\theta}$ is the polar form of the complex number $x_1 + iy_1$.

Proposition 4.14. Suppose $x_1, x_2, y_1, y_2, \theta_1, \theta_2$ are real numbers, r_1 and r_2 are positive nummbers,

$$z_1 = x_1 + iy_1 = r_1 e^{i\theta_1} \quad \text{and} \quad z_2 = x_2 + iy_2 = r_2 e^{i\theta_2}.$$

Then

$$<x_1, y_1> \cdot <x_2, y_2> = \text{Re}(z_1 \overline{z_2}) = \text{Re}(r_1 r_2 e^{i(\theta_1 - \theta_2)}).$$

Proof:

$$z_1 \overline{z_2} = (x_1 + iy_1)(x_2 - iy_2) = (x_1x_2 + y_1y_2) + i(y_1x_2 - x_1y_2);$$

also,

$$z_1 \overline{z_2} = r_1 e^{i\theta_1} \overline{r_2 e^{i\theta_2}} = (r_1 r_2 e^{i(\theta_1 - \theta_2)}),$$

by Theorem 1.12. $\square$

We have not identified the measure of the angle between orthogonal vectors, or even shown that it is unique. Corollary 2.14(c) asserts that the counterclockwise angle from $<x_1, y_1>$ to $<-y_1, x_1>$ is $\frac{\pi}{2}$. Since $<x_1, y_1> \cdot <-y_1, x_1> = 0$ (that is, $<x_1, y_1>$ and $<-y_1, x_1>$ are orthogonal), $\frac{\pi}{2}$ is looking like a viable candidate for the measure of the angle between orthogonal vectors.

Proposition 4.15. Suppose $\vec{a}$ and $\vec{b}$ are two nontrivial vectors. Then the following are equivalent.

(a) $\vec{a}$ and $\vec{b}$ are orthogonal.

(b) (Pythagorean theorem) $\|\vec{a}+\vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2$.

(c) $\|\vec{a}+s\vec{b}\| \geq \|\vec{a}\|$ for all real s.

(d) The measure of the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{2}$.

Proof: The equivalence of (a) and (b) follows from 4.5(d) (see Definition 4.3).

(a) $\iff$ (d). As we discussed before the proof of Proposition 4.14, denote $\vec{a} = r_1 e^{i\theta_1}, \vec{b} = r_2 e^{i\theta_2}$, with $0 \leq \theta_1 \leq \theta_2 \leq \theta_1 + 2\pi, r_1 > 0, r_2 > 0$. By Definition 4.3 and Proposition 4.14,

$$\text{(a)} \iff \vec{b}\cdot\vec{a} = 0 \iff \mathrm{Re}\left(r_2 r_1 e^{i(\theta_2-\theta_1)}\right) = 0 \iff (\theta_2-\theta_1) = \frac{\pi}{2} \text{ or } \frac{3\pi}{2},$$

which is equivalent to (d), since the counterclockwise measure from $\vec{a}$ to $\vec{b}$ is $(\theta_2-\theta_1)$ (see Definitions 2.10).

Regarding the equivalence of (a) and (c), define a function $F : \mathbf{R} \to \mathbf{R}$ by

$$F(s) \equiv \|\vec{a}+s\vec{b}\|^2 - \|\vec{a}\|^2.$$

Assertion (c) is equivalent to $F(s) \geq 0$ for all real s.

Apply 4.5(d), and complete the square:

$$F(s) = s^2\|\vec{b}\|^2 + 2s(\vec{a}\cdot\vec{b}) = \|\vec{b}\|^2\left[s^2 + 2s\left(\frac{a\cdot\vec{b}}{\|\vec{b}\|^2}\right) + \left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\right)^2 - \left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\right)^2\right] = \|\vec{b}\|^2\left(s + \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\right)^2 - \left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|}\right)^2$$

Assertion (a) implies that $F(s) = s^2\|\vec{b}\|^2 \geq 0$, which gives us (c). If (a) is not true, then

$$F\left(-\frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\right) = -\left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|}\right)^2 < 0,$$

so that (c) does not hold. This shows the equivalence of (a) and (c). $\square$

Remark 4.16. The Pythagorean theorem is ultimately a consequence of our definition of norm (Definitions 2.4(i)). For example, if we chose the norm

$$\| < x, y > \| \equiv |x| + |y|,$$

it can be shown that there is no dot product with the properties of 4.5, so that defining orthogonal vectors is problematic.

Examples 4.17. Which of the following pairs of vectors are orthogonal?

(a) $\{< 1, 2 >, < -2, 1 >\}$.

(b) $\{< 4, -2 >, < -2, 1 >\}$.

(c) $\{< 1, 2 >, < 3, 4 >\}$.

(d) $\{< 5, 17 >, < 0, 0 >\}$.

Solutions. The pairs in (a) and (d) are orthogonal, since their dot products are zero. The other pairs are not orthogonal, since their dot products are not zero. Notice that the pair in (c) is neither orthogonal nor parallel (see Examples 1.17).

As in Examples 1.17(d), many people are uncomfortable with $< 0, 0 >$ being orthogonal to anything, since it implies that $< 0, 0 >$ has a direction.

Since this chapter has been devoted to the geometric idea of orthogonality (being perpendicular), we cannot resist finishing with a believable relationship between perpendicular lines and parallel lines

(Corollary 4.20; see Proposition 4.18 for a special case), since parallel may be considered a sort of opposite of perpendicular.

See Theorem 5.2 for a more complete list of relationships between parallel and perpendicular.

Notice that the proof of the very geometric result in Proposition 4.18 is entirely algebraic; more explicitly, it relies on our dot product characterization of orthogonality (Definition 4.3) and our algebraic definition of parallel (Definitions 1.5).

Proposition 4.18. If $\vec{a}, \vec{b}$, and $\vec{c}$ are vectors, with $\vec{c}$ nontrivial, $\vec{a} \perp \vec{c}$ and $\vec{b} \perp \vec{c}$, then $\vec{a}$ and $\vec{b}$ are parallel.

Proof: Denote

$$\vec{a} \equiv < a_1, a_2 >, \ \vec{b} \equiv < b_1, b_2 >, \ \vec{c} \equiv < c_1, c_2 > .$$

First suppose neither c_1 nor c_2 equals zero. Then

$$0 = \vec{a} \cdot \vec{c} = a_1 c_1 + a_2 c_2$$

implies that

$$(*) \ \ a_1 = \frac{-a_2 c_2}{c_1}.$$

Identically, $0 = \vec{b} \cdot \vec{c}$ implies that

$$(**) \ \ b_1 = \frac{-b_2 c_2}{c_1}.$$

If either $\vec{a}$ or $\vec{b}$ is the trivial vector $\vec{0}$, then $\vec{a}$ and $\vec{b}$ are trivially parallel.

If neither $\vec{a}$ nor $\vec{b}$ is the trivial vector, then (*) implies that both a_1 and a_2 are nonzero, while (**) implies that both b_1 and b_2 are nontrivial. Then, again by (*) and (**),

$$\gamma \equiv \frac{a_1}{b_1} = \frac{\left(\frac{-a_2 c_2}{c_1}\right)}{\left(\frac{-b_2 c_2}{c_1}\right)} = \frac{a_2}{b_2},$$

so that

$$\vec{a} = \gamma \vec{b};$$

that is, $\vec{a}$ and $\vec{b}$ are parallel.

We leave it to the reader to finish this proof by showing

$$(1) \ \ c_1 = 0 \rightarrow \vec{a} =< a_1, 0 > \ \text{ and } \vec{b} =< b_1, 0 >$$

and

$$(2) \ \ c_2 = 0 \rightarrow \vec{a} =< 0, a_2 > \ \text{ and } \vec{b} =< 0, b_2 >;$$

in either (1) or (2), $\vec{a}$ and $\vec{b}$ are parallel. □

Remark 4.19. We could also prove Proposition 4.18 "by contradiction" (see Appendix 0); that is, by hypothesizing that $\vec{a}$ and $\vec{b}$ are *not* parallel. Denoting by ℓ_c a line with direction vector $\vec{c}$, by ℓ_a a line with direction vector $\vec{a}$ that intersects ℓ_c, and by ℓ_b a line with direction vector $\vec{b}$ that intersects ℓ_c at a different point than ℓ_a, as drawn in DRAWING 4.6 at the end of this chapter, we see that the shaded triangle would then violate Proposition 3.9.

Corollary 4.20 (same as Corollary 5.3). Suppose $\ell_1, \ell_2, \ell_3, \ell_4$ are lines, ℓ_1 and ℓ_2 are parallel, and both ℓ_3 and ℓ_4 are perpendicular to ℓ_2 (see DRAWING 5.4 at the end of Chapter V). Then both ℓ_3 and ℓ_4 are perpendicular to ℓ_1, ℓ_3 is parallel to ℓ_4, the line segments from ℓ_1 to ℓ_2 are of equal length, and the line segments from ℓ_3 to ℓ_4 are of equal length. See DRAWING 4.7 at the end of this chapter.

Proof: Denote by A the intersection of ℓ_3 with ℓ_2, by B the intersection of ℓ_4 with ℓ_2, by C the intersection of ℓ_4 with ℓ_1, and by D the intersection of ℓ_3 with ℓ_1, as in DRAWING 4.7 at the end of this chapter. Proposition 4.18 implies that ℓ_3 is parallel to ℓ_4. Thus the quadrilateral ABCD is a parallelogram. Proposition 3.3 now implies that $\|\overrightarrow{AB}\| = \|\overrightarrow{DC}\|$ and $\|\overrightarrow{AD}\| = \|\overrightarrow{BC}\|$. $\square$

See Corollary 5.3 for a different proof (using Theorem 5.2). Note that Proposition 4.18 is the special case of Corollary 4.20 when $\ell_1 = \ell_2$.

HOMEWORK

HWIV.1. Get $< 2, -3 > \cdot < -4, 1 >$.

HWIV.2. Which of the following pairs of vectors are orthogonal?

(a) $< 1, -3 >, < 3, -9 >$.

(b) $< 1, -3 >, < -3, 9 >$.

(c) $< 1, -3 >, < 0, 0 >$.

(d) $< 1, -3 >, < 6, 2 >$.

HWIV.3. (a) Find the orthogonal projection of $< 3, 4 >$ onto $< 1, 0 >$.

(b) Find the orthogonal projection of $< 3, 4 >$ onto $< 0, 1 >$.

(c) Find the orthogonal projection of $< 1, 0 >$ onto $< 3, 4 >$.

(d) Find the orthogonal projection of $< 0, 1 >$ onto $< 3, 4 >$.

(e) Find the orthogonal projection of $< 1, 2 >$ onto $< 3, 4 >$. How is this related to (c) and (d)?

HWIV.4. Find the orthogonal projection of $(3, 1)$ onto the line $y = 2x + 5$.

HWIV.5. Find the orthogonal projection of $(1, -1)$ onto the line (expressed parametrically) $P = (2, 1) + t < -1, 2 >$.

HWIV.6. For arbitrary real numbers a, b, c, d, e, get the orthogonal projection of (d, e) onto

$$ax + by = c$$

in terms of a, b, c, d, e.

HWIV.7. Suppose a line ℓ has slope m. Use dot product and a convenient direction vector for ℓ (HWII.1 and HWII.2) to get the slope of any line orthogonal to ℓ.

HWIV.8. If $\vec{a}$ and $\vec{b}$ are both parallel and orthogonal, what can be said about $\vec{a}$ and $\vec{b}$? Prove your assertion.

HWIV.9. For what real number α is $< 1, \alpha > \perp < 2, -3 >$?

HWIV.10. Find a real number s so that

$$\| < 1, 2 > + s < -1, 1 > \| < \| < 1, 2 > \|.$$

It can be shown that such an s must exist, because $< 1, 2 >$ and $< -1, 1 >$ are not perpendicular. Stated more positively, it can be shown that two vectors $\vec{a}$ and $\vec{b}$ are orthogonal if and only if $\|\vec{a} + s\vec{b}\| \geq \|\vec{a}\|$, for any real s.

HWIV.11. Show that, for any vectors $\vec{a}, \vec{b}$, $\|\vec{a} + \vec{b}\| = \|\vec{a} - \vec{b}\| \iff \vec{a} \perp \vec{b}$.

HWIV.12. Show that, for any vectors $\vec{a}, \vec{b}$, $(\vec{a} + \vec{b}) \perp (\vec{a} - \vec{b}) \iff \|\vec{a}\| = \|\vec{b}\|$.

HOMEWORK ANSWERS

HWIV.1. $2(-4) + (-3)(1) = -11$.

HWIV.2. Each of the pairs in (c) and (d) are orthogonal, because of having zero dot product. Each of the pairs in (a) and (b) fail to be orthogonal, because of having nonzero dot product.

HWIV.3. (a) $< 3,4 > \cdot < 1,0 >= 3, \| < 1,0 > \| = 1$, thus our projection is $\frac{3}{1^2} < 1,0 >=< 3,0 >$.
(b) As in (a), our projection is $< 0,4 >$.
(c) $< 1,0 > \cdot < 3,4 >= 3, \| < 3,4 > \| = 5$, thus our projection is $\frac{3}{5^2} < 3,4 >= \frac{3}{25} < 3,4 >$.
(d) As in (c), our projection is $\frac{4}{25} < 3,4 >$.
(e) $< 1,2 > \cdot < 3,4 >= 11, \| < 3,4 > \|$ still equals 5, so now we want $\frac{11}{25} < 3,4 >$.
This is

$$[(\text{answer to (c)}) \times 1] + [(\text{answer to (d)}) \times 2] .$$

Note that

$$< 1,2 >= [< 1,0 > \times 1] + [< 0,1 > \times 2] .$$

HWIV.4. Using Theorem 4.11, $\vec{x}_0 = (0,5), \vec{b} =< 1,2 >$, gives us $(-1,3)$.

HWIV.5. Using Theorem 4.11, $\vec{x}_0 = (2,1), \vec{b} =< -1,2 >$, gives us $\frac{1}{5}(13,-1)$.

HWIV.6. Using Theorem 4.11, with $\vec{x}_0 = (0, \frac{c}{b})$ and $\vec{b} =< b, -a >$, gives us

$$(0, \frac{c}{b}) + \frac{\left[db + (e - \frac{c}{b})(-a)\right]}{(b^2 + a^2)} < b, -a > .$$

HWIV.7. First assume m is finite and nonzero.

From HWII.1, use $< 1, m >$ as a direction vector for ℓ. Suppose ℓ_0 is a line orthogonal to ℓ. Denote by m_0 its slope.

If m_0 were infinite, then the allegedly orthogonal line would have direction vector $< 0,1 >$, whose dot product with $< 1, m >$ is m, assumed to be nonzero. Thus m_0 is finite, so that it has a direction vector $< 1, m_0 >$. Since the direction vectors of the orthogonal lines are orthogonal,

$$0 =< 1, m > \cdot < 1, m_0 >= 1 + mm_0,$$

so that $m_0 = -\frac{1}{m}$.

If m is infinite, then ℓ has direction vector $< 0,1 >$, so that direction vectors for a line orthogonal to ℓ have the form $< s, 0 >$, for any real s. This implies that the slope of any orthogonal line is 0.

If $m = 0$, then ℓ has direction vector $< 1,0 >$, clearly orthogonal to $< 0,1 >$, a direction vector for a vertical line. Thus a line orthogonal to ℓ now has infinite slope.

HWIV.8. Denote $\vec{a} =< a_1, a_2 >, \vec{b} =< b_1, b_2 >$, with a_1, a_2, b_1, b_2 real numbers. Assume, at least for the sake of contradiction, as in proof by contradiction (see Appendix 0), neither $\vec{a}$ nor $\vec{b}$ are $\vec{0}$; that is, $\vec{a}$ and $\vec{b}$ are nontrivial.

Since $\vec{a}$ and $\vec{b}$ are parallel and nontrivial, there's real nonzero s so that

$$< a_1, a_2 >= \vec{a} = s\vec{b} =< sb_1, sb_2 >, \text{ so that } a_1 = sb_1 \text{ and } a_2 = sb_2.$$

Since $\vec{a}$ and $\vec{b}$ are orthogonal, we now have

$$0 = \vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 = sb_1^2 + sb_2^2 = s(b_1^2 + b_2^2),$$

so that $(b_1^2 + b_2^2) = 0$, which implies that $b_1 = b_2 = 0$; that is, $\vec{b}$ is trivial. Since $\vec{a} = s\vec{b}$, $\vec{a}$ is also trivial.

We conclude that either $\vec{a}$ or $\vec{b}$ must be $\vec{0}$.

HWIV.9. $0 =< 1, \alpha > \cdot < 2, -3 >= 2 - 3\alpha$, thus $\alpha = \frac{2}{3}$.
Compare to HWII.8.

HWIV.10. Let's calculate:

$$\| < 1, 2 > +s < -1, 1 > \| < \| < 1, 2 > \| \iff \| < 1-s, 2+s > \|^2 < \| < 1, 2 > \|^2 \iff (1-s)^2+(2+s)^2 < 5$$
$$\iff (1 - 2s + s^2) + (4 + 4s + s^2) < 5 \iff 2s^2 + 2s + 5 < 5 \iff 2s^2 + 2s < 0 \iff s(s + 1) < 0.$$

We need $s < 0$ and $(s + 1) > 0$; for example, $s = -\frac{1}{2}$ will do what we want.

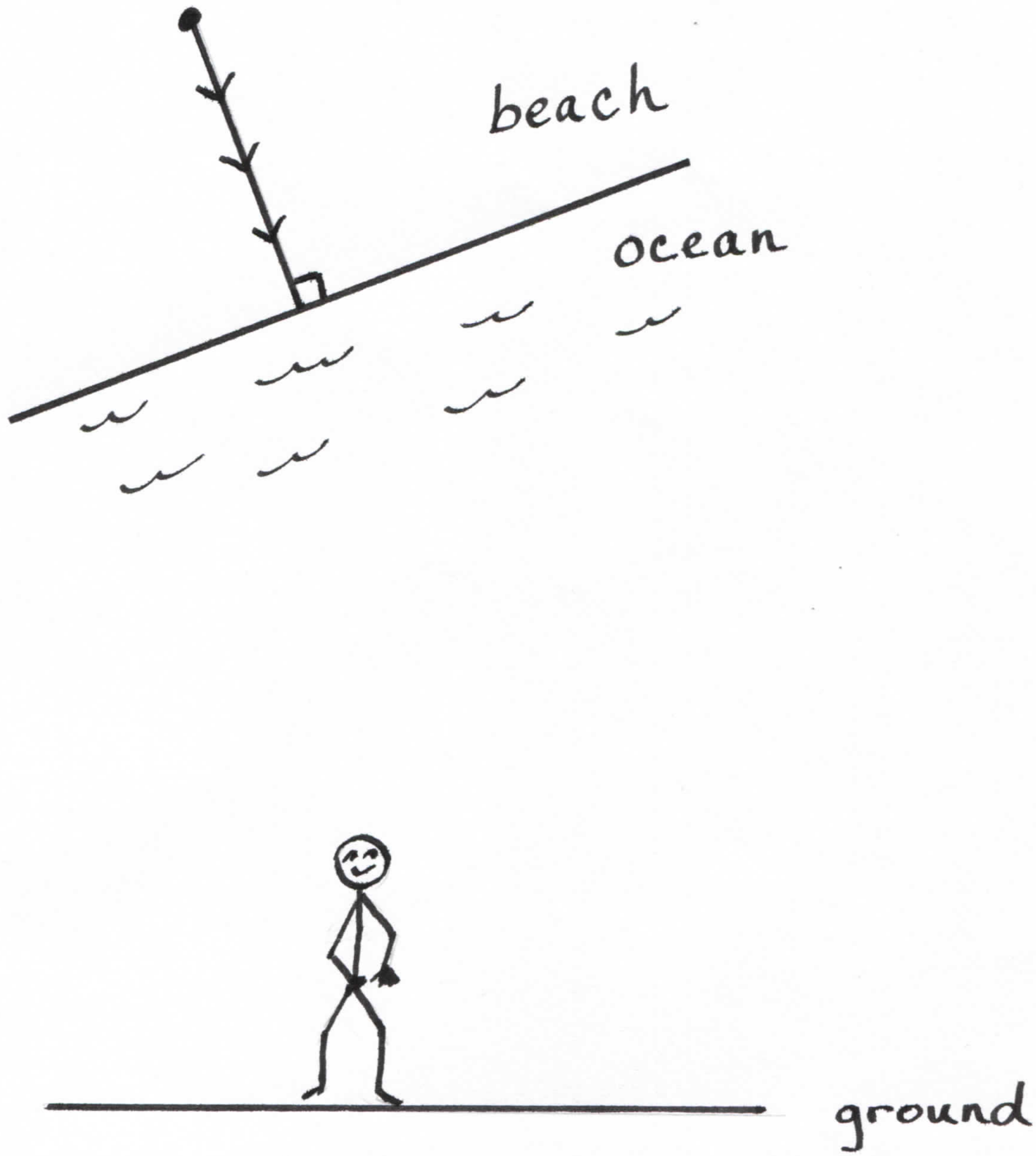
HWIV.11. By 4.5(d),

$$\|\vec{a}+\vec{b}\| = \|\vec{a}-\vec{b}\| \iff \|\vec{a}+\vec{b}\|^2 = \|\vec{a}-\vec{b}\|^2 \iff \|\vec{a}\|^2+\|\vec{b}\|^2+2(\vec{a}\cdot\vec{b}) = \|\vec{a}\|^2+\|\vec{b}\|^2-2(\vec{a}\cdot\vec{b}) \iff \vec{a}\cdot\vec{b} = 0 \iff \vec{a} \perp \vec{b}.$$

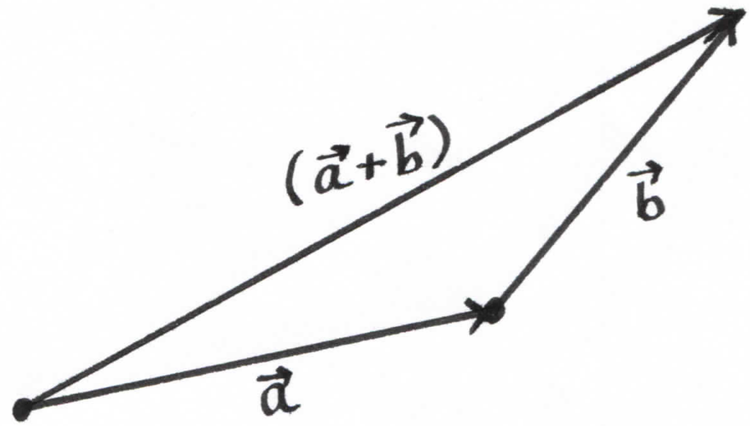
HWIV.12.

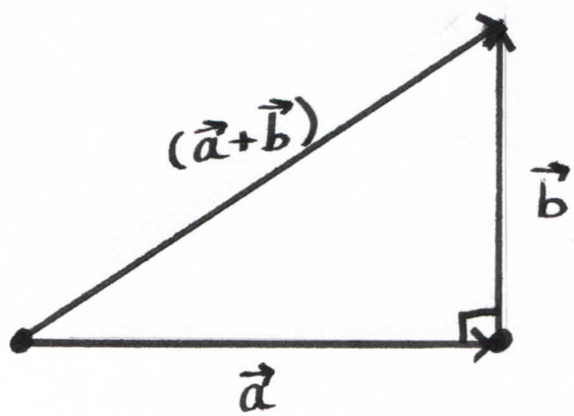
$$(\vec{a}+\vec{b}) \perp (\vec{a}-\vec{b}) \iff (\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = 0 \iff (\vec{a}\cdot\vec{a})-(\vec{a}\cdot\vec{b})+(\vec{b}\cdot\vec{a})-(\vec{b}\cdot\vec{b}) = 0 \iff (\vec{a}\cdot\vec{a})-(\vec{b}\cdot\vec{b}) = 0$$
$$\iff \|\vec{a}\|^2 - \|\vec{b}\|^2 = 0 \iff \|\vec{a}\| = \|\vec{b}\|.$$

DRAWING 4.1



DRAWING 4.2



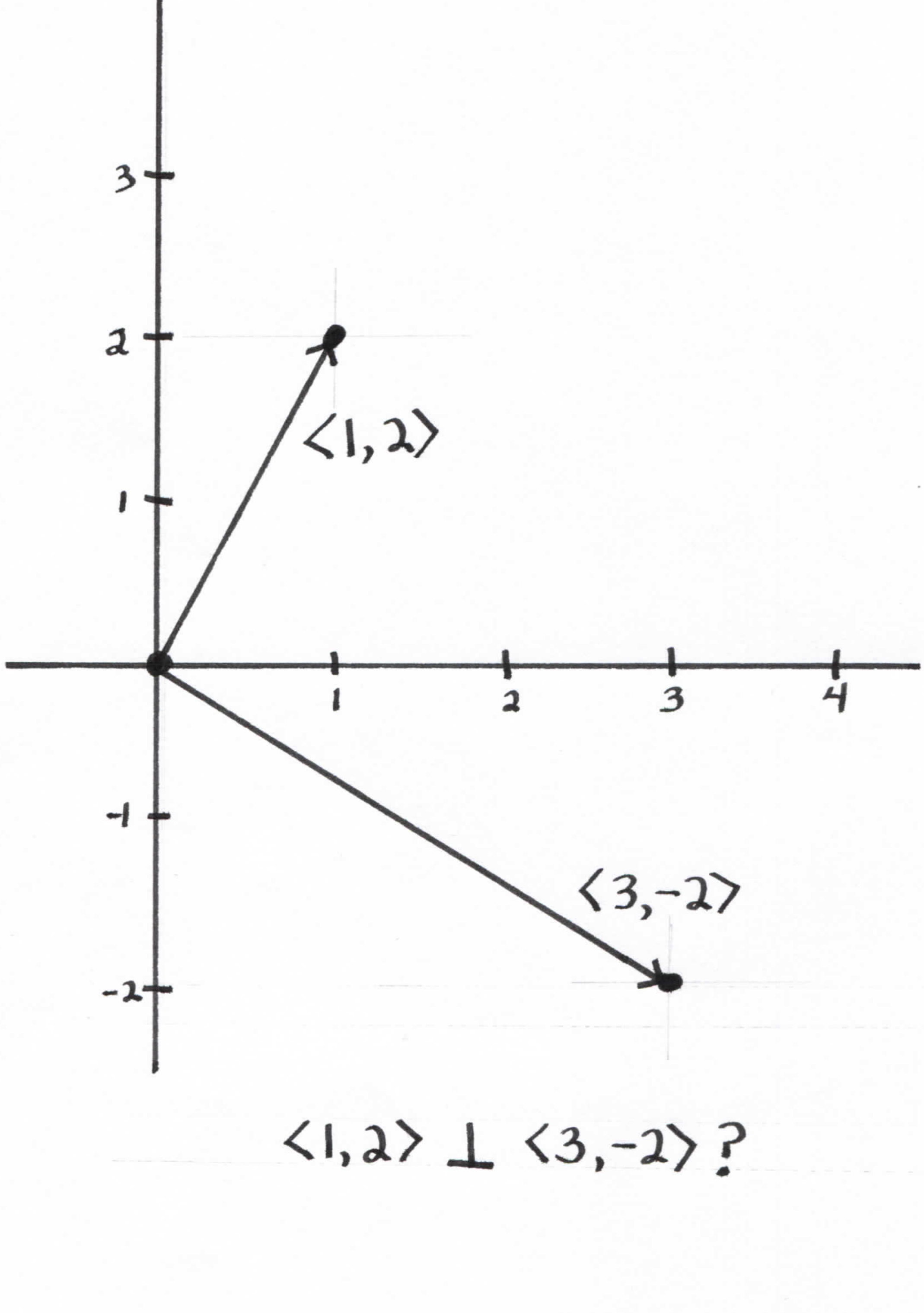


$$\|\vec{a}+\vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2$$

<u>IF</u> $\vec{a}$ and $\vec{b}$ are perpendicular

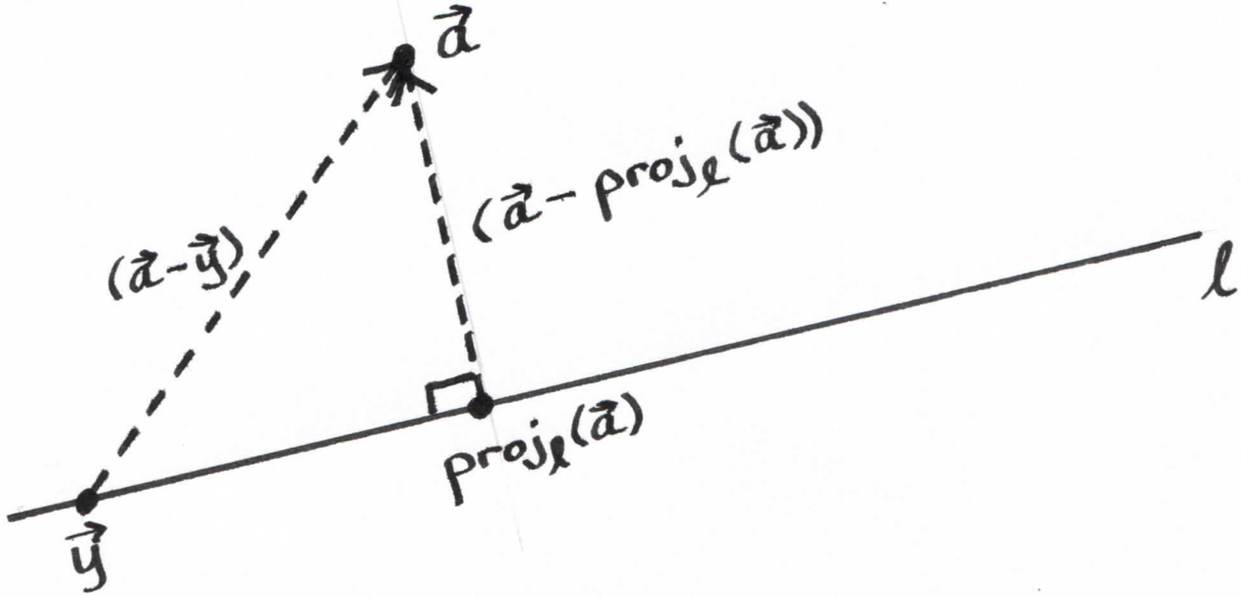
(Pythagorean Theorem)

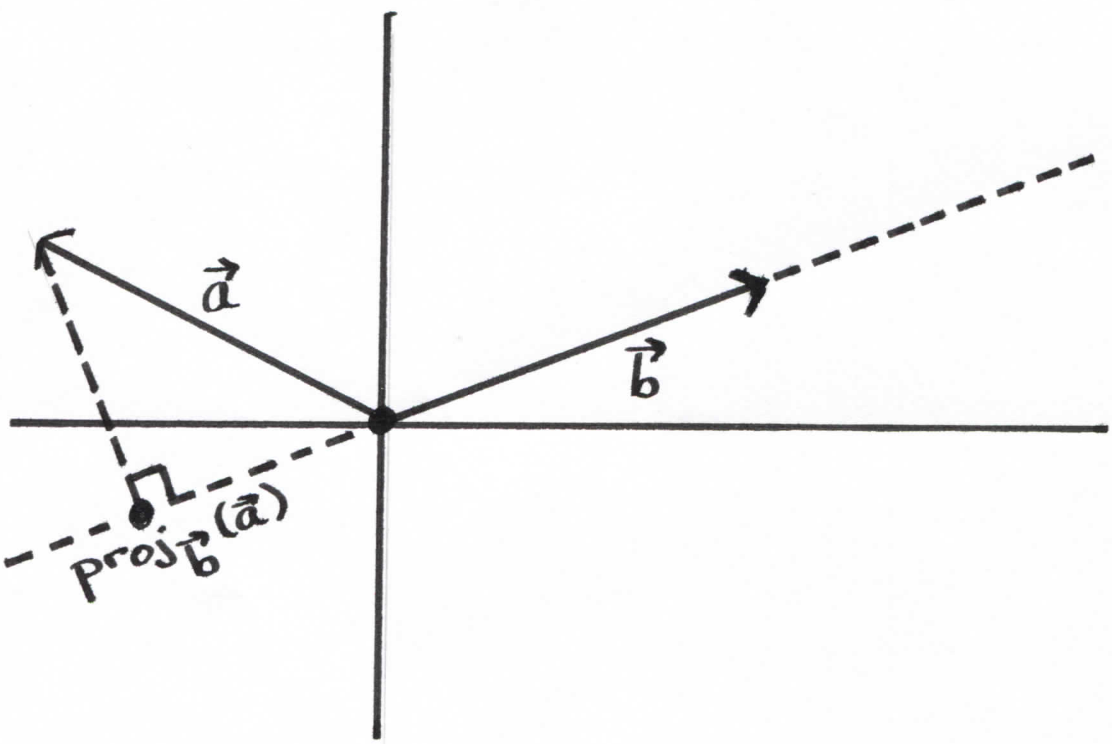
DRAWING 4.3



$\langle 1, 2\rangle \perp \langle 3, -2\rangle$?

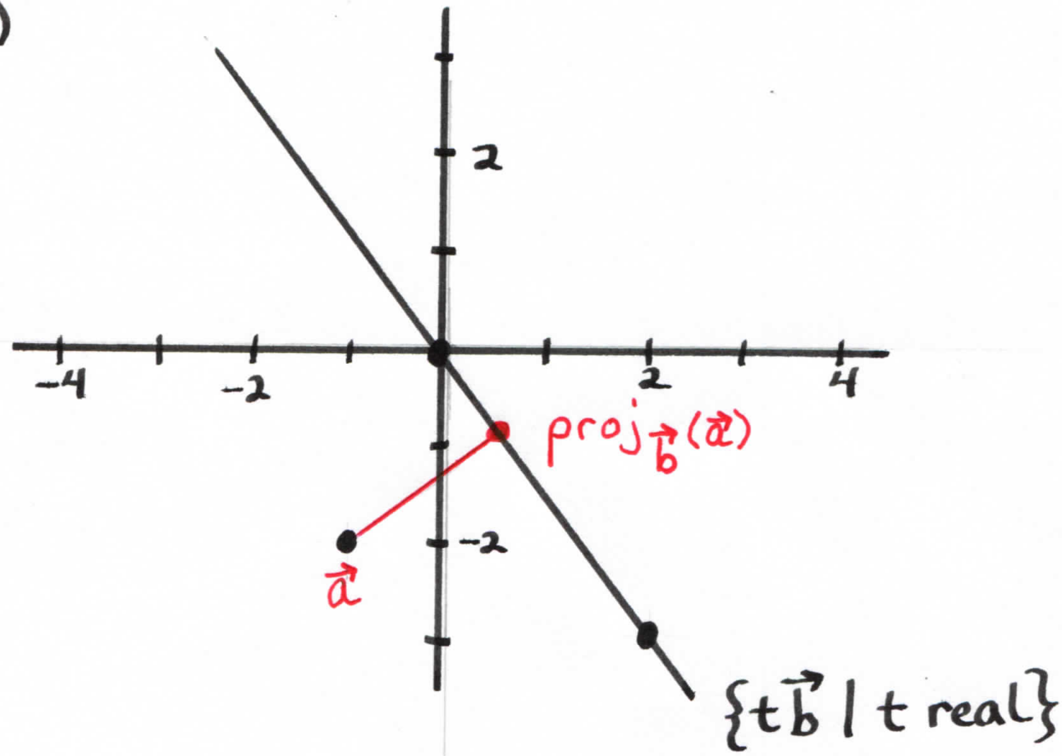
DRAWING 4.4


$\vec{a}$
$(\vec{a} - \text{proj}_\ell(\vec{a}))$
$(\vec{a} - \vec{y})$
ℓ
$\text{proj}_\ell(\vec{a})$
$\vec{y}$

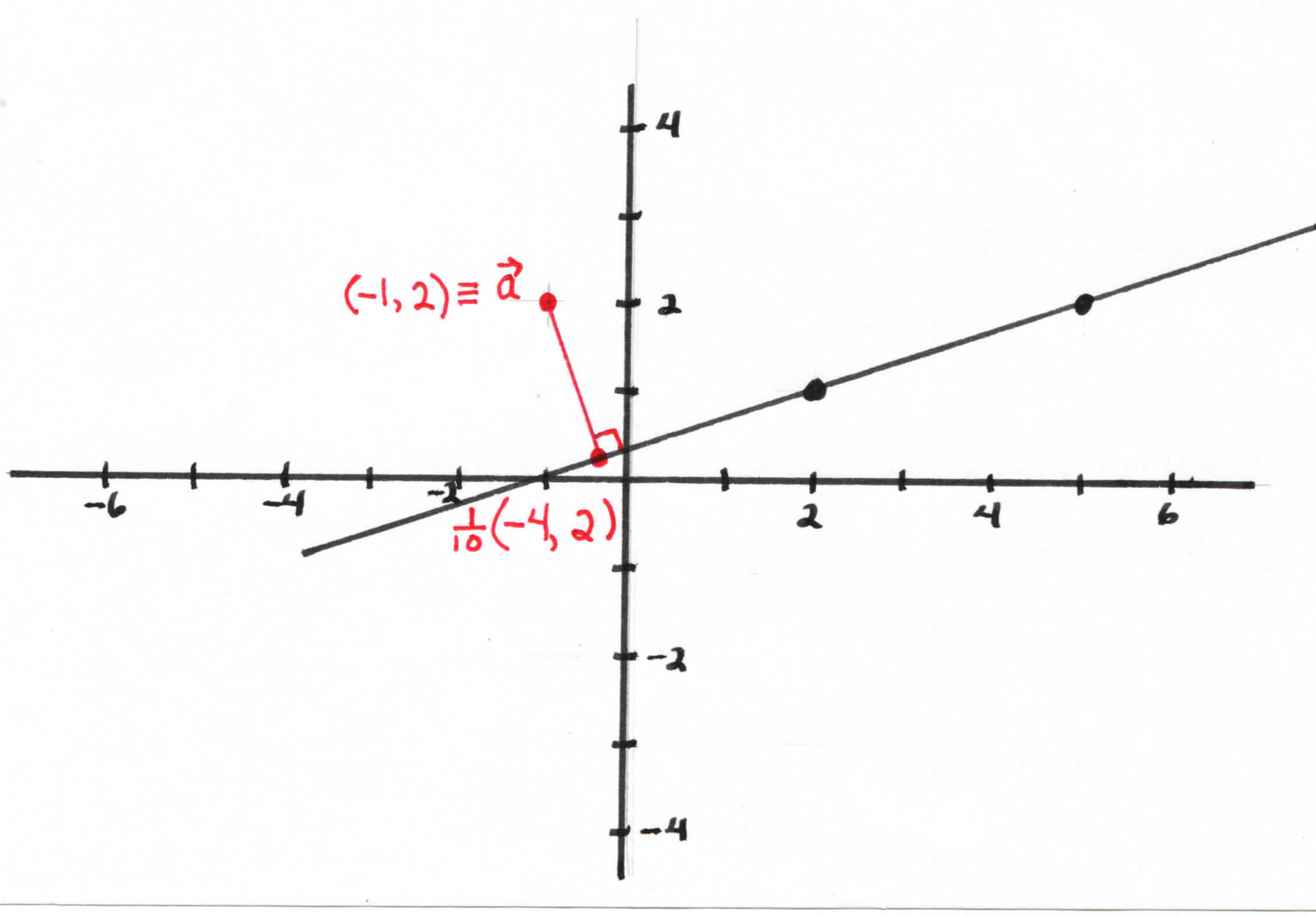

$\vec{a}$
$\vec{b}$
$\text{proj}_{\vec{b}}(\vec{a})$

DRAWING 4.5

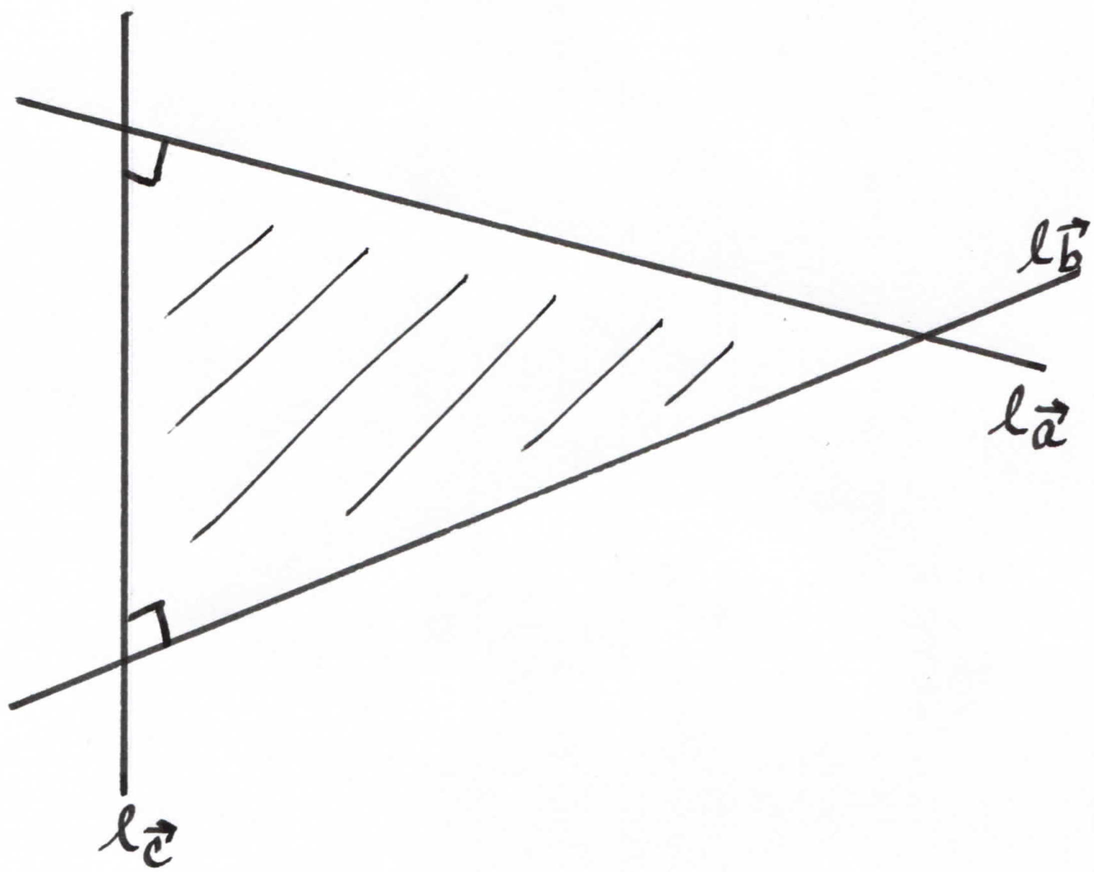
(a)



(b)

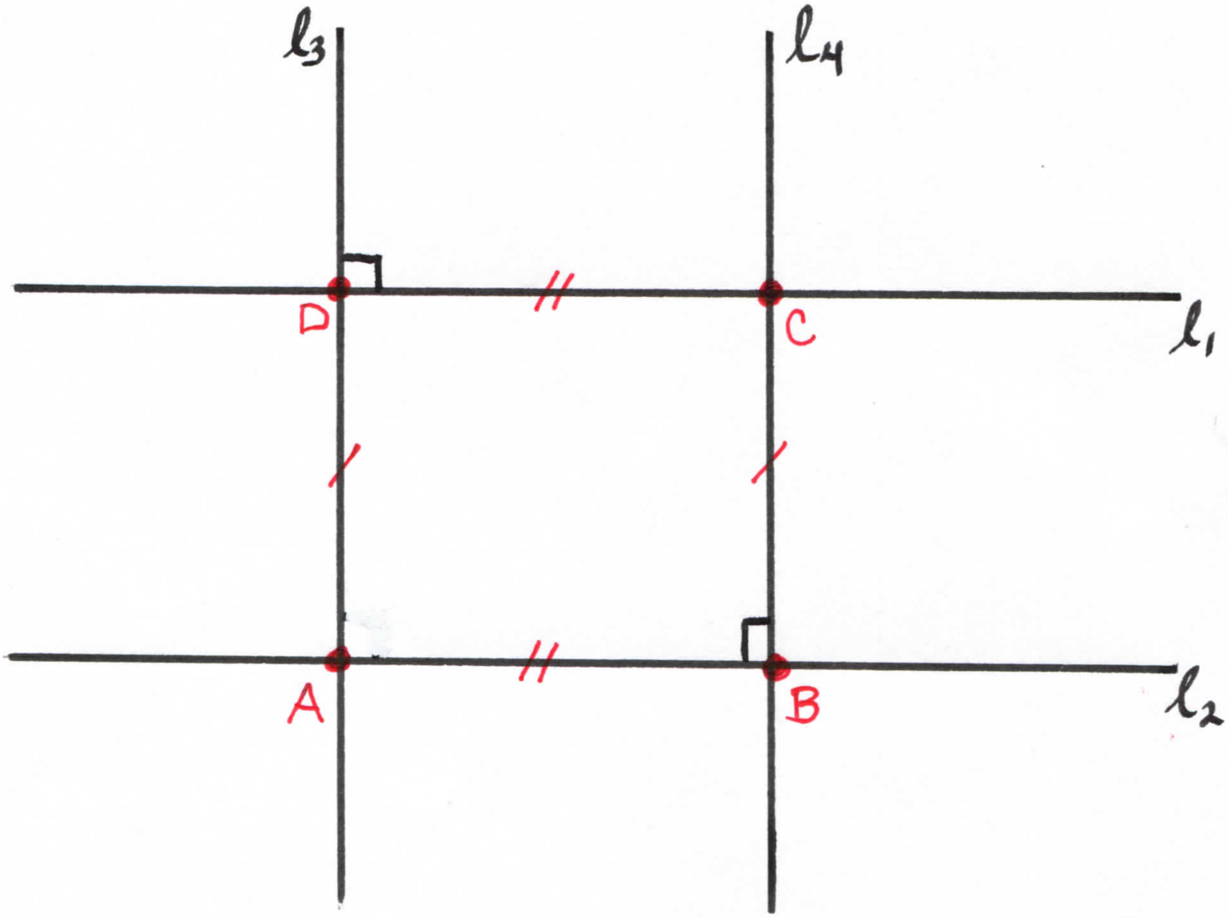


DRAWING 4.6



DRAWING 4.7

GIVEN: $\ell_3 \perp \ell_1$, $\ell_4 \perp \ell_2$,
ℓ_1 parallel to ℓ_2



ℓ_3 parallel to ℓ_4

CHAPTER V: Some Perpendicular Geometry.

This section will present some geometric consequences of the concept of being *orthogonal* or *perpendicular* (see Definition 4.3).

Recall the dot product, Definition 4.1.

Here are the key dot product factoids for this chapter, for any vectors $\vec{a}, \vec{b}$:

$$\|\vec{a}\|^2 = \vec{a} \cdot \vec{a} \quad \text{and} \quad \vec{a} \perp \vec{b} \iff \vec{a} \cdot \vec{b} = 0.$$

Definition 5.1. The **distance** from a point $\vec{x}$ to a line ℓ, denoted $d(\vec{x}, \ell)$, is the minimum distance from $\vec{x}$ to points on ℓ :

$$d(\vec{x}, \ell) \equiv \text{minimum } \{\|\vec{x} - \vec{y}\| \,|\, \vec{y} \text{ is in } \ell\}.$$

By Corollary 4.10,

$$d(\vec{x}, \ell) = \|\vec{x} - \text{proj}_\ell(\vec{x})\|;$$

see Definition 4.6 and DRAWING 5.1 at the end of this chapter.

In terms of distance from a point to a line, we may further characterize being parallel (see Theorem 3.1).

To those not familiar with the extent to which math grows back on itself, it might seem strange or even ironic to characterize being parallel in terms of an idea equivalent to orthogonality, a sort of opposite to being parallel.

See Proposition 4.15 for characterizations of orthogonality. See Proposition 4.18 for a preliminary result in the spirit of Theorem 5.2.

Theorem 5.2. Suppose ℓ_1 and ℓ_2 are two different lines. Then the following are equivalent.

(a) ℓ_1 and ℓ_2 are parallel.

(b) $d(\vec{x}, \ell_2)$ is constant, for $\vec{x}$ in ℓ_1.

(c) $\left(\vec{x} - \text{proj}_{\ell_2}(\vec{x})\right)$ is constant, that is, equals the same vector, for $\vec{x}$ in ℓ_1. See DRAWING 5.2 at the end of this chapter.

(d) A line is perpendicular to ℓ_1 if and only if it is perpendicular to ℓ_2.

(e) There are two (different) points $\vec{x}$ and $\vec{y}$ in ℓ_1 such that $d(\vec{x}, \ell_2) = d(\vec{y}, \ell_2)$.

Proof: (a) $\to$ (c). Fix $\vec{x}$ and $\vec{y}$ in ℓ_1.

Since $\left(\vec{x} - \text{proj}_{\ell_2}(\vec{x})\right)$ and $\left(\vec{y} - \text{proj}_{\ell_2}(\vec{y})\right)$ are both orthogonal to ℓ_2, Proposition 4.18 implies they are parallel. Thus the quadrilateral with vertices $\vec{x}, \vec{y}, \text{proj}_{\ell_2}(\vec{x})$, and $\text{proj}_{\ell_2}(\vec{y})$ (see DRAWING 5.3(a) at the end of this chapter) is a parallelogram.

Proposition 3.3 now implies that the vectors $\left(\vec{x} - \text{proj}_{\ell_2}(\vec{x})\right)$ and $\left(\vec{y} - \text{proj}_{\ell_2}(\vec{y})\right)$ are equal. Since $\vec{x}$ and $\vec{y}$ are arbitrary, this proves (c).

(c) $\to$ (b) follows from (see comment after Definition 5.1) $d(\vec{x}, \ell) = \|\vec{x} - \text{proj}_\ell(\vec{x})\|$, for any line ℓ, point $\vec{x}$.

(b) $\to$ (e) is clear.

(e) $\to$ (a). Consider the quadrilateral with vertices $\vec{x}, \vec{y}, \text{proj}_{\ell_2}(\vec{x}), \text{proj}_{\ell_2}(\vec{y})$ (see DRAWING 5.3(b) at the end of this chapter). Since $\left(\vec{x} - \text{proj}_{\ell_2}(\vec{x})\right)$ and $\left(\vec{y} - \text{proj}_{\ell_2}(\vec{y})\right)$ are both orthogonal to ℓ_2, Proposition 4.18 implies that they are parallel. By hypothesis,

$$\|\left(\vec{x} - \text{proj}_{\ell_2}(\vec{x})\right)\| = d(\vec{x}, \ell_2) = d(\vec{y}, \ell_2) = \|\left(\vec{y} - \text{proj}_{\ell_2}(\vec{y})\right)\|,$$

thus Proposition 3.5 implies that our quadrilaterial is a parallelogram; in particular, ℓ_1 is parallel to ℓ_2.

(a) $\rightarrow$ (d). Suppose a line ℓ_3 is perpendicular to ℓ_1 (see DRAWING 5.3(c) at the end of this chapter). Proposition 3.6 (ℓ_3 is called a *transversal* there) implies that ℓ_3 is perpendicular to ℓ_2.

The same argument shows that

$$(\ell_3 \perp \ell_2) \rightarrow (\ell_3 \perp \ell_1).$$

(d) $\rightarrow$ (a). Fix $\vec{x}$ in ℓ_1 and let ℓ_3 be the line through $\vec{x}$ and $(\mathrm{proj}_{\ell_2}(\vec{x}))$ (see DRAWING 5.3(d) at the end of this chapter).

By the definition of projection, ℓ_3 is perpendicular to ℓ_2. By (d), ℓ_3 is also perpendicular to ℓ_1 (see DRAWING 5.3(e) at the end of this chapter).

By Proposition 4.18, ℓ_1 and ℓ_2 are parallel. $\square$

Corollary 5.3 (same as Corollary 4.20). Suppose $\ell_1, \ell_2, \ell_3, \ell_4$ are lines, ℓ_1 and ℓ_2 are parallel, and both ℓ_3 and ℓ_4 are perpendicular to ℓ_2 (see DRAWING 5.4(a) at the end of this chapter). Then both ℓ_3 and ℓ_4 are perpendicular to ℓ_1, ℓ_3 is parallel to ℓ_4, the line segments from ℓ_1 to ℓ_2 are of equal length, and the line segments from ℓ_3 to ℓ_4 are of equal length. See DRAWING 5.4(b) at the end of this chapter.

Proof: The orthogonality conclusions follow from Theorem 5.2(a) $\iff$ (d). Proposition 4.18 implies that ℓ_3 and ℓ_4 are parallel. Theorem 5.2(a) $\iff$ (c) implies that the indicated line segments (see DRAWING 5.4(b) at the end of this chapter) are of equal length. $\square$

See Corollary 4.20 for a different proof.

The next three results involve diagonals in a parallelogram. We have already observed (Corollary 3.4 and DRAWING 2.8(b) at the end of Chapter II) that any parallelogram is formed by two vectors, call them $\vec{a}, \vec{b}$; in DRAWING 5.5 at the end of this chapter we have included diagonals $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$.

Propositions 5.5 and 5.6 present an interesting duality between length and orthogonality: perpendicular diagonals correspond to sides of equal length (Proposition 5.5), while perpendicular sides correspond to diagonals of equal length (Proposition 5.6).

Definitions 5.4. A **rhombus** is a parallelogram whose sides all have equal length. A **rectangle** is a parallelogram formed by orthogonal vectors $\vec{a}, \vec{b}$, as in Corollary 3.4 with $\vec{a} \perp \vec{b}$. A **square** is both a rectangle and a rhombus.

Proposition 5.5. In any parallelogram, the diagonals are perpendicular if and only if the parallelogram is a rhombus.

Proof: By Properties 4.5, after cancellation,

$$(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = \|\vec{a}\|^2 - \|\vec{b}\|^2,$$

thus

$$(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 0 \iff \|\vec{a}\| = \|\vec{b}\|.$$

Since the diagonals are $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$, and the lengths of the sides are $\|\vec{a}\|$ and $\|\vec{b}\|$ (see DRAWING 5.5 at the end of this chapter), the result follows from Definition 4.3. $\square$

Proposition 5.6. In a parallelogram, the diagonals are of equal length if and only if the parallelogram is a rectangle.

Proof: See DRAWING 5.5 at the end of this chapter. By 4.5(d),

$$\|\vec{a}+\vec{b}\| = \|\vec{a}-\vec{b}\| \iff \|\vec{a}+\vec{b}\|^2 = \|\vec{a}-\vec{b}\|^2 \iff \|\vec{a}\|^2+\|\vec{b}\|^2+2(\vec{a}\cdot\vec{b}) = \|\vec{a}\|^2+\|\vec{b}\|^2-2(\vec{a}\cdot\vec{b}) \iff \vec{a}\cdot\vec{b} = 0,$$

thus the result follows from Definition 4.3. □

Proposition 5.7. In a parallelogram, the sum of the squares of the lengths of the sides equals the sum of the squares of the lengths of the diagonals.

Proof: See DRAWING 5.5 at the end of this chapter. By 4.5(d),

$$\|\vec{a}+\vec{b}\|^2 + \|\vec{a}-\vec{b}\|^2 = \left(\|\vec{a}\|^2 + \|\vec{b}\|^2 + 2(\vec{a}\cdot\vec{b})\right) + \left(\|\vec{a}\|^2 + \|\vec{b}\|^2 - 2(\vec{a}\cdot\vec{b})\right) = 2\left(\|\vec{a}\|^2 + \|\vec{b}\|^2\right).$$

□

Definition 5.8. A line or line segment ℓ_2 is a **perpendicular bisector** of the line segment ℓ_1 from A to B if ℓ_2 bisects ℓ_1 at the midpoint C (see Definitions 3.11) and ℓ_2 is orthogonal to ℓ_1.

It is not hard to construct (hence show that it always exists) the perpendicular bisector ℓ_2. Writing the coordinates of the directed line segment $\overrightarrow{AB} =< v_1, v_2 >$, the line ℓ_2 is all points of the form

$$P = A + \frac{1}{2} < v_1, v_2 > +t < -v_2, v_1 > \quad (t \text{ real}).$$

See DRAWING 5.6 at the end of this chapter.

Definition 5.9. A triangle is **isosceles** if two sides are of equal length.

The following theorem relates being isosceles to the presence of a particular perpendicular bisector, from the vertex where the sides of equal length meet, to the side opposite said vertex.

Theorem 5.10. Let P, Q, R be the three vertices of a triangle. Then the following are equivalent.

(a) $\|\overrightarrow{PR}\| = \|\overrightarrow{QR}\|$.

(b) The line segment between the vertex R and the midpoint of the opposite side $\overrightarrow{PQ}$ is orthogonal to $\overrightarrow{PQ}$.

(c) The orthogonal projection of the vertex R onto the opposite side $\overrightarrow{PQ}$ is the midpoint of $\overrightarrow{PQ}$.

Proof: See DRAWING 5.7 at the end of this chapter. Let $S \equiv \text{proj}_{\overrightarrow{PQ}}(R)$. By the Pythagorean theorem (Proposition 4.15) applied twice,

$$\|\overrightarrow{PR}\|^2 - \|\overrightarrow{PS}\|^2 = \|\overrightarrow{SR}\|^2 = \|\overrightarrow{QR}\|^2 - \|\overrightarrow{SQ}\|^2,$$

so that

$$\|\overrightarrow{PR}\| = \|\overrightarrow{QR}\| \iff \|\overrightarrow{PS}\| = \|\overrightarrow{SQ}\|;$$

the latter equality is equivalent to both (b) and (c). □

Definition 5.11. Let P be a point on a circle centered at C. A **tangent line** to the circle at P is a line through P orthogonal to $\overrightarrow{CP}$. See DRAWING 5.8 at the end of this chapter.

A tangent line is not hard to construct. If $\overrightarrow{CP} =< v_1, v_2 >$, let $\vec{v} \equiv< -v_2, v_1 >$, orthogonal to $\overrightarrow{CP}$ since their dot product is 0; then

$$\{P + t\vec{v} \,|\, t \text{ is real})\}$$

is the desired line.

Physically, imagine the disc enclosed by the circle is a merry-go-round in motion, with you clinging to the point P on the edge of the merry-go-round. The tangent line at P is the route your body would take if you stepped off the merry-go-round, until gravity brought you to earth.

Example 5.12. Find the tangent line to the circle centered at $(3, 1)$ at the point $(3+\sqrt{2}, 1-\sqrt{2})$ on the circle.

Solution. In the language of Definition 5.11 and the discussion succeeding it, $C = (3, 1), P = (3+\sqrt{2}, 1-\sqrt{2})$, and $< v_1, v_2 >\equiv \overrightarrow{CP} = \sqrt{2} < 1, -1 >$. The desired line is

$$\{(3+\sqrt{2}, 1-\sqrt{2}) + t < 1, 1 > \mid t \text{ is real}\}.$$

See DRAWING 5.9 at the end of this chapter.

Examples 5.13. In each of the parallelograms in DRAWINGS 5.10 at the end of this chapter, use the results of Chapters III–V to fill in any lengths of sides or measures of angles, whenever possible. Do not assume anything is drawn to scale.

Solutions. See DRAWINGS 5.11 at the end of this chapter.

(a) Propositions 3.3 and 5.5. (b) Propositions 3.3 and 5.5. (c) Propositions 3.3, 3.13, and 5.7. (d) Propositions 3.3, 3.13, 5.6, and 5.7. (e) Propositions 3.3, 3.13, Pythagorean theorem and 5.6. (f) Propositions 3.3, 3.13, 5.5 and 5.6 and the Pythagorean theorem. (g) Propositions 3.13 and 5.6. (h) Propositions 3.3, 3.13, 5.5 and 5.6 and the Pythagorean theorem. (i) Propositions 3.3, 3.13, and 5.7.

Examples 5.14. In each of the triangles in DRAWINGS 5.12 at the end of this chapter, use the results of Chapters III–V to fill in any lengths of sides or measures of angles, whenever possible. Do not assume anything is drawn to scale.

Solutions. See DRAWINGS 5.13 at the end of this chapter. Each part uses Theorem 5.10 and the Pythagorean theorem.

Examples 5.15. In each of the figures in DRAWINGS 5.14 at the end of this chapter, find x and y. All quadrilaterals are parallelograms.

Solutions. (a) By Theorem 5.2, $(2x + 25) = (5x - 11)$, so $x = 12$. By Corollary 5.3, $(y + 10) = 2y$, so $y = 10$.

(b) By Proposition 5.5, $(2x - 10) = 18$, so $x = 14$. By Proposition 3.3, $18 = (y + 8)$, so $y = 10$.

(c) By Propositions 5.6 and 3.13, $2x = 10 = (y - 6)$, so $x = 5$ and $y = 16$.

(d) By Theorem 5.10, $(3x - 5) = (x + 10)$, so $x = \frac{15}{2}$.

(e) By Theorem 5.10, $2(2x + 1) = (x + 20)$, so $x = 6$.

HOMEWORK

HWV.1. Find the tangent line to the circle centered at $(1, 2)$ at the point $(2, 2+\sqrt{3})$ on the circle.

HWV.2. In each of the parallelograms in DRAWINGS 5.15 at the end of this chapter, use the results of Chapters III–V to fill in any lengths of sides or measures of angles, whenever possible. Do not assume anything is drawn to scale.

HWV.3. In each of the triangles in DRAWINGS 5.16 at the end of this chapter, use the results of Chapters III–V to fill in any lengths of sides or measures of angles, whenever possible. Do not assume anything is drawn to scale.

HWV.4. In each of the figures in DRAWINGS 15.17 at the end of this chapter, find x and y. All quadrilaterals are parallelograms. Do not assume anything is drawn to scale.

HWV.5. Let P be a point on a circle, as in Definition 5.11. Show that no other point on the tangent line to the circle at P is on the circle.

HWV.6. Suppose P, Q, R are points such that $\overrightarrow{QP} \perp \overrightarrow{QR}$. Show that $\|\overrightarrow{PQ}\| \leq \|\overrightarrow{PR}\|$.

HWV.7. Given two points P, Q, show that the perpendicular bisector (Definition 5.8) of $\overrightarrow{PQ}$ equals the set of all points equidistant from P and Q.

HOMEWORK ANSWERS

HWV.1. $P = (2, 2+\sqrt{3}) + t < -\sqrt{3}, 1 >$ (t real).

HWV.2. See DRAWINGS 5.18 at the end of this chapter.

HWV.3. See DRAWINGS 5.19 at the end of this chapter.

HWV.4. (a) $x = 7$ (b) $x = 1$ (c) $x = 2, y = \frac{5}{4}$ (d) $x = 6, y = 3$ (e) $x = 2, y = \frac{7}{2}$.

HWV.5. Let C be the center of the circle. If Q (not equal to P) is on the line, then $\|\overrightarrow{CQ}\|^2 = \|\overrightarrow{CP}\|^2 + \|\overrightarrow{PQ}\|^2$, so $\|\overrightarrow{CQ}\| > \|\overrightarrow{CP}\|$, the radius of the circle. Thus Q cannot be on the circle.

HWV.6. By Proposition 4.15,

$$\|\overrightarrow{PR}\|^2 = \|\overrightarrow{PQ} + \overrightarrow{QR}\|^2 = \|\overrightarrow{PQ}\|^2 + \|\overrightarrow{QR}\|^2 \geq \|\overrightarrow{PQ}\|^2.$$

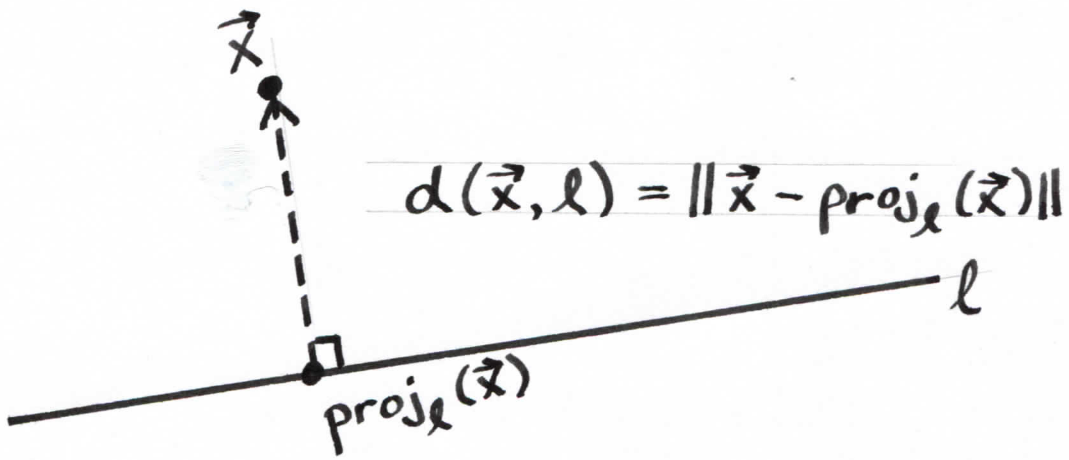
□

HWV.7. Suppose R is a point equidistant from P and Q. By Theorem 5.10, the line from R to the midpoint of $\overrightarrow{PQ}$ is perpendicular to $\overrightarrow{PQ}$; this is saying that R is on the perpendicular bisector of $\overrightarrow{PQ}$. See DRAWING 5.20 at the end of this chapter.

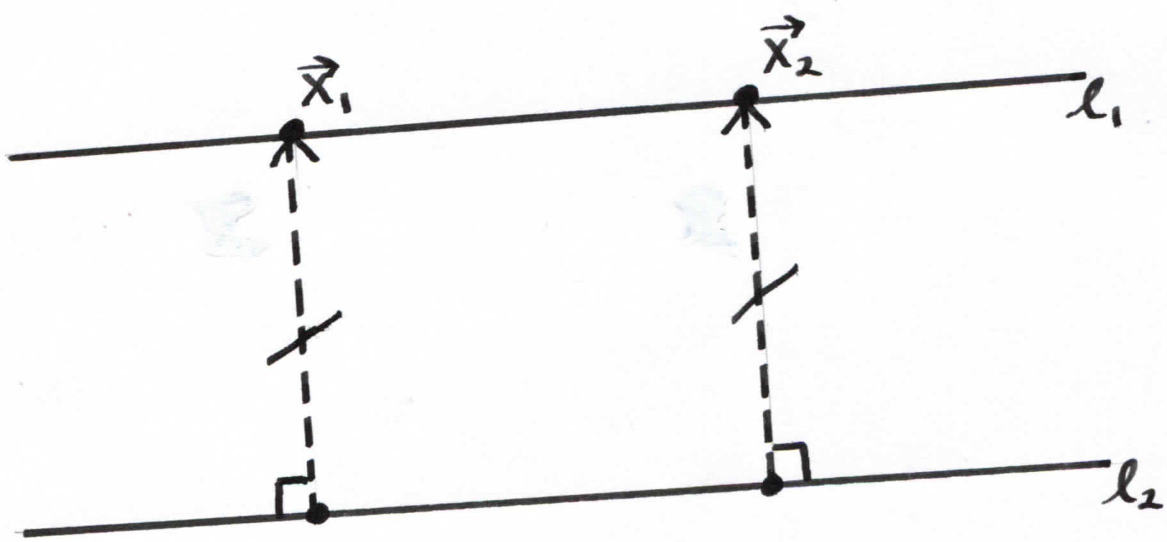
Conversely, if R is on the perpendicular bisector of $\overrightarrow{PQ}$, then by the Pythagorean theorem (see DRAWING 5.21 at the end of this chapter),

$$\|\overrightarrow{RP}\|^2 = \|\frac{1}{2}\overrightarrow{PQ}\|^2 + \|\overrightarrow{RS}\|^2 = \|\overrightarrow{RQ}\|^2.$$

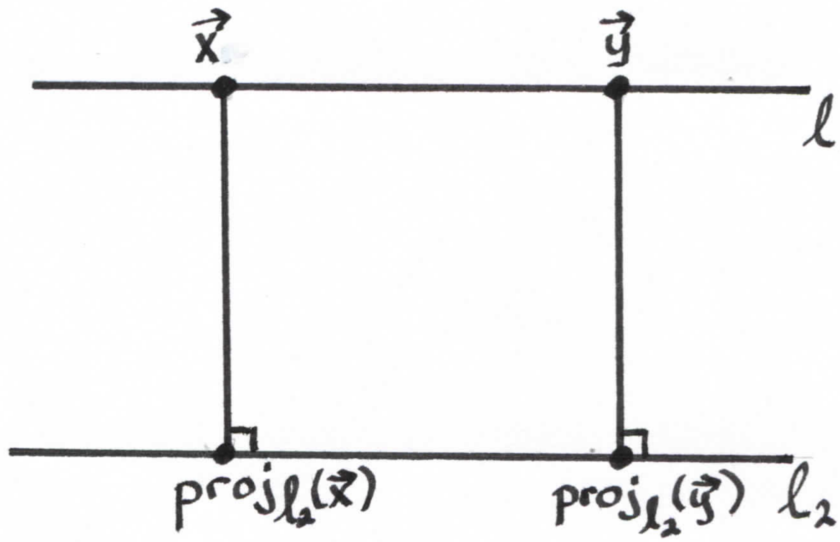
DRAWING 5.1



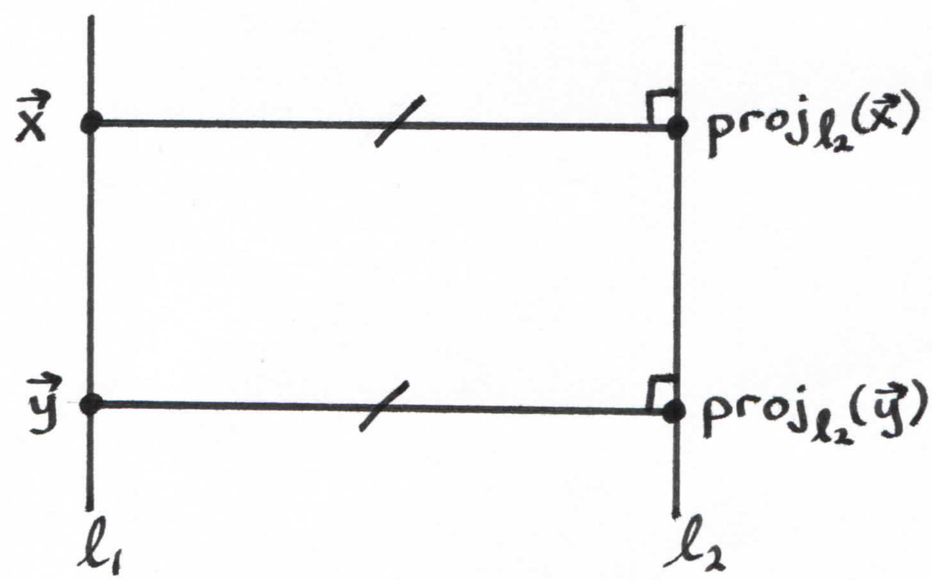
DRAWING 5.2



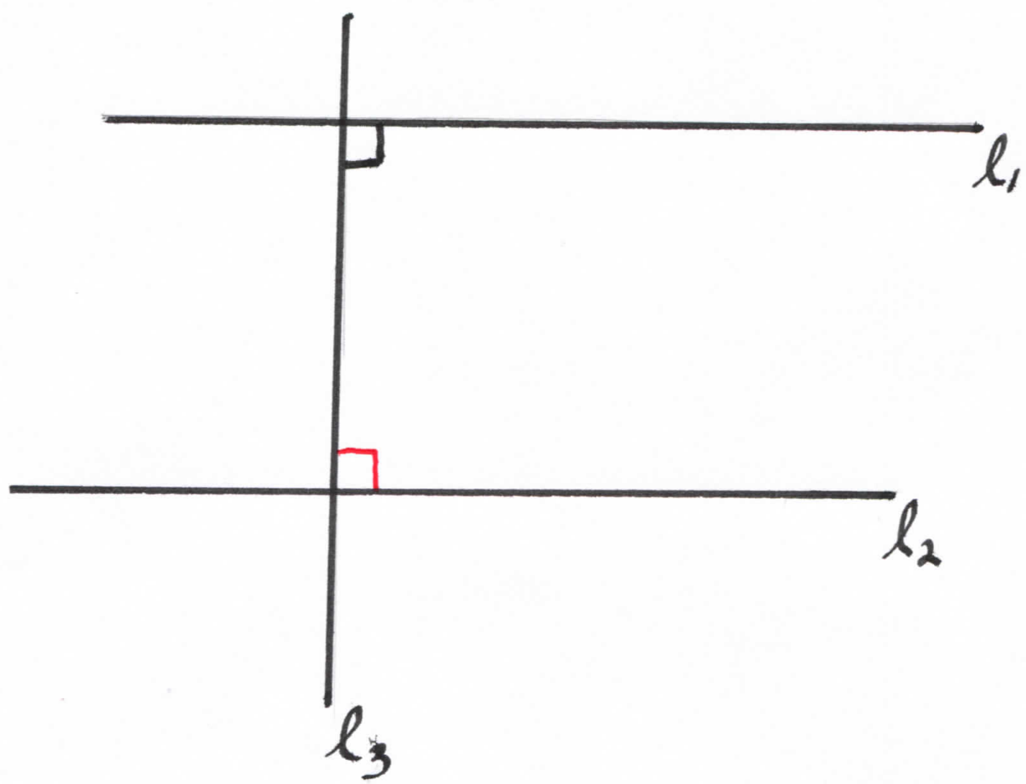
DRAWING 5.3(a)



DRAWING 5.3(b)

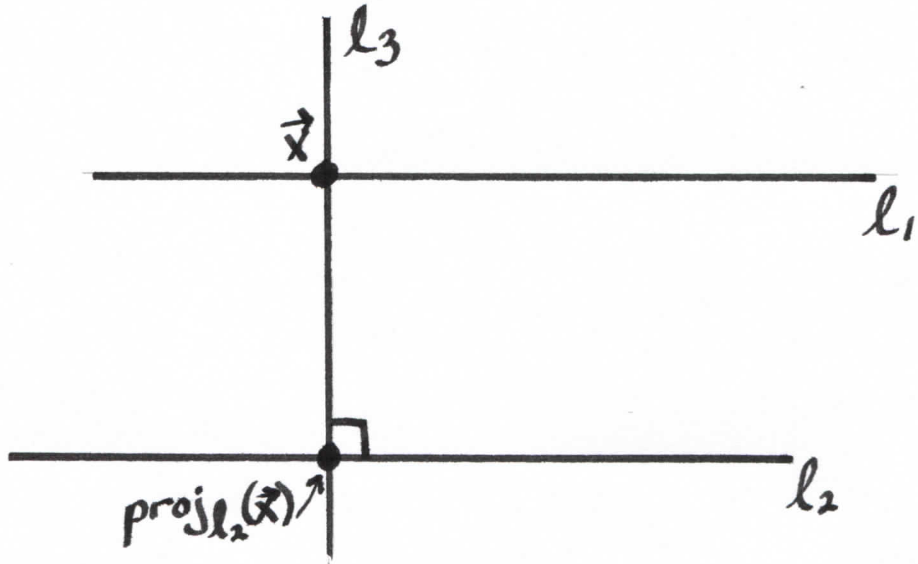


DRAWING 5.3(c)

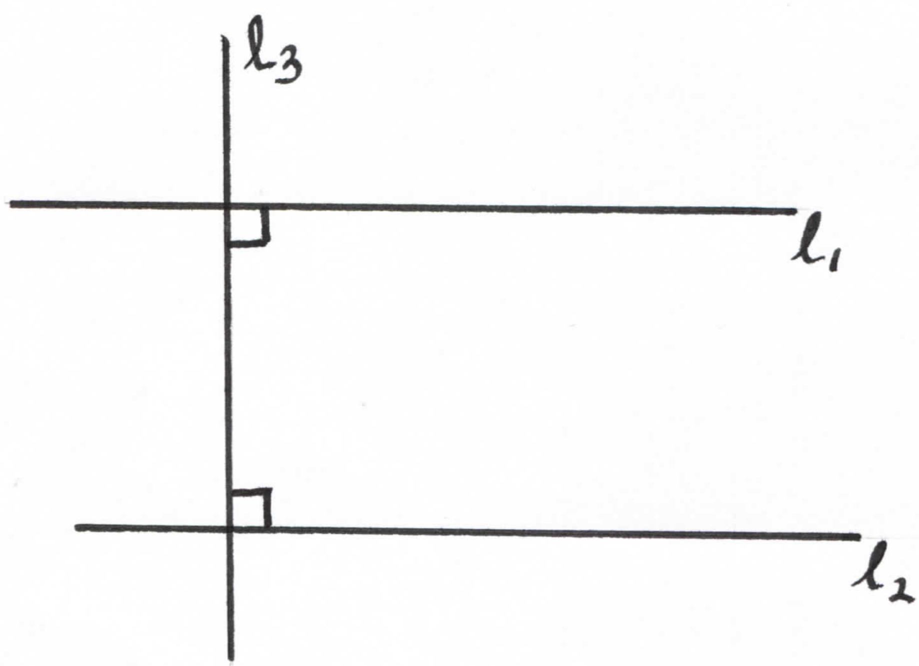


"⅂" is from Proposition 3.6

DRAWING 5.3(d)

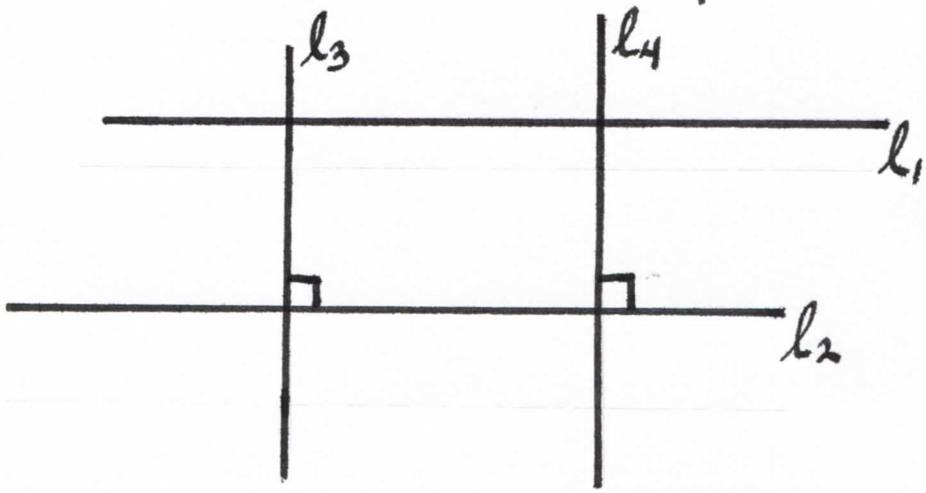


DRAWING 5.3(e)

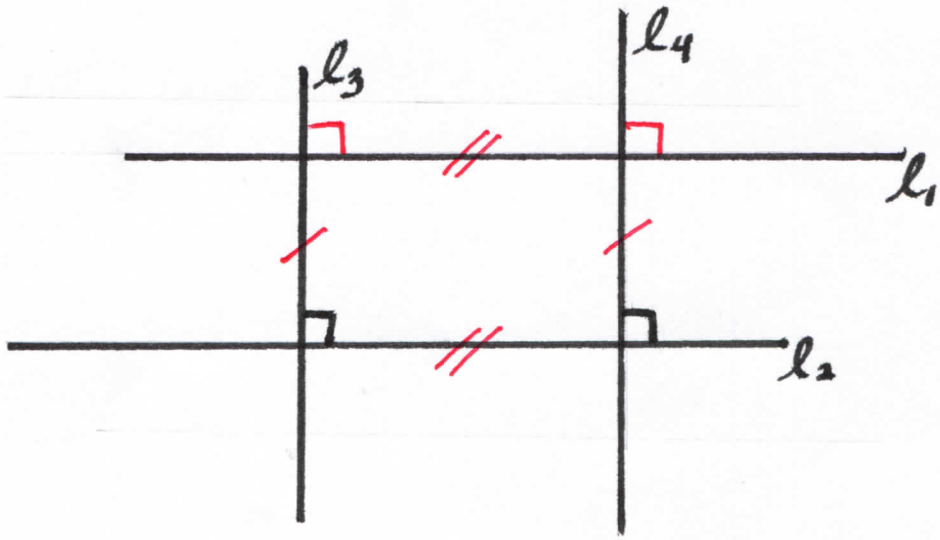


DRAWING 5.4(a)

GIVEN: $l_3 \perp l_2$, $l_4 \perp l_2$, l_1 and l_2 parallel

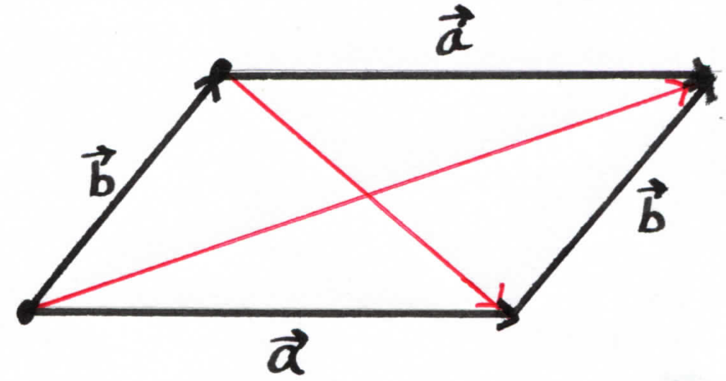


DRAWING 5.4(b)



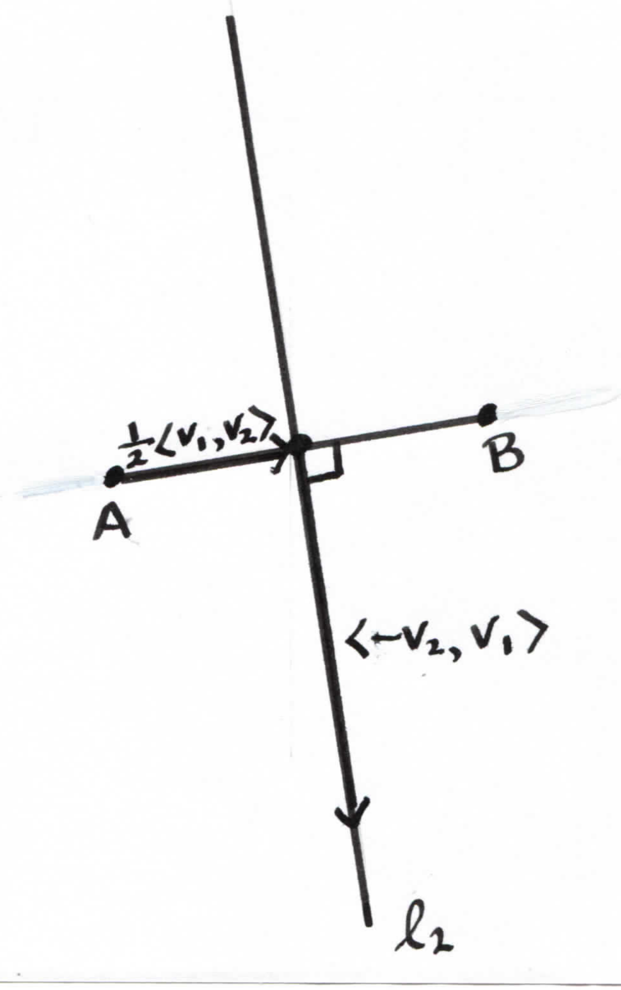
l_3 parallel to l_4

DRAWING 5.5

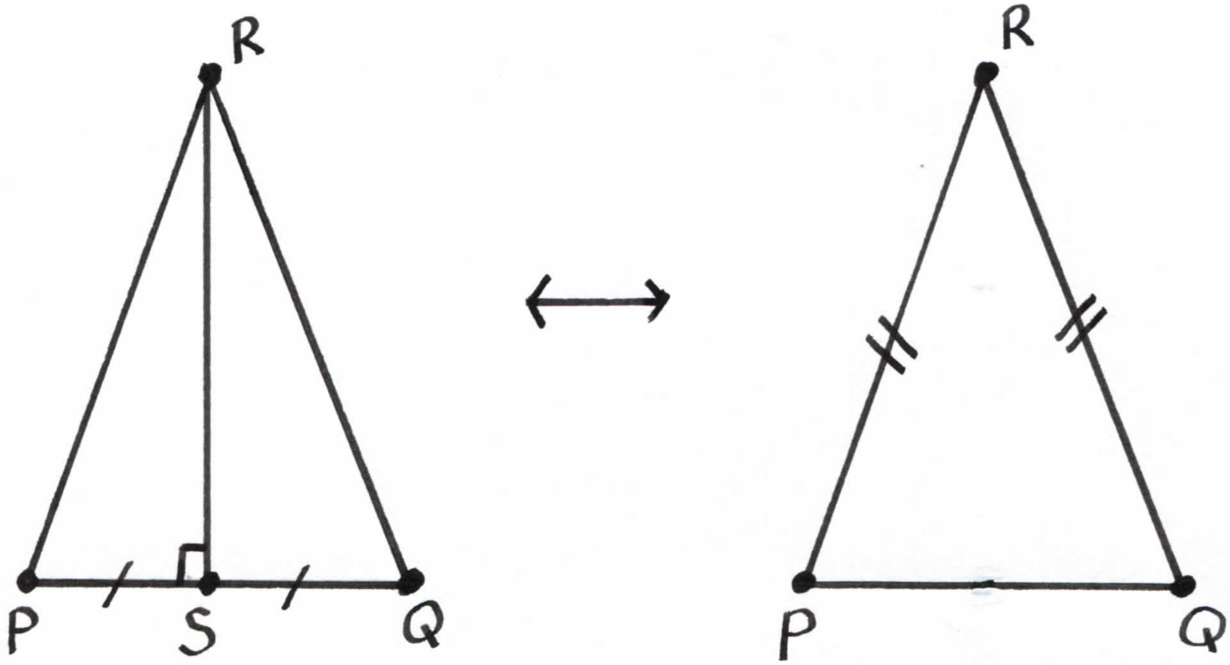


diagonals $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$

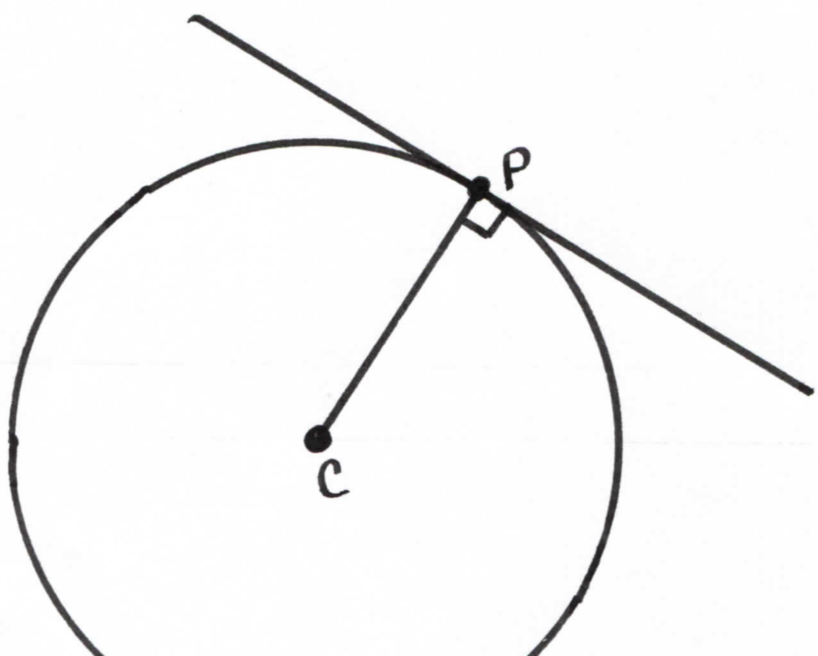
DRAWING 5.6



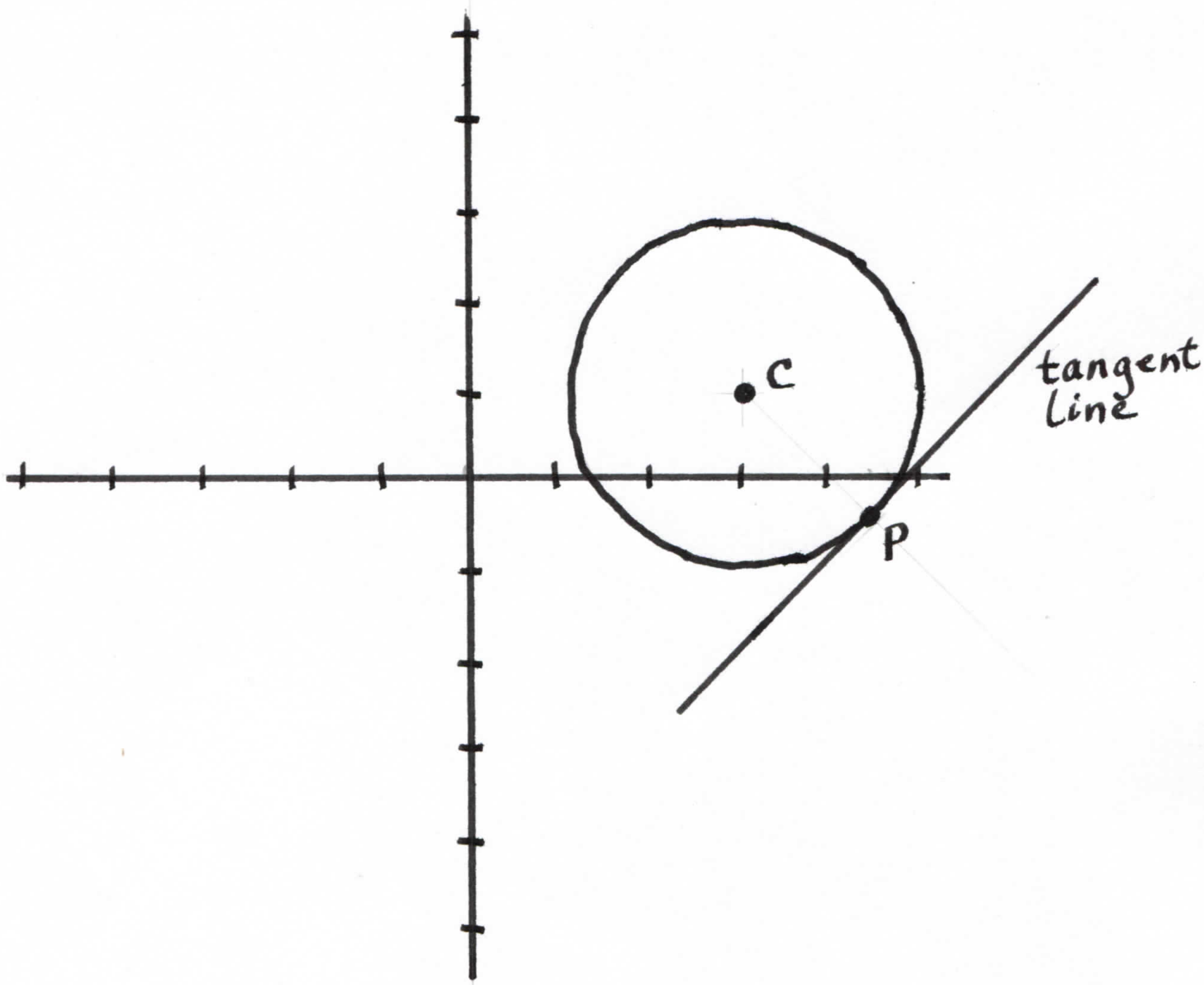
DRAWING 5.7



DRAWING 5.8

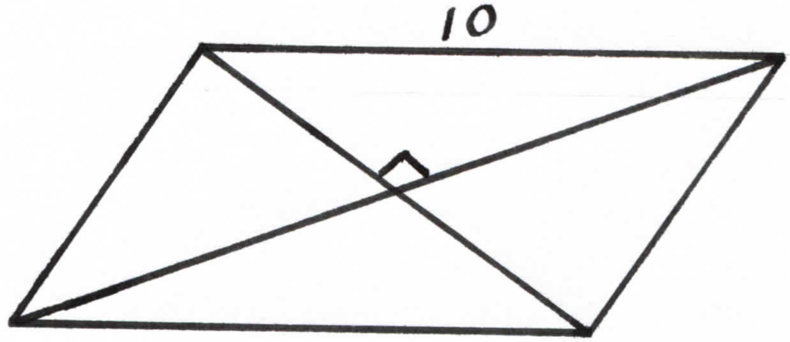


DRAWING 5.9

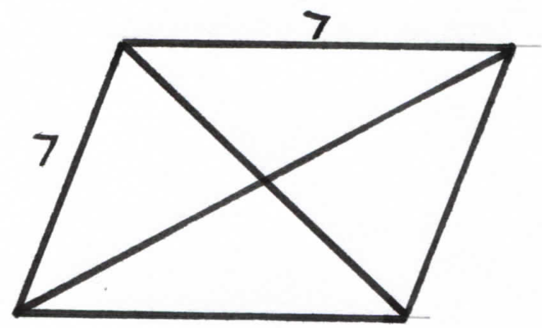


DRAWINGS 5.10

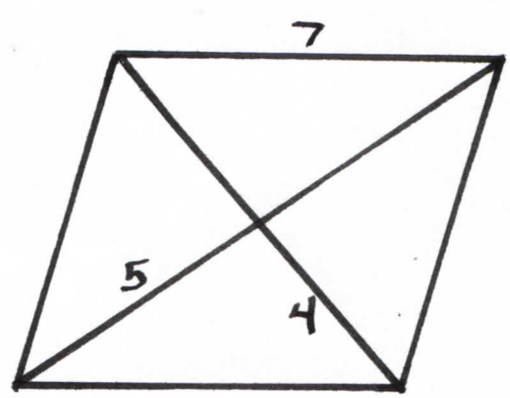
(a)



(b)

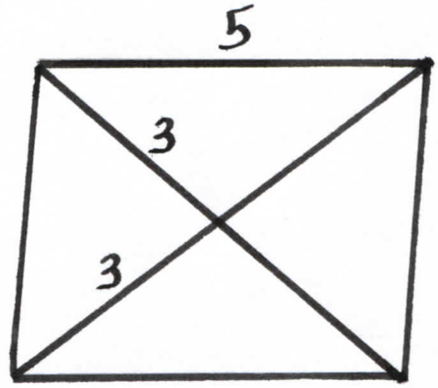


(c)

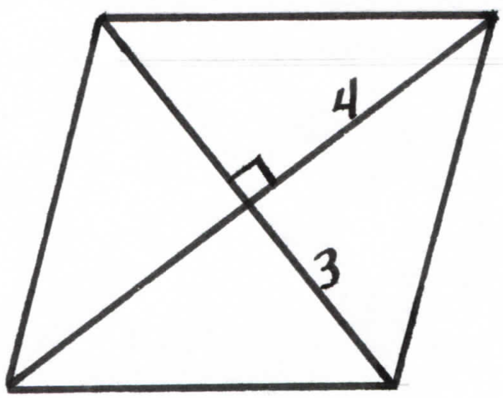


DRAWINGS 5.10 (continued)

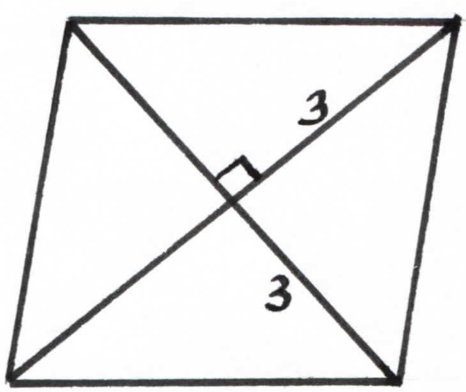
(d)



(e)

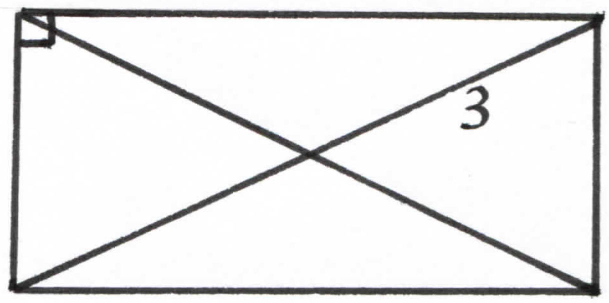


(f)

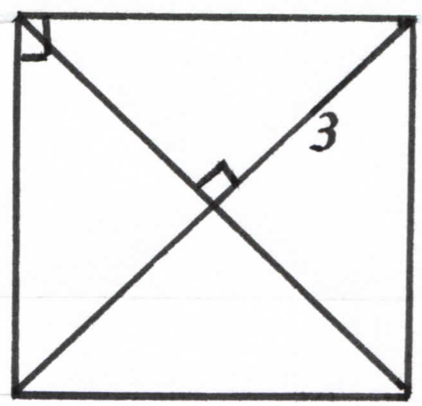


DRAWINGS 5.10 (continued)

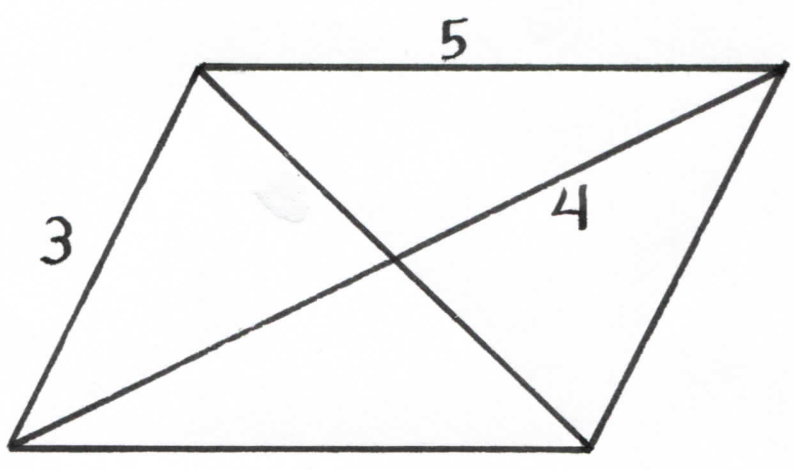
(g)



(h)

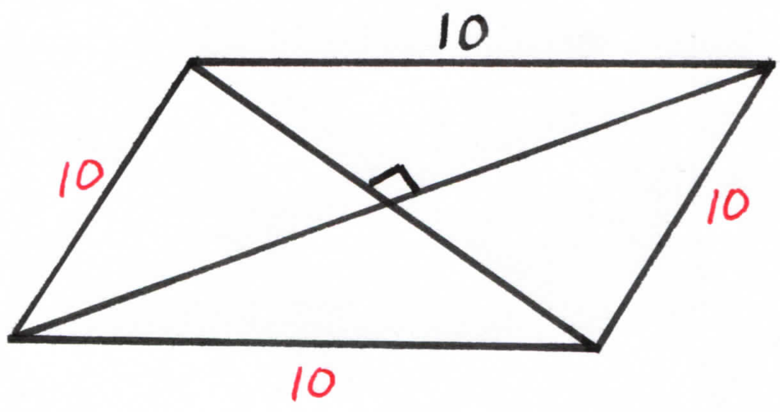


(i)

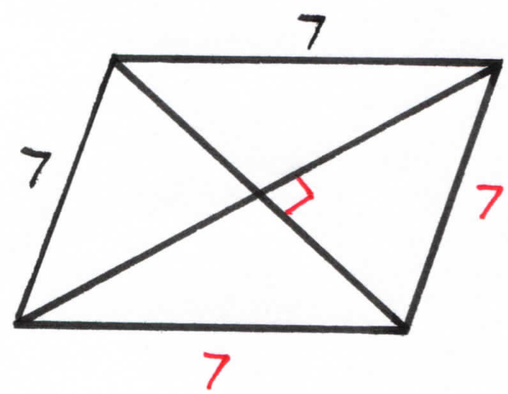


DRAWINGS 5.11

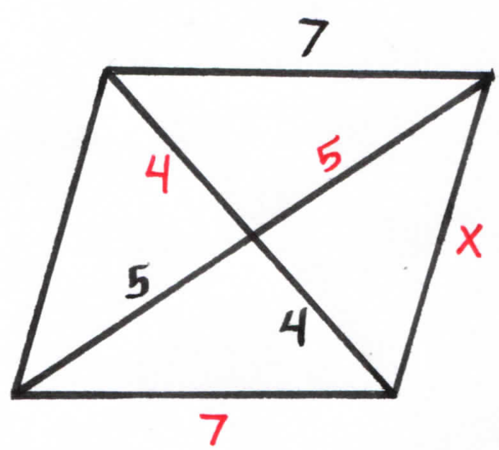
(a)



(b)



(c)

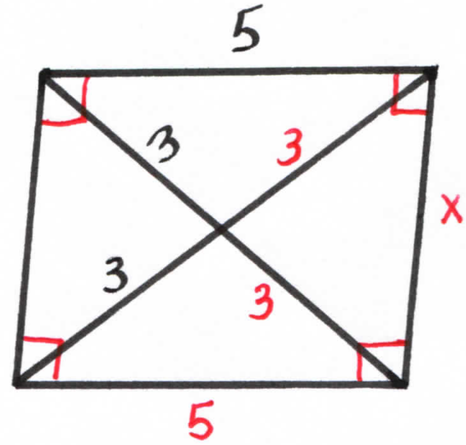


$2(7^2) + 2x^2 = 8^2 + 10^2$

$\longrightarrow \quad x = \sqrt{33}$

DRAWINGS 5.11 (continued)

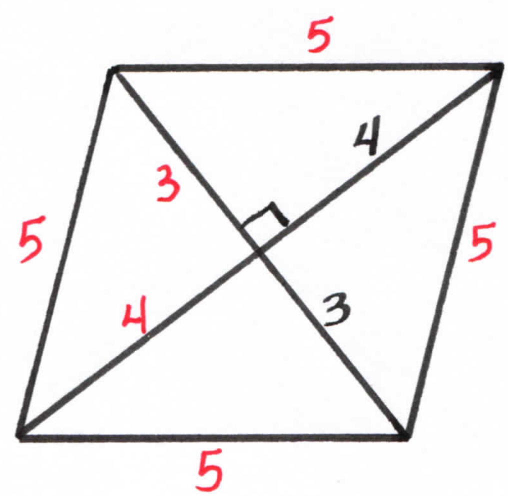
(d)



$2(5^2)+2x^2 = 2(6^2)$

$\rightarrow x = \sqrt{11}$

(e)



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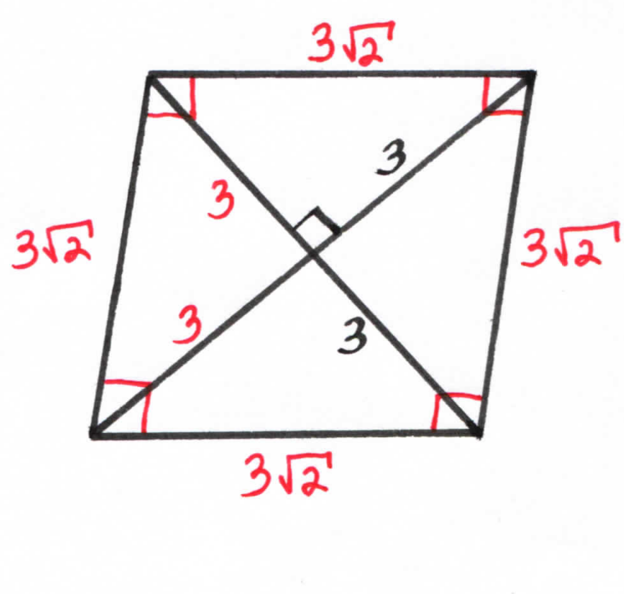
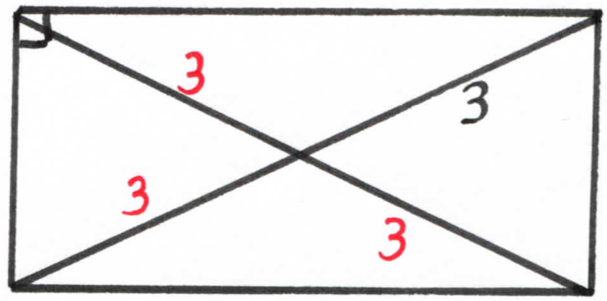


(g)



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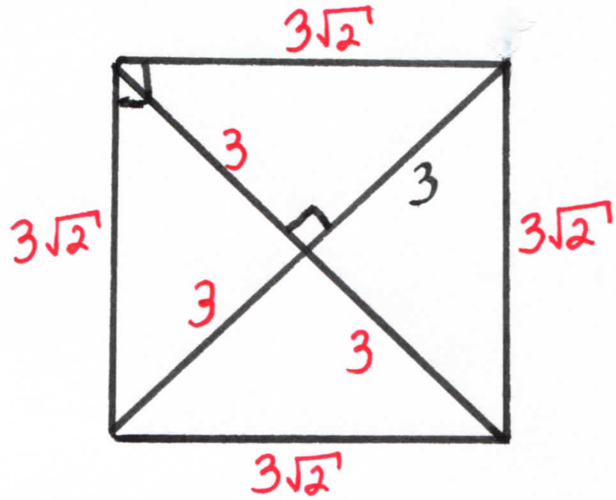
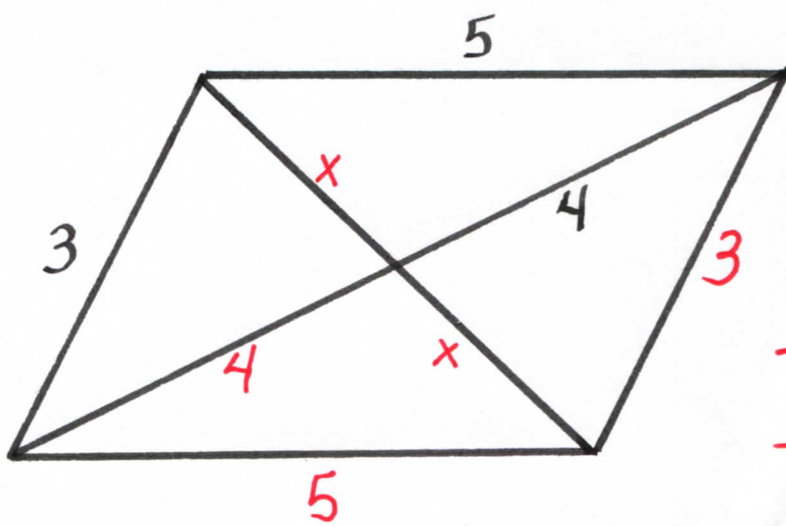


(i)



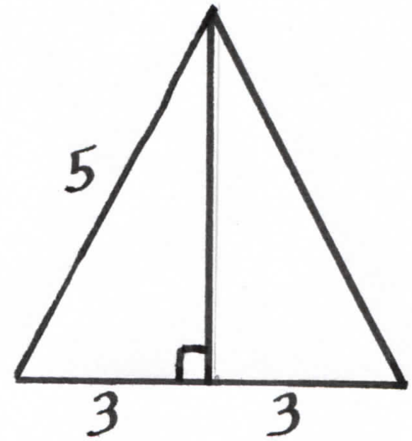
$(2x)^2 + 8^2 = 2(5^2) + 2(3^2)$

$\rightarrow 4x^2 + 64 = 68$

$\rightarrow x = 1$

DRAWINGS 5.12

(a)



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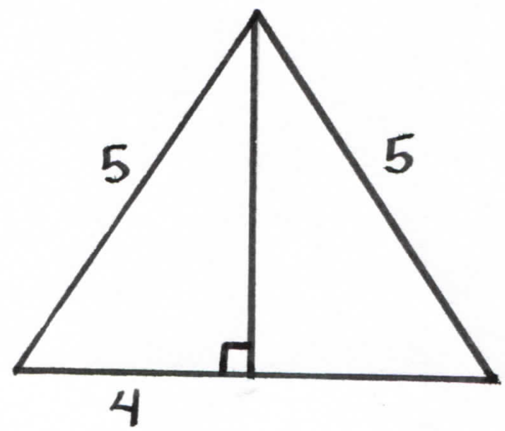

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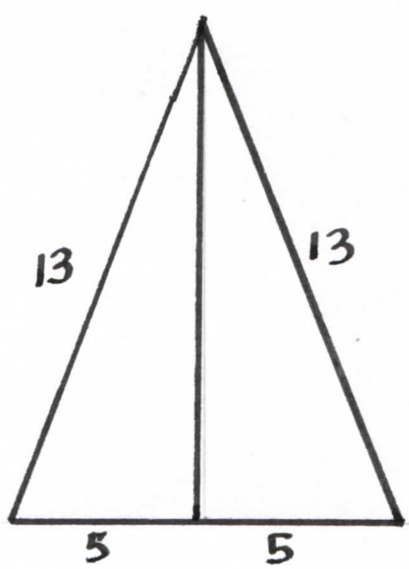
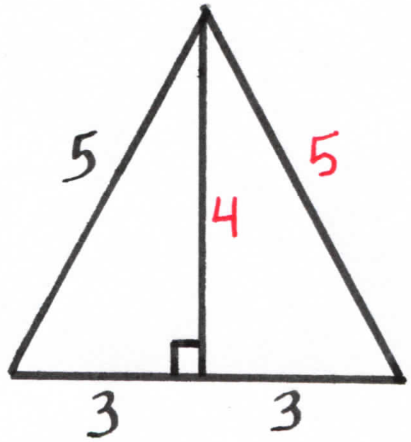


DRAWINGS 5.13

(a)



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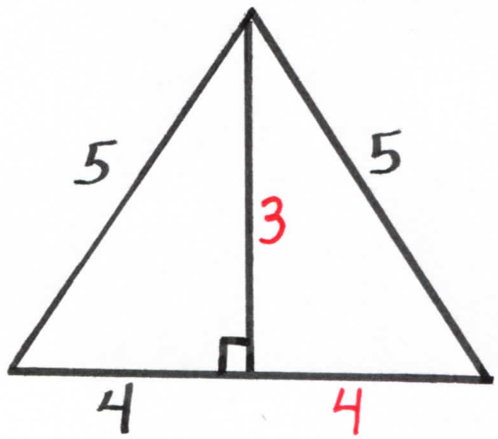

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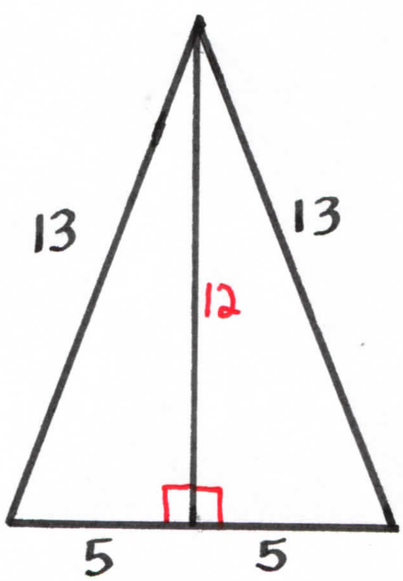
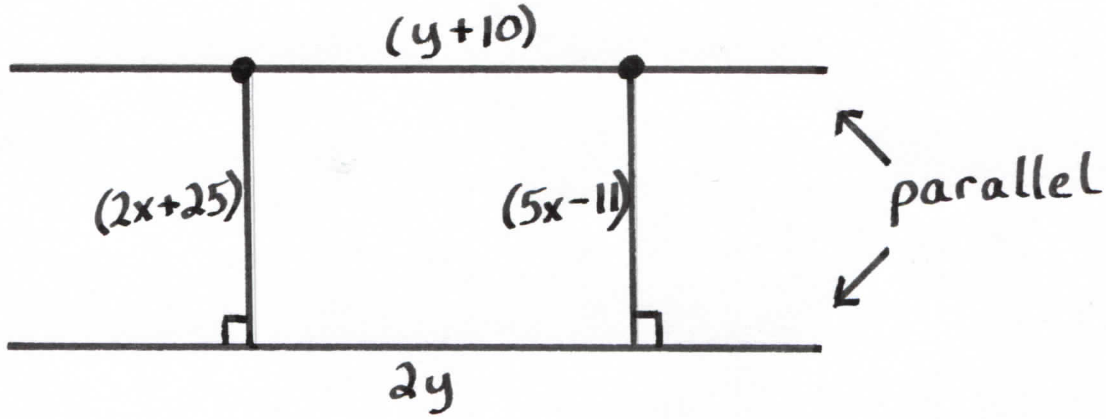


DRAWINGS 5.14

(a)



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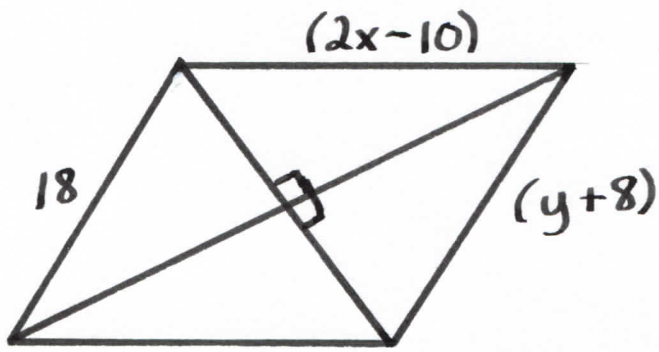

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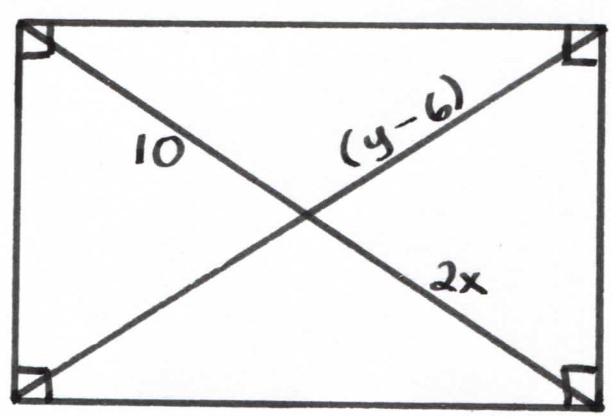
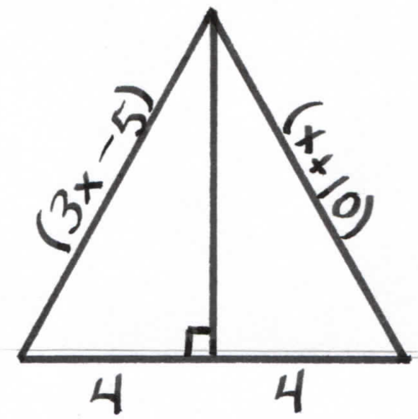
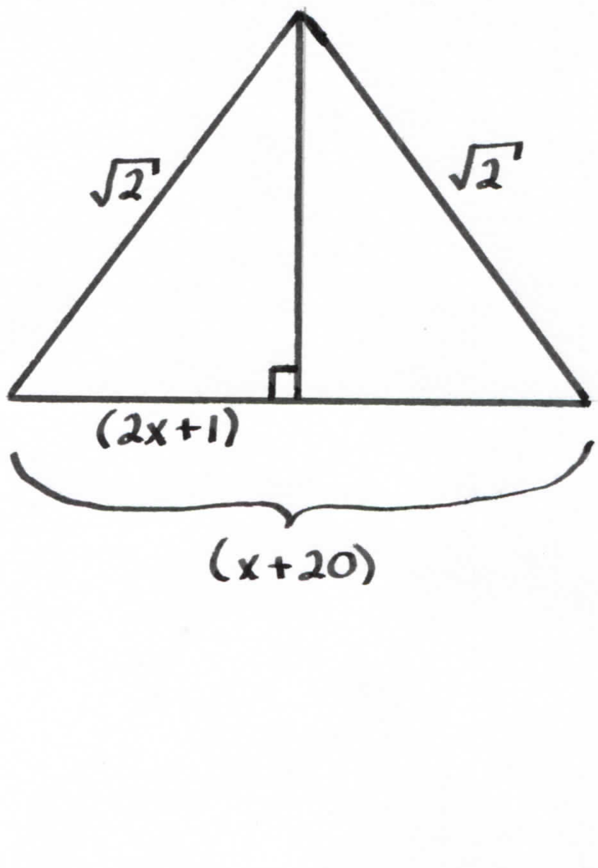


DRAWINGS 5.14 (continued)

(d)

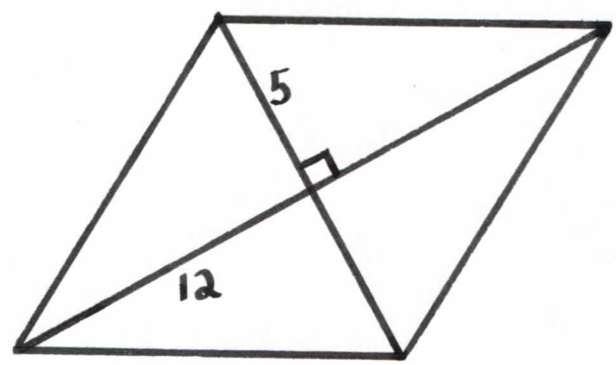


(e)



DRAWINGS 5.15

(a)



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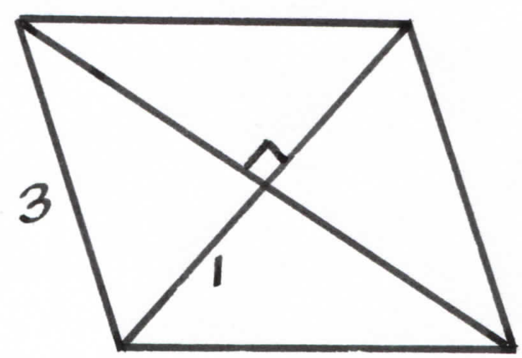

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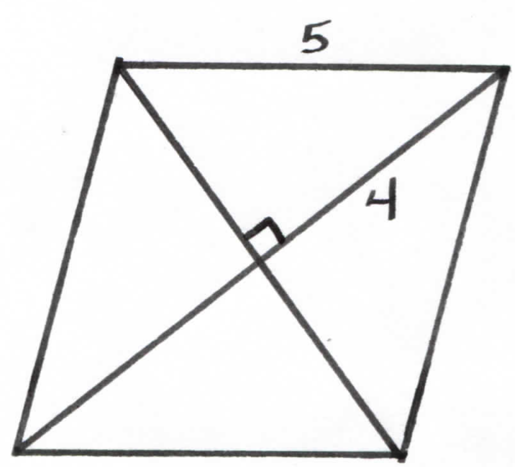
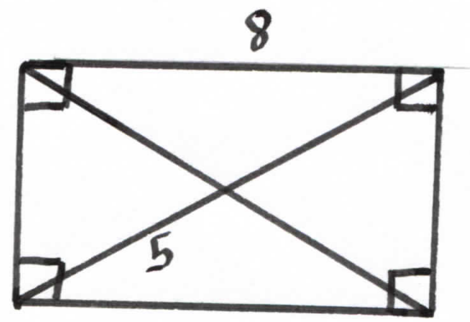


DRAWINGS 5.15 (continued)

(d)



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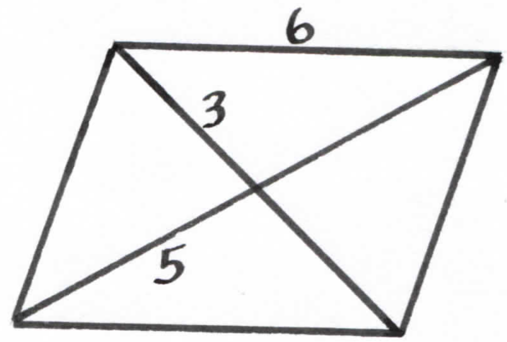
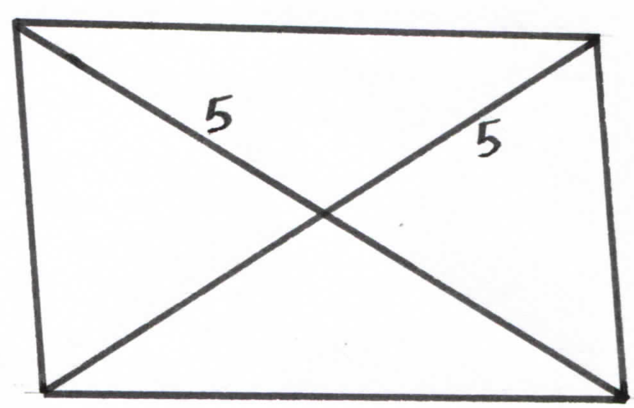
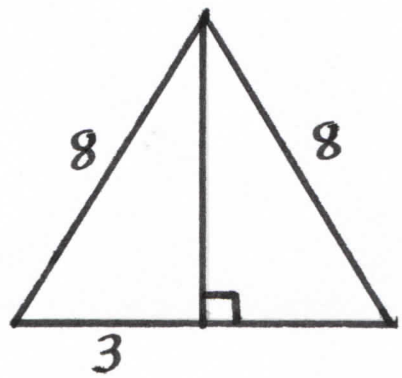


(f)



DRAWINGS 5.16

(a)



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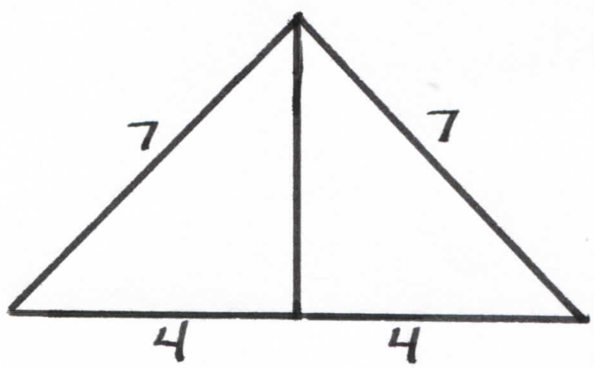

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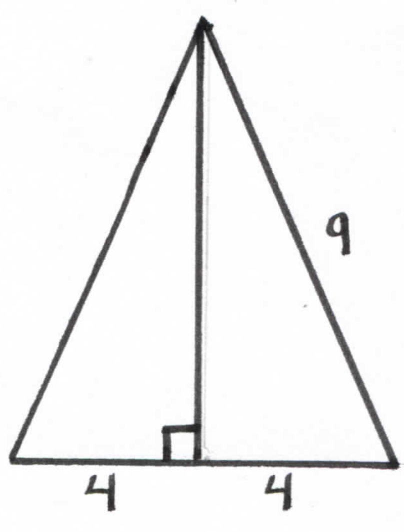
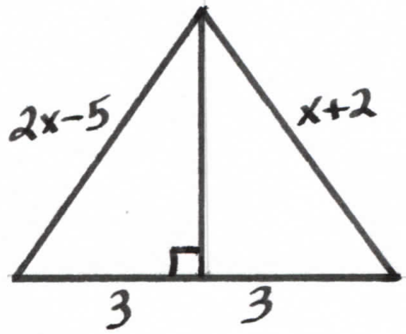


DRAWINGS 5.17

(a)



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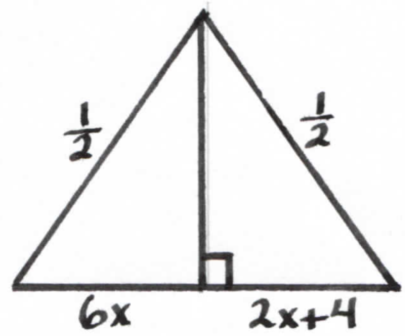

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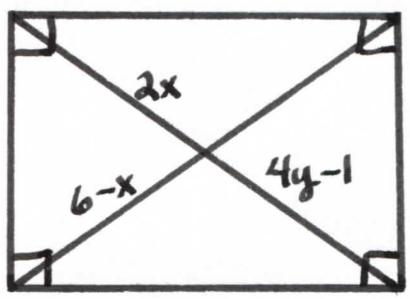
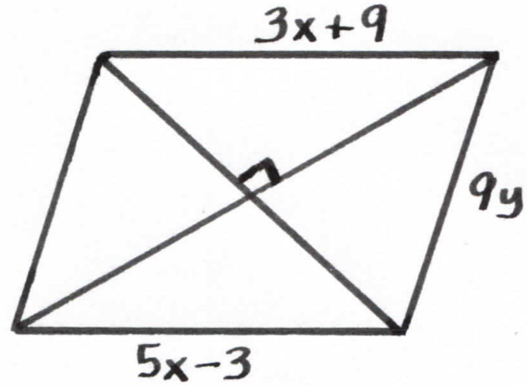


(d)



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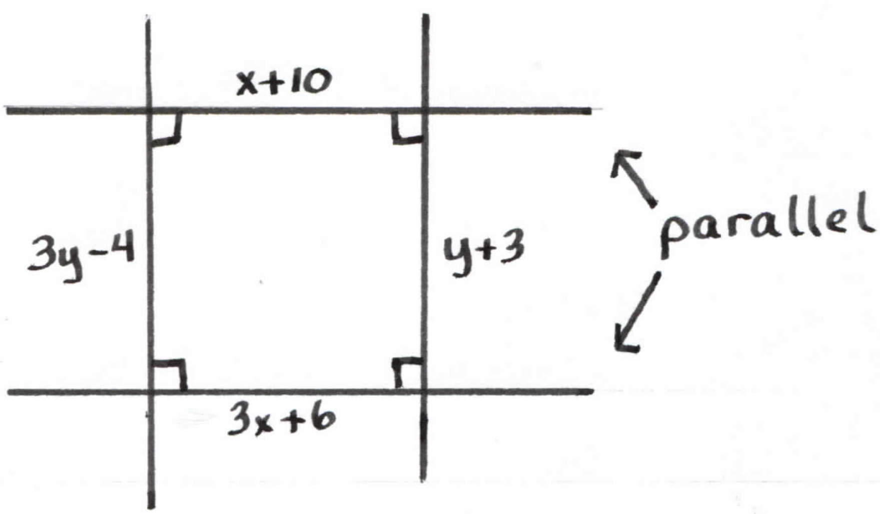
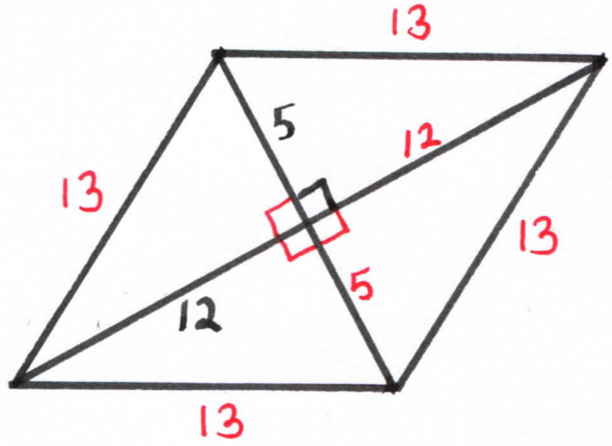


DRAWINGS 5.18

(a)



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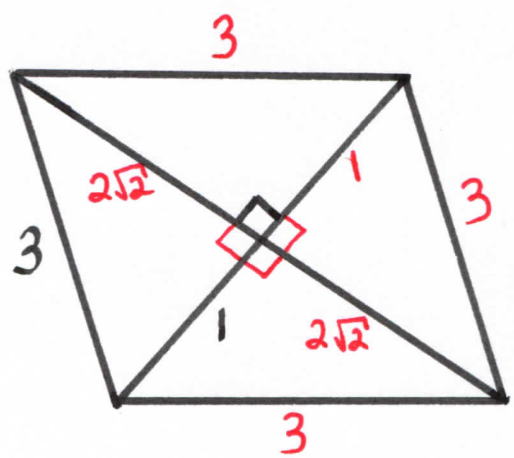

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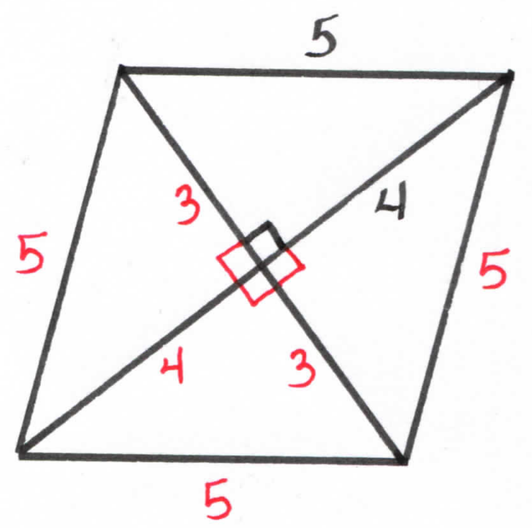
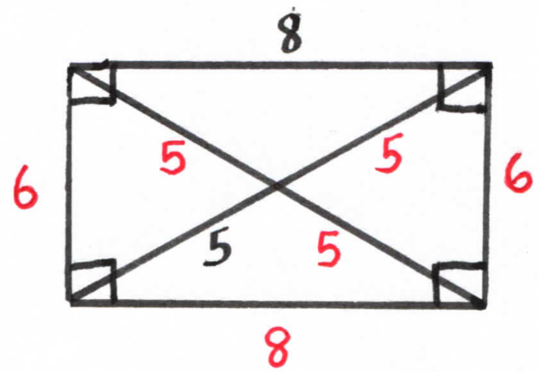
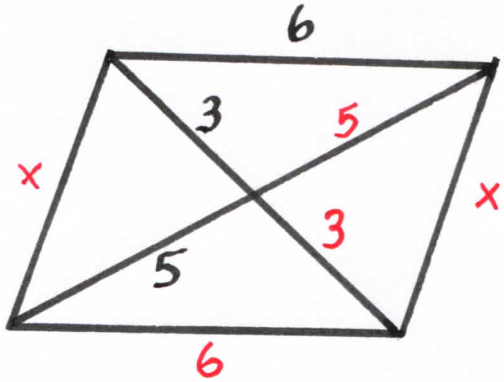


DRAWINGS 5.18 (continued)

(d)



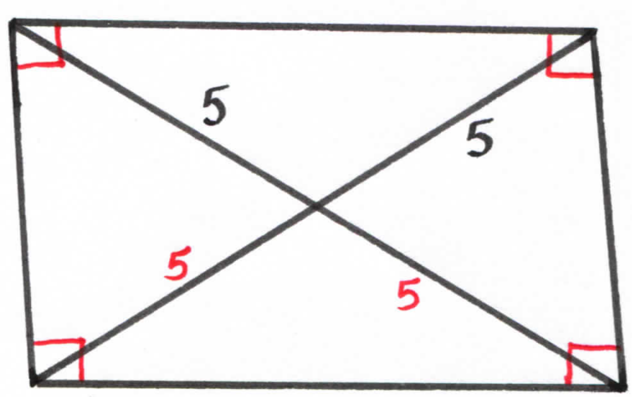
(e)



$2x^2+2(6^2)=10^2+6^2$

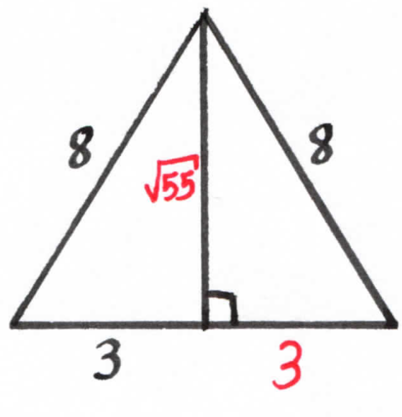
$\rightarrow x=\sqrt{32}$

(f)

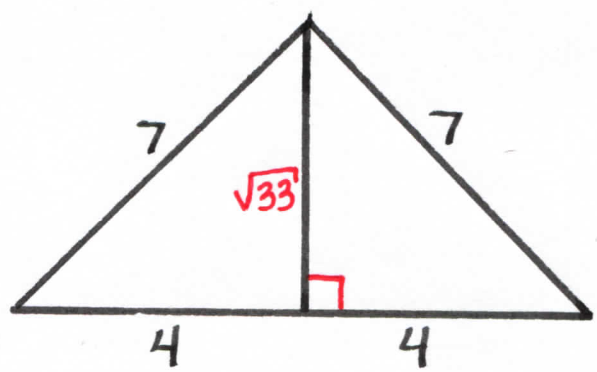


DRAWINGS 5.19

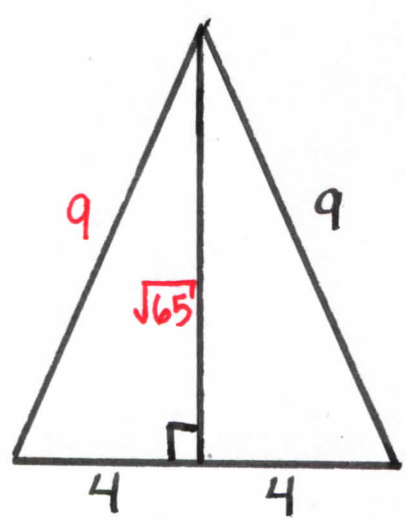
(a)



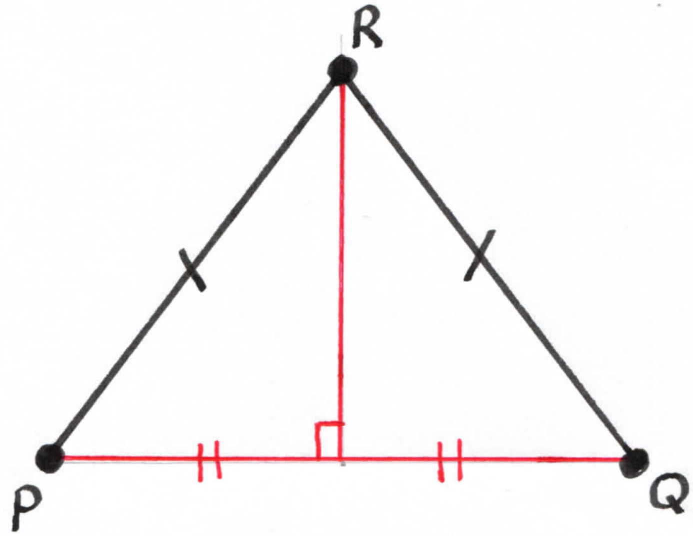
(b)



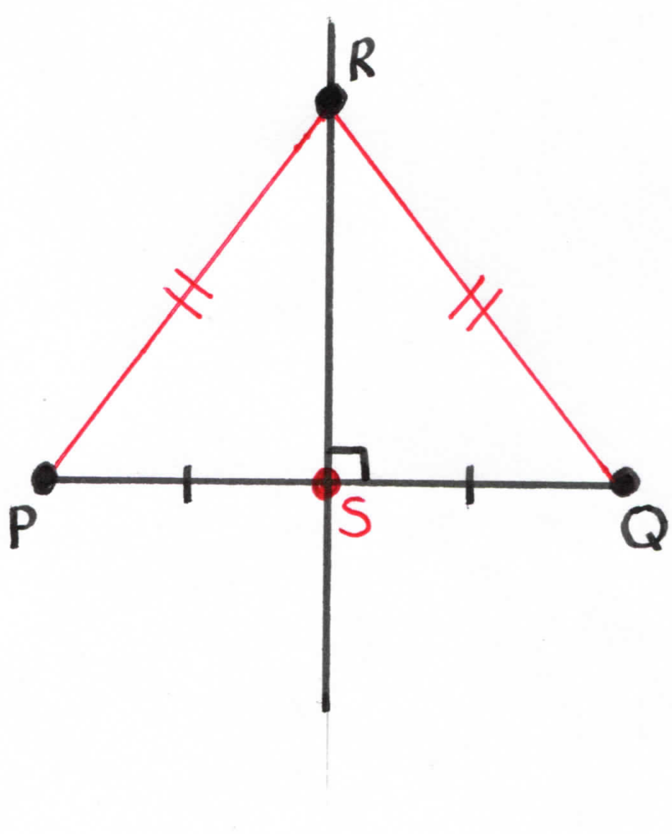
(c)



DRAWING 5.20



DRAWING 5.21



CHAPTER VI: Exponential, Cosine and Sine; Angle Revisited.

We used the dot product to characterize *orthogonality*, which turned out to correspond to an angle of measure $\frac{\pi}{2}$ (see Proposition 4.15). We shall see in this chapter that *any* angle measure may be described with dot product. This will be accomplished after using the polar form of complex numbers (Definition 1.15) to introduce what is traditionally taught as a completely separate and disjoint subject, in subject matter and style, what is known as *trigonometry* (see Definition 6.1). We shall see that the partitioning separation of geometry and trigonometry is unnatural, dependent on fundamental misunderstandings of math: both trigonometry and geometry are intimately involved with triangles and their relevant parameters (length, area, and angle).

The key results of this chapter are described by DRAWING 6.1 at the end of this chapter, where the fundamental trigonometric functions sine and cosine are defined, and Theorem 6.9, where the angle between vectors is measured with dot product and norm.

Complex numbers (from Chapter I) will give a straightforward definition of the trigonometric functions sine and cosine (Definition 6.1) and a very useful way to calculate angle measure (Theorem 6.9).

The reader should, before proceeding, review the exponential function described in 1.12–1.15, Lemma 2.8, and, most importantly, DRAWING 2.11 at the end of Chapter II.

All you need to memorize about trig (short for "trigonometry") is in the following definition and DRAWING 6.1 at the end of this chapter; compare to DRAWING 2.11 at the end of Chapter II. Everything in trig follows from DRAWING 6.1; Definitions 6.1 is the *first principle* that any serious thinker tries to identify.

Definitions 6.1. Let θ be any real number. Let "cos" be short for **cosine**, "sin" for **sine**. Define the **trigonometric functions**

$$\cos(\theta) \equiv \mathrm{Re}(e^{i\theta}), \quad \sin(\theta) \equiv \mathrm{Im}(e^{i\theta}).$$

Thus, for $0 \leq \theta \leq 2\pi$,

$$< \cos\theta, \sin\theta >$$

is the unit vector such that the counterclockwise angle (Definitions 2.10) from the unit vector $< 1, 0 >$ to $< \cos\theta, \sin\theta >$ has measure θ. See DRAWING 6.1 at the end of this chapter, and recall that angles corresponds to arcs of the unit circle and angle measure corresponds to arclength, as we drew in a curved way.

Read out loud, "$\cos(\theta)$" reads "cosine of θ," "$\sin(\theta)$" reads "sine of θ."

Definition 6.2. Stated explicitly as

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

Definition 6.1 is called **Euler's formula**.

Compare the easily identified (by symmetry of the unit circle) sines and cosines in DRAWING 6.1 at the end of this chapter to DRAWING 2.13 at the end of Chapter II.

Some Properties 6.3. From staring at the picture of $(\cos\theta, \sin\theta)$ in DRAWING 6.1 at the end of this chapter and using the symmetry of the unit circle, the following properties seem believable, for any real θ (see DRAWINGS 6.3 at the end of this chapter). Euler's formula provides straightforward proofs.

(i) $\cos(-\theta) = \cos\theta$.

(ii) $\sin(-\theta) = -\sin\theta$.

(iii) $\cos(\theta+\pi) = -\cos\theta$.

(iv) $\sin(\theta+\pi) = -\sin\theta$.

(v) $(\cos\theta)^2 + (\sin\theta)^2 = 1$.

(vi) $|\sin\theta| \le 1$ and $|\cos\theta| \le 1$.

(vii) $\cos(\frac{\pi}{2}+\theta) = -\sin\theta = -\cos(\frac{\pi}{2}-\theta)$ and $\sin(\frac{\pi}{2}+\theta) = \cos\theta = \sin(\frac{\pi}{2}-\theta)$.

(viii) $\cos(\theta+2k\pi) = \cos\theta$, $\sin(\theta+2k\pi) = \sin\theta$, for any real θ, integer k (this is called **periodicity** of sine and cosine).

Proof: See Theorem 1.12 for relevant properties of the exponential. We will make extensive use of Euler's formula in Definition 6.2.

The calculation

$$\cos(-\theta) + i\sin(-\theta) = e^{-i\theta} = e^{\overline{i\theta}} = \overline{e^{i\theta}} = \cos(\theta) + i(-\sin(\theta))$$

implies Properties 6.3(i) and (ii).

$$\cos(\theta+\pi) + i\sin(\theta+\pi) = e^{i(\theta+\pi)} = e^{i\theta}e^{i\pi} = e^{i\theta}(-1) = (-\cos(\theta)) + i(-\sin(\theta))$$

implies Properties 6.3(iii) and (iv).

Property 6.3(v) follows from Euler's formula, since $1 = |e^{i\theta}|^2$, by Corollary 1.13, while (vi) follows immediately from (v).

For (vii), make two calculations:

$$\cos(\frac{\pi}{2}-\theta)+i\sin(\frac{\pi}{2}-\theta) = e^{i(\frac{\pi}{2}-\theta)} = e^{i\frac{\pi}{2}}e^{-i\theta} = ie^{\overline{i\theta}} = i\overline{e^{i\theta}} = i(\overline{\cos\theta + i\sin\theta}) = i(\cos\theta - i\sin\theta) = \sin\theta + i\cos\theta$$

and

$$\cos(\frac{\pi}{2}+\theta) + i\sin(\frac{\pi}{2}+\theta) = e^{i(\frac{\pi}{2}+\theta)} = e^{i\frac{\pi}{2}}e^{i\theta} = i(\cos\theta + i\sin\theta) = (-\sin\theta) + i\cos\theta,$$

thus, equating real and imaginary parts gives the results.

For the periodicity (viii), write

$$\cos(\theta+2k\pi) + i\sin(\theta+2k\pi) = e^{i(\theta+2k\pi)} = e^{i\theta}e^{2k\pi i} = e^{i\theta}\left(e^{2\pi i}\right)^k = e^{i\theta}1^k = e^{i\theta} = \cos\theta + i\sin\theta,$$

as desired. $\square$

We similarly get quick algebraic proofs of less-believable formulas for sine and cosine.

Proposition 6.4. Let θ, ψ be arbitrary real numbers.

(i) $\cos(\theta+\psi) = \cos\theta\cos\psi - \sin\theta\sin\psi$.

(ii) $\sin(\theta+\psi) = \sin\theta\cos\psi + \sin\psi\cos\theta$.

(iii) $(\cos\theta)(\cos\psi) = \frac{1}{2}(\cos(\theta+\psi) + \cos(\theta-\psi))$.

(iv) $(\sin\theta)(\sin\psi) = \frac{1}{2}(\cos(\theta-\psi) - \cos(\theta+\psi))$.

(v) $(\sin\theta)(\cos\psi) = \frac{1}{2}(\sin(\theta+\psi) + \sin(\theta-\psi))$.

(vi) $(\cos\theta)^2 = \frac{1}{2}(1+\cos(2\theta))$.

(vii) $(\sin\theta)^2 = \frac{1}{2}(1-\cos(2\theta))$.

(viii) $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$.

(ix) $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$.

Proof: For (i) and (ii), calculate

$$\cos(\theta+\psi) + i\sin(\theta+\psi) = e^{i(\theta+\psi)} = e^{i\theta}e^{i\psi} = (\cos\theta + i\sin\theta)(\cos\psi + i\sin\psi)$$
$$= (\cos\theta\cos\psi - \sin\theta\sin\psi) + i\,(\cos\theta\sin\psi + \sin\theta\cos\psi)\,,$$

so that equating the real and imaginary parts gives both *sum-of-angles* results simultaneously.

(iii) and (iv) follow from (i) and 6.3(i) and (ii); (v) follows from (ii) and 6.3(i) and (ii). (vi) and (vii) are (iii) and (iv) with $\theta = \psi$. (viii) and (ix) follow from Euler's formula. □

Examples 6.5. Let's get some explicit sines and cosines.

From DRAWING 6.2 at the end of this chapter,

$$\cos(0) = 1 = \cos(2\pi), \sin(0) = 0 = \sin(2\pi), \cos(\frac{\pi}{2}) = 0, \sin(\frac{\pi}{2}) = 1, \cos(\pi) = -1, \sin(\pi) = 0, \cos(\frac{3\pi}{2}) = 0, \sin(\frac{3\pi}{2}) = -$$

From Examples 2.9, $e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}}(1+i)$, thus

$$\cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}} = \sin(\frac{\pi}{4}).$$

By 6.3(vii),

$$\cos(\frac{3\pi}{4}) = -\frac{1}{\sqrt{2}},\ \ \sin(\frac{3\pi}{4}) = \frac{1}{\sqrt{2}}.$$

By 6.3(i) and (ii),

$$\cos(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}},\ \ \sin(-\frac{\pi}{4}) = -\frac{1}{\sqrt{2}}.$$

The sines and cosines of $\frac{\pi}{6}$ and $\frac{\pi}{3}$ will follow from 6.3 and Theorem 1.12, as follows.

Denote $a \equiv \cos(\frac{\pi}{6})$ and $b \equiv \sin(\frac{\pi}{6})$.

Since $\frac{\pi}{3} = \left(\frac{\pi}{2} - \frac{\pi}{6}\right)$, 6.3(vii) implies that

$$\cos(\frac{\pi}{3}) = b \quad \text{and} \quad \sin(\frac{\pi}{3}) = a,$$

thus, by Theorem 1.12,

$$(a^2 - b^2) + i(2ab) = (a+ib)^2 = \left(e^{i\frac{\pi}{6}}\right)^2 = e^{i\frac{\pi}{3}} = (b+ia),$$

so that $2ab = a$, thus

$$\sin(\frac{\pi}{6}) = \cos(\frac{\pi}{3}) = \frac{1}{2};$$

6.3(v) now implies that

$$\sin(\frac{\pi}{3}) = \cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}.$$

See also Examples 7.11(e) and HWII.11(g).

Let's expand our limited knowledge a little more:

$$\cos(\frac{\pi}{8}) = \sqrt{\frac{1}{2}(1+\cos(\frac{\pi}{4}))} = \sqrt{\frac{1}{2}(1+\frac{1}{\sqrt{2}})} \ \text{ and } \ \sin(\frac{\pi}{8}) = \sqrt{\frac{1}{2}(1-\cos(\frac{\pi}{4}))} = \sqrt{\frac{1}{2}(1-\frac{1}{\sqrt{2}})},$$

from 6.4 (vi) and (vii).

$$\cos(\frac{\pi}{12}) = \cos(\frac{\pi}{3} - \frac{\pi}{4}) = \cos(\frac{\pi}{3})\cos(-\frac{\pi}{4}) - \sin(\frac{\pi}{3})\sin(-\frac{\pi}{4}) = \frac{1}{2} \times \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \times (-\frac{1}{\sqrt{2}}) = \frac{1+\sqrt{3}}{2\sqrt{2}},$$

$$\sin(\frac{\pi}{12}) = \sin(\frac{\pi}{3} - \frac{\pi}{4}) = \sin(\frac{\pi}{3})\cos(-\frac{\pi}{4}) + \cos(\frac{\pi}{3})\sin(-\frac{\pi}{4}) = \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times (-\frac{1}{\sqrt{2}}) = \frac{\sqrt{3}-1}{2\sqrt{2}},$$

by 6.4 (i) and (ii) and 6.3 (i) and (ii).

We could also get cosine and sine of $\frac{\pi}{12}$ as we did with $\frac{\pi}{8}$:

$$\cos(\frac{\pi}{12}) = \sqrt{\frac{1}{2}(1+\cos(\frac{\pi}{6}))} = \sqrt{\frac{1}{2}(1+\frac{\sqrt{3}}{2})} \quad \text{and} \quad \sin(\frac{\pi}{12}) = \sqrt{\frac{1}{2}(1-\cos(\frac{\pi}{6}))} = \sqrt{\frac{1}{2}(1-\frac{\sqrt{3}}{2})},$$

from 6.4 (vi) and (vii).

Notice how different *looking* our two different expressions for sine and cosine of $\frac{\pi}{12}$ are.

Recall the equating of vectors and complex numbers, as in 1.16 and the comments before Proposition 4.14.

Proposition 6.6. The dot product of the vectors $r_1 e^{i\theta_1}$ and $r_2 e^{i\theta_2}$ is $r_1 r_2 \cos(\theta_1 - \theta_2)$.

Proof: Proposition 4.14 and Euler's formula 6.2. □

Definition 6.7. $\cos^{-1}$, the **inverse cosine** function is defined by

$$\cos^{-1} y = x \quad \iff \quad \cos x = y \text{ and } 0 \le x \le \pi;$$

that is, $\cos^{-1} y$ is the number between 0 and π whose cosine is y.

More succinctly (see Appendix 0.3 and DRAWING 6.4 at the end of this chapter),

$$\cos^{-1} \equiv \left(\cos|_{[0,\pi]}\right)^{-1}.$$

Remarks 6.8. See Appendix Four for a different definition of $\cos^{-1}, \cos, \sin$, and exponential, in that order, via integration.

Most calculators have cos and $\cos^{-1}$ buttons, for decimal approximations, along with a choice of degrees or radians.

Theorem 6.9. The measure of the angle between vectors $\vec{a}$ and $\vec{b}$ is given by

$$\cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|}\right).$$

Proof: By Lemma 2.8 and the periodicity of $e^{i\theta}$, we may write $\vec{a}$ and $\vec{b}$ in polar form $r_1 e^{i\theta_1}$ and $r_2 e^{i\theta_2}$, with $0 \le \theta_1 \le \theta_2 \le \theta_1 + 2\pi$, as in Definitions 2.10.

From Proposition 6.6, with $r_1 \equiv \|\vec{a}\|, r_2 \equiv \|\vec{b}\|$,

$$\left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|}\right) = \cos(\theta_2 - \theta_1) = \cos(2\pi - (\theta_2 - \theta_1)).$$

By Definitions 2.10, $(\theta_2 - \theta_1)$ is the measure of the counterclockwise angle from $\vec{a}$ to $\vec{b}$ and $(2\pi - (\theta_2 - \theta_1))$ is the measure of the clockwise angle from $\vec{a}$ to $\vec{b}$ (see DRAWING 2.14 at the end of Chapter II).

Since those angle measures add up to 2π, the smaller of them is the angle between 0 and π whose cosine is $\left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|}\right)$; that is,

$$\cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|}\right)$$

is the smaller of those two angle measures. $\square$

The following should be compared to Proposition 4.15.

Corollary 6.10. If $\vec{a}$ and $\vec{b}$ are two nontrivial vectors, then $(\vec{a}\cdot\vec{b}) > 0$ if and only if the angle between $\vec{a}$ and $\vec{b}$ has measure less than $\frac{\pi}{2}$.

Proof: For $0 \le \theta \le \pi$, the definition of cosine implies that

$$(*) \quad \cos\theta > 0 \iff \theta < \frac{\pi}{2}.$$

Denote $y \equiv \cos\theta$, then $\theta = \cos^{-1} y$, so (*) becomes

$$(**) \quad y > 0 \iff \cos^{-1} y < \frac{\pi}{2}.$$

Thus, for θ defined to be the measure of the angle between $\vec{a}$ and $\vec{b}$, by (**),

$$\vec{a}\cdot\vec{b} > 0 \iff y \equiv \left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|}\right) > 0 \iff \theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|}\right) < \frac{\pi}{2}.$$

$\square$

Examples 6.11. (a) Find the measure of the angle between $< 1, 2 >$ and $< -3, 1 >$.

(b) Find the measure of the angle between $< -1, 1 >$ and $< 0, -1 >$.

Solutions. (a) We need norms and dot products:
$< 1, 2 > \cdot < -3, 1 >= -3+2 = -1, \quad \| < 1, 2 > \| = \sqrt{1+4} = \sqrt{5}, \quad \| < -3, 1 > \| = \sqrt{9+1} = \sqrt{10},$
thus, by Theorem 6.9, the angle measure is $\cos^{-1}\left(\frac{-1}{\sqrt{5}\sqrt{10}}\right) = \cos^{-1}\left(\frac{-1}{\sqrt{50}}\right)$.

From a calculator, a decimal approximation is 1.713 radians (notice that this is greater than $\frac{\pi}{2}$, since the dot product is negative) or 98.13 degrees (notice that this is greater than 90 degrees).

(b) $\frac{<-1,1>\cdot<0,-1>}{\|<-1,1>\|\|<0,-1>\|} = -\frac{1}{\sqrt{2}}$. We already know, from Examples 6.5, that $\cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$; reflecting through the y axis (see 6.3(vii) and DRAWING 6.5 at the end of this chapter) tells us that $\cos(\frac{3\pi}{4}) = -\frac{1}{\sqrt{2}}$, so that our angle measure is

$$\cos^{-1}(-\frac{1}{\sqrt{2}}) = \frac{3\pi}{4}.$$

Remark 6.12. Theorem 6.9 provides another proof of Proposition 4.15 (a) $\iff$ (d), equating orthogonality of vectors $\vec{a}$ and $\vec{b}$ with the angle between $\vec{a}$ and $\vec{b}$ being $\frac{\pi}{2}$.

Remarks 6.13. (a) See Proposition 4.15 for other characterizations of orthogonality.

(b) The periodicity of cosine and sine (6.3(viii)) causes them to be good models of *waves*, such as sound waves and electromagnetic waves, including light and radio. See DRAWING 6.6 at the end of this chapter for cosine.

A general function describing waves is

$$f(x) \equiv A\cos(\gamma x - \psi),$$

where A and γ are positive real numbers and ψ is real. A is called **amplitude**, γ corresponds to **frequency**, and ψ is a phase shift. For example, if $f(x)$ is describing sound, A corresponds to volume and γ to pitch.

HOMEWORK

HWVI.1. Use the expression from the proof of Theorem 6.9:

$$\left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|}\right) = \cos\left(\theta_1 - \theta_2\right)$$

to prove the **Cauchy-Schwarz inequality**

$$|\vec{a}\cdot\vec{b}| \leq \|\vec{a}\|\|\vec{b}\|$$

for any vectors $\vec{a}, \vec{b}$.

HWVI.2. Use the Cauchy-Schwarz inequality in HWVI.1 to prove the **triangle inequality**

$$\|\vec{a}+\vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$$

for any vectors $\vec{a}, \vec{b}$. See DRAWING 1.5 at the end of Chapter I for the "triangle" part of this inequality.

The triangle inequality is stating the popular cliche about the shortest distance between two points being a straight line.

HWVI.3. For each of the following, find the sine and cosine of the given angle measure θ. See HWII.11.

(a) 4π.

(b) 5π.

(c) $\frac{13\pi}{4}$.

(d) $\frac{17\pi}{4}$.

(e) $\frac{3\pi}{2}$.

(f) $\frac{18\pi}{2}$.

HWVI.4. For arbitrary integers k, get a general expression for the sine and cosine of $k\frac{\pi}{2}$. Your answer should have ks in it.

HWVI.5. Get exact expressions (no calculator appproximations) for the sines and cosines of the following angles.

(a) $\frac{2\pi}{3}$.

(b) $\frac{7\pi}{6}$.

(c) $\frac{7\pi}{12}$.

(d) $\frac{5\pi}{12}$.

(e) $-\frac{7\pi}{12}$.

(f) $\frac{13\pi}{12}$.

HWVI.6. For each of the following pairs of vectors, find the measure of the angle between them.

(a) $< -1, 1 >$ and $< 1, 1 >$ (compare with HWII.12).

(b) $< 1, 2 >$ and $< -3, 1 >$.

(c) $< -1, -2 >$ and $< -1 + 2\sqrt{3}, -\sqrt{3} - 2 >$.

HWVI.7. (a) Use the Pythagorean theorem and Definition 4.6 to show that, for any vectors $\vec{a}, \vec{b}$,

$$\|\vec{a}\| \geq \|\text{proj}_{\vec{b}}(\vec{a})\|.$$

Don't use the Cauchy-Schwarz inequality or Corollary 4.12.

(b) Use (a) and our algebraic expression Corollary 4.12 for $\text{proj}_{\vec{b}}(\vec{a})$ to prove the Cauchy-Schwarz inequality in HWVI.1.

(This proof works in more than two dimensions; see "Linear Algebra, or E Pluribus Unum," http://teacherscholarinstitute.com/FreeLinearAlgebraBook.html (2017), 6.26, page 428.)

HWVI.8. Suppose $\|\vec{a} + \vec{b}\| > \|\vec{a} - \vec{b}\|$. What can be said about the measure of the angle between $\vec{a}$ and $\vec{b}$? Compare this with Proposition 5.6 and HWIV.11, then restate our result in terms of parallelograms.

HWVI.9. Norms and dot products are numbers associated to vectors or pairs of vectors. Here we'll look at how much a pair of vectors $\vec{a}, \vec{b}$ is specified by identifying $\|\vec{a}\|, \|\vec{b}\|$, and $\vec{a} \cdot \vec{b}$. Informally, this result says that $\vec{a}$ and $\vec{b}$ are determined up to rotation, which is the effect of multiplying by $e^{i\theta}$ (see Theorem 2.13).

Prove the following. Suppose a, b, and c are real numbers, with a and b nonzero, and $\vec{a}$ and $\vec{b}$ are vectors such that

$$\|\vec{a}\| = a, \quad \|\vec{b}\| = b, \quad \text{and} \quad \vec{a} \cdot \vec{b} = c.$$

Then there's real θ so that

$$\vec{b} = be^{i\theta} \quad \text{and} \quad \vec{a} = ae^{i(\theta \pm \theta_1)},$$

where $\theta_1 \equiv \cos^{-1}(\frac{c}{ab})$.

HOMEWORK ANSWERS

HWVI.1.

$$\left|\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|}\right| = |\cos(\theta_1-\theta_2)| \le 1,$$

by 6.3(vi).

HWVI.2.

$$\|\vec{a}+\vec{b}\|^2 = \|\vec{a}\|^2+\|\vec{b}\|^2+2(\vec{a}\cdot\vec{b}) \le \|\vec{a}\|^2+\|\vec{b}\|^2+2(\|\vec{a}\|\|\vec{b}\|) = \Big(\|\vec{a}\|+\|\vec{b}\|\Big)^2,$$

by 4.5(d) and the Cauchy inequality, in that order.

HWVI.3. (a) cosine is 1, sine is 0.

(b) cosine is -1, sine is 0.

(c) cosine and sine are both $-\frac{1}{\sqrt{2}}$.

(d) cosine and sine are both $\frac{1}{\sqrt{2}}$.

(e) cosine is 0, sine is -1.

(f) cosine is -1, sine is 0.

HWVI.4. In the following, m is an arbitrary integer.
The cosine equals

$$1,\ \text{for } k=4m,\ 0,\ \text{for } k=1+4m \text{ or } k=3+4m,\ -1 \text{ for } k=2+4m.$$

The sine equals

$$0,\ \text{for } k=4m \text{ or } k=2+4m,\ 1,\ \text{for } k=1+4m,\ -1 \text{ for } k=3+4m.$$

HWVI.5. (a) 6.3(vii) or 6.4(i) and (ii) lead to cosine equal to $-\frac{1}{2}$ and sine equal to $\frac{\sqrt{3}}{2}$.

(b) from (a) and 6.3(vii) or 6.4(i) and (ii) or 6.3(iii) and (iv), cosine equals $-\frac{\sqrt{3}}{2}$, sine equals $-\frac{1}{2}$.

(c) from 6.4(vi), cosine equals $-\sqrt{\frac{1}{4}(2-\sqrt{3})}$, sine equals $\sqrt{\frac{1}{4}(2+\sqrt{3})}$, OR, by 6.4(i), cosine equals $\frac{1}{2\sqrt{2}}(1-\sqrt{3})$ and sine equals $\frac{1}{2\sqrt{2}}(1+\sqrt{3})$.

(d) from 6.3(vii) and (c), OR 6.4(i) and (ii), cosine equals $\frac{1}{2\sqrt{2}}(\sqrt{3}-1)$ and sine equals $\frac{1}{2\sqrt{2}}(1+\sqrt{3})$.

(e) from 6.3(i) and (ii), and (c), cosine equals $-\sqrt{\frac{1}{4}(2-\sqrt{3})}$ and sine equals $-\sqrt{\frac{1}{4}(2+\sqrt{3})}$.

(f) from 6.3(iii) and (iv) and Examples 6.5, cosine equals $-\frac{(1+\sqrt{3})}{2\sqrt{2}}$ and sine equals $\frac{(1-\sqrt{3})}{2\sqrt{2}}$.

HWVI.6. (a) $\frac{\pi}{2}$ (b) $\cos^{-1}(\frac{-1}{5\sqrt{2}}) \sim 98.1$ degrees (c) $\frac{\pi}{3}$ or 60 degrees.

HWVI.7. (a) By definition of projection, $\vec{b}$ is orthogonal to $\left(\vec{a}-\mathrm{proj}_{\vec{b}}(\vec{a})\right)$, thus, by the Pythagorean theorem,

$$\|\vec{a}\|^2 = \|\vec{a}-\mathrm{proj}_{\vec{b}}(\vec{a})\|^2 + \|\mathrm{proj}_{\vec{b}}(\vec{a})\|^2 \ge \|\mathrm{proj}_{\vec{b}}(\vec{a})\|^2.$$

(b) By Corollary 4.12 and (a),

$$\|\vec{a}\| \ge \|\mathrm{proj}_{\vec{b}}(\vec{a})\| = \left\|\left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\right)\vec{b}\right\| = \frac{|\vec{a}\cdot\vec{b}|}{\|\vec{b}\|^2}\|\vec{b}\| = \frac{|\vec{a}\cdot\vec{b}|}{\|\vec{b}\|}.$$

HWVI.8. By 4.5(b) and (d),

$$\|\vec{a}+\vec{b}\| > \|\vec{a}-\vec{b}\| \iff \|\vec{a}+\vec{b}\|^2 > \|\vec{a}-\vec{b}\|^2 \iff \|\vec{a}\|^2+\|\vec{b}\|^2+2(\vec{a}\cdot\vec{b}) > \|\vec{a}\|^2+\|\vec{b}\|^2-2(\vec{a}\cdot\vec{b}) \iff (\vec{a}\cdot\vec{b}) > 0,$$

which, by Corollary 6.10, is equivalent to the angle between $\vec{a}$ and $\vec{b}$ measuring less than $\frac{\pi}{2}$.

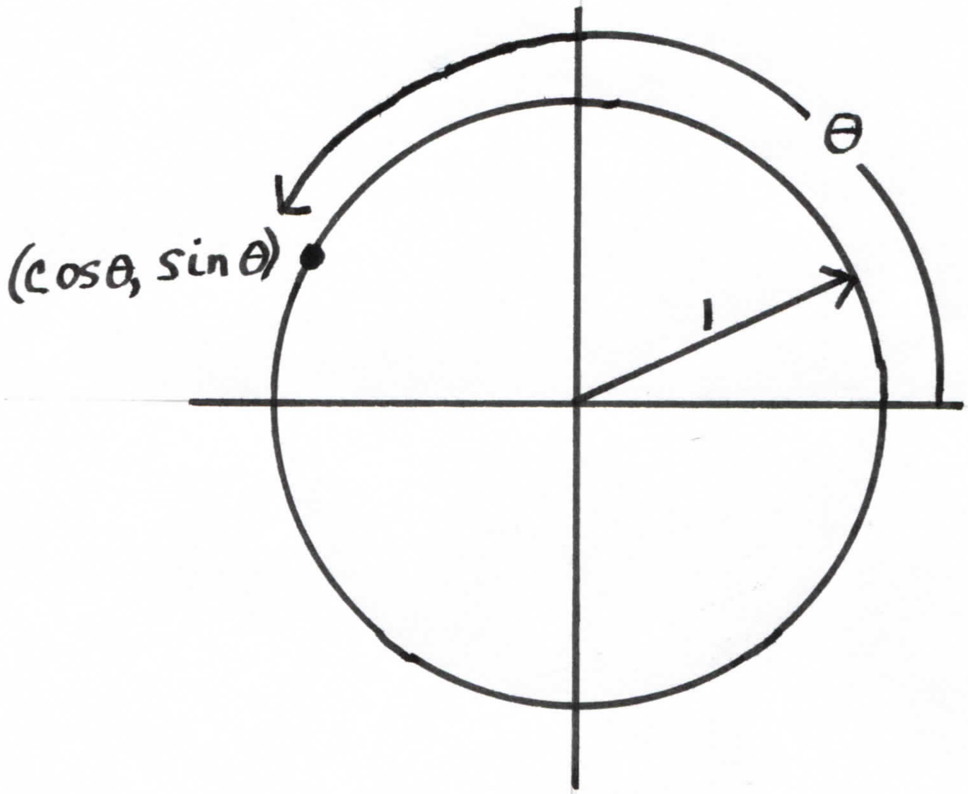
This is similar to Proposition 5.6 and HWIV.11, which says that $\|\vec{a}+\vec{b}\|$ *equal* to $\|\vec{a}-\vec{b}\|$ is equivalent to the angle measure between $\vec{a}$ and $\vec{b}$ being *equal* to $\frac{\pi}{2}$.

Proposition 5.6 is actually a statement about diagonals in a parallelogram. Here is the result of this problem stated in terms of diagonals in a parallelogram: In a parallelogram, the longer diagonal goes between the interior angles of smaller measure. See DRAWING 5.5 at the end of Chapter V for a picture of the diagonals of a parallelogram.

HWVI.9. Lemma 2.8 gives us $\vec{b} = be^{i\theta}$ and $\vec{a} = ae^{i\psi}$, for some real θ and ψ. By Proposition 6.6,

$$c \equiv \vec{a}\cdot\vec{b} = ab\cos(\theta-\psi)$$

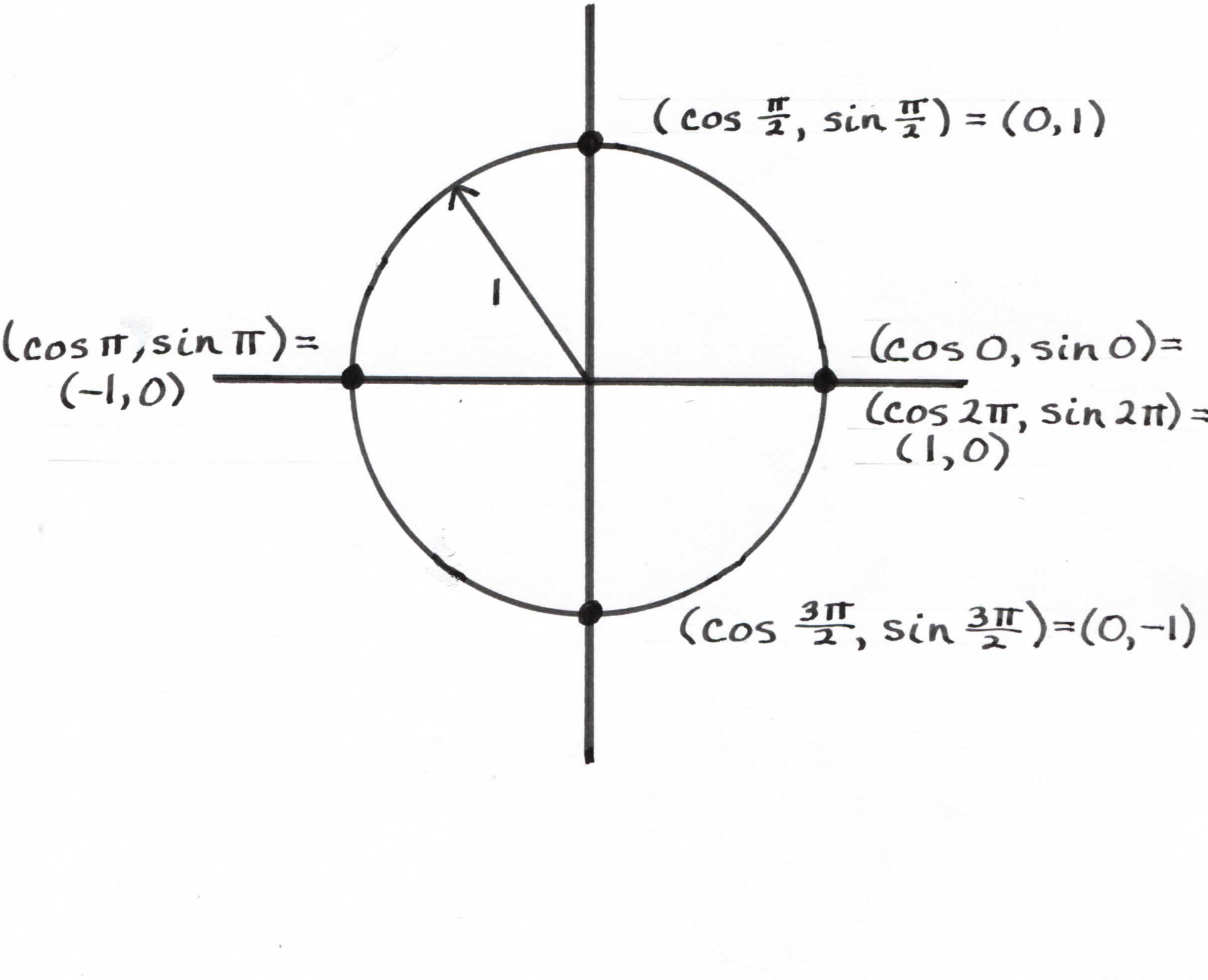
DRAWING 6.1



unit circle $x^2 + y^2 = 1$

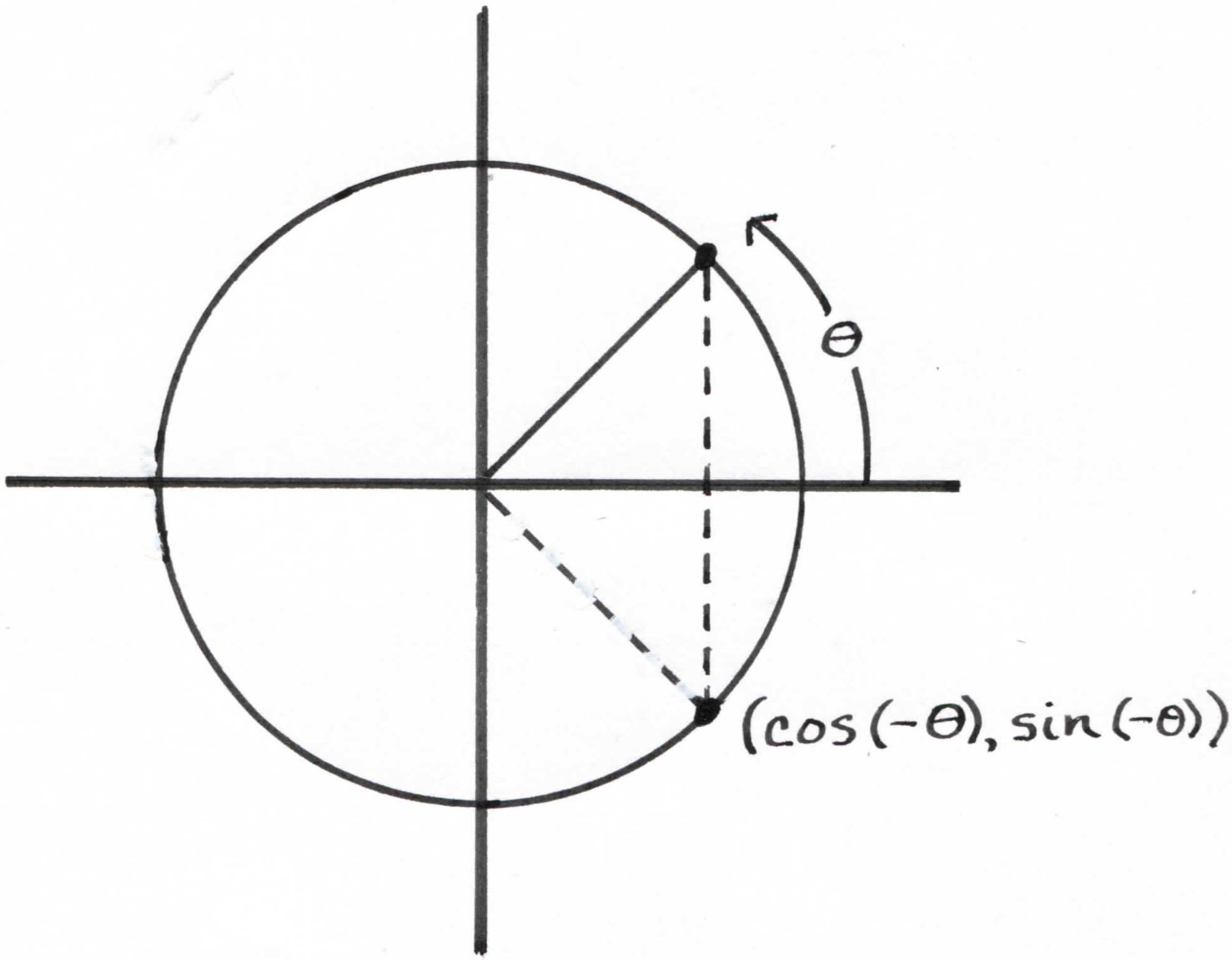
DRAWING 6.2

(compare to Drawing 2.13)



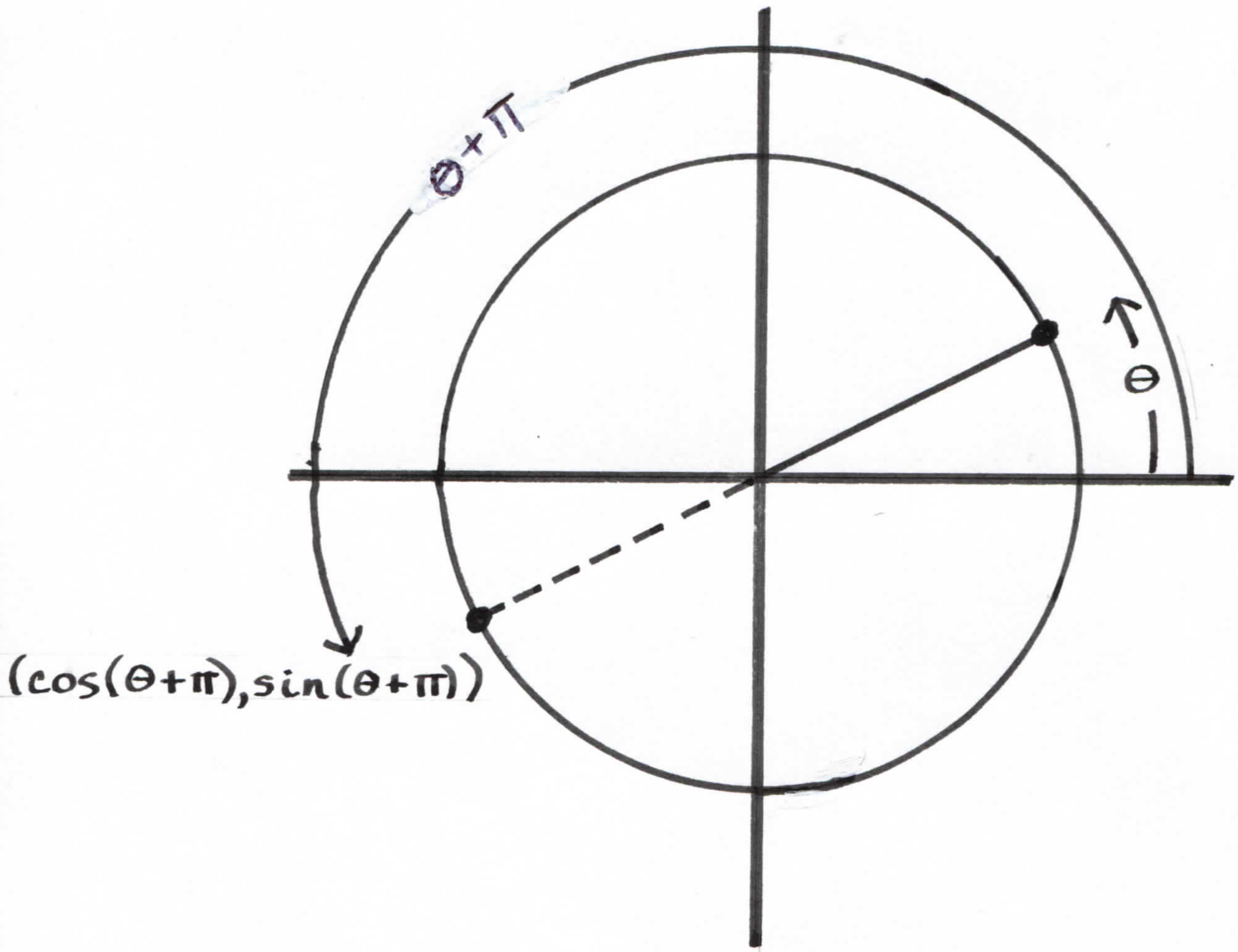
DRAWINGS 6.3

(i) and (ii):



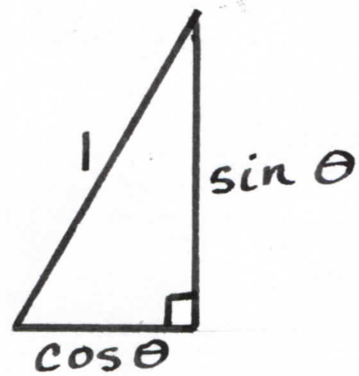
DRAWINGS 6.3 (continued)

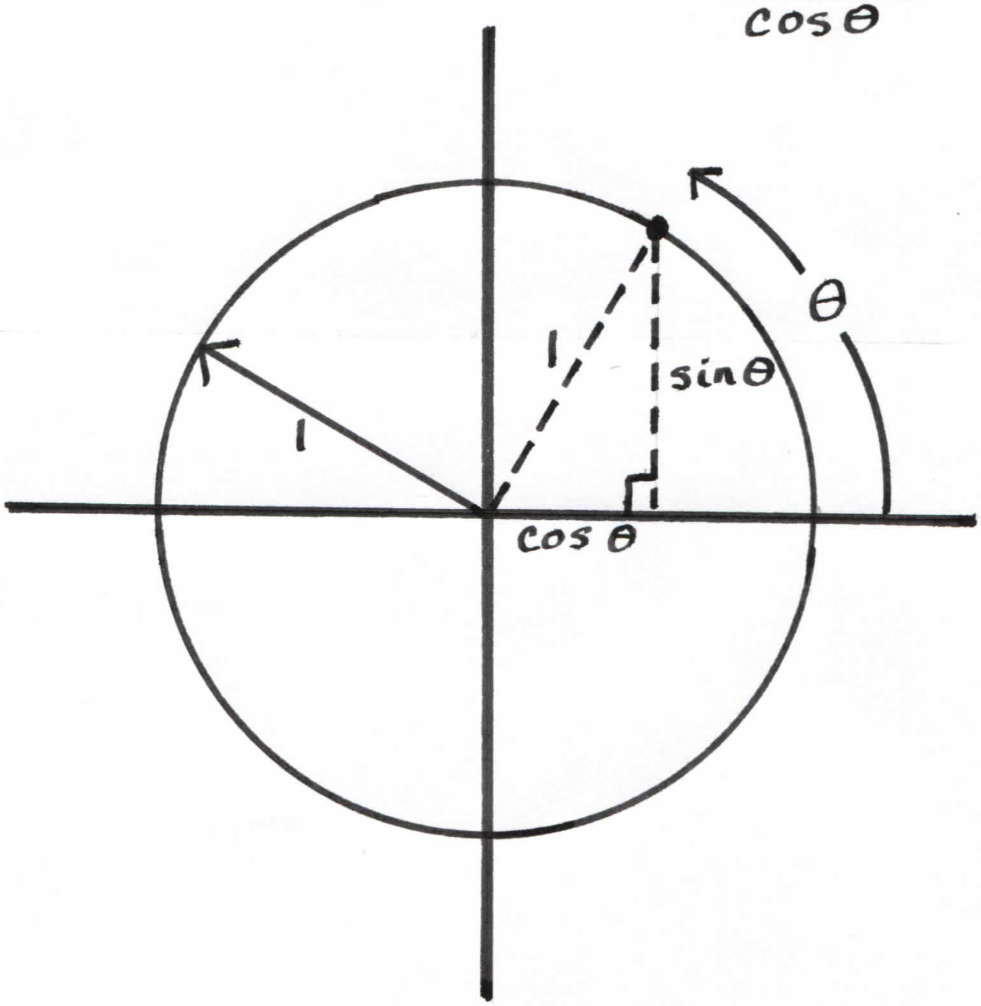
(iii) and (iv):



DRAWINGS 6.3 (continued)

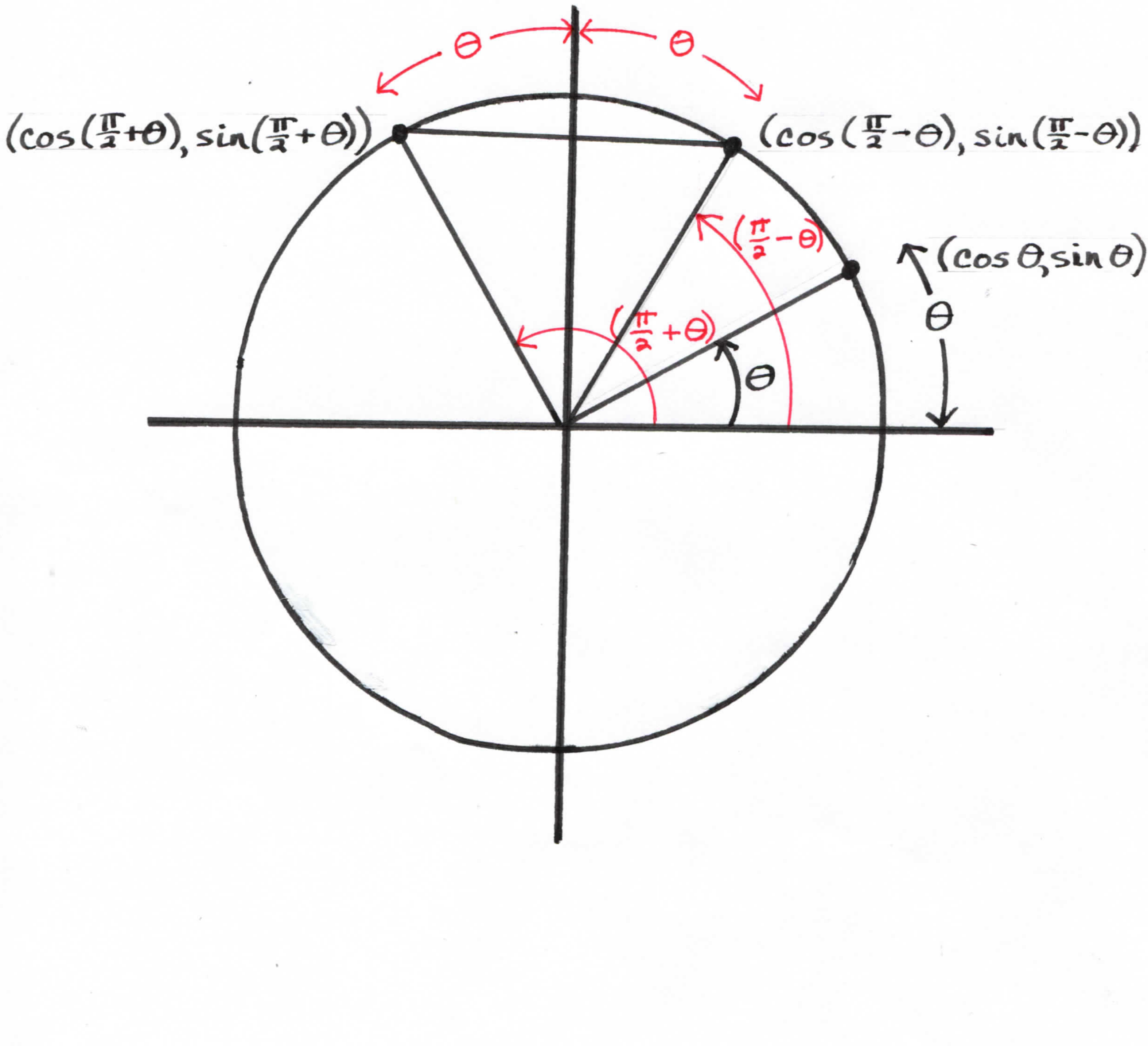
(v) and (vi)



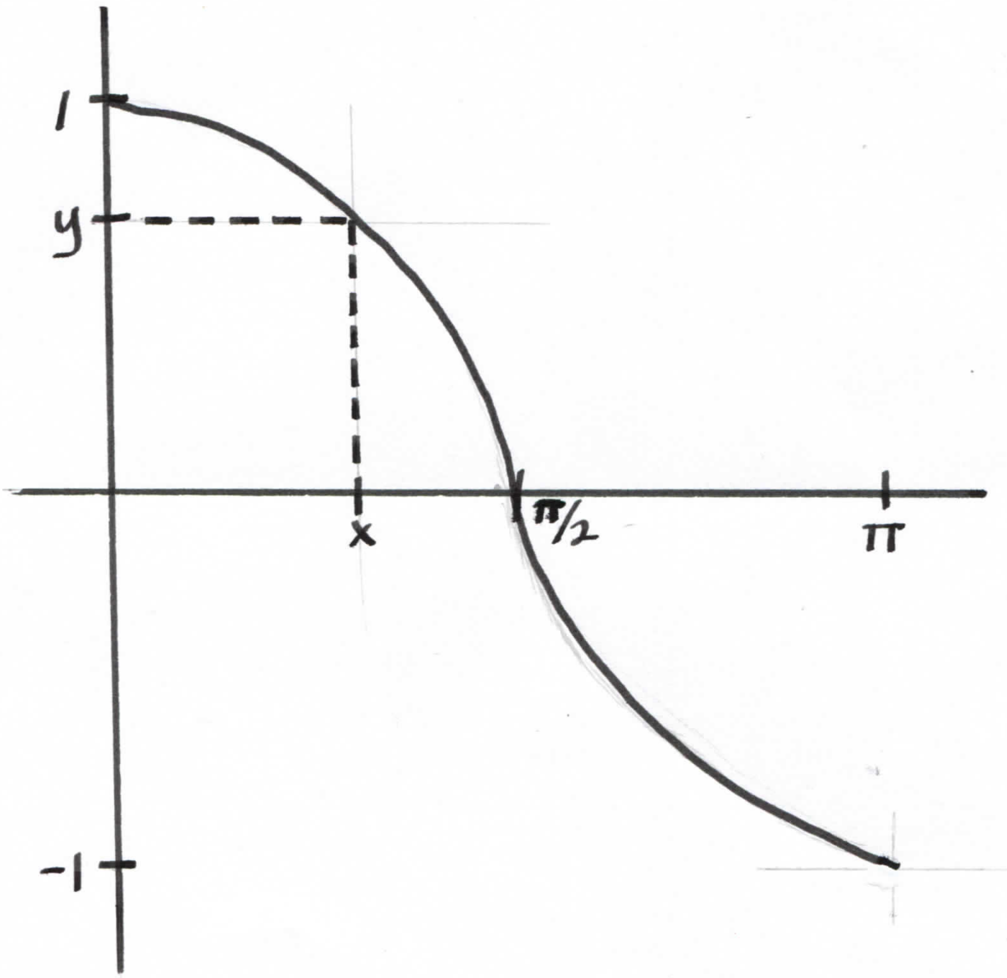


DRAWINGS 6.3 (continued)

(vii)



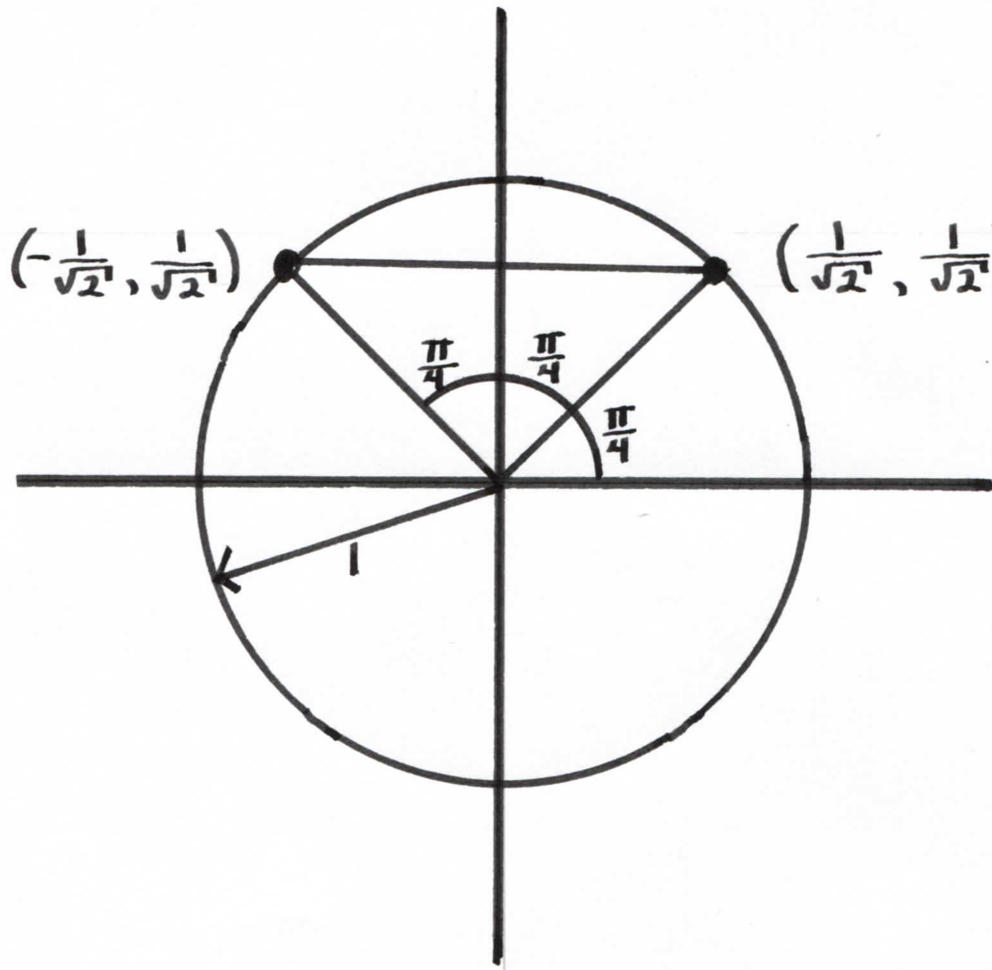
DRAWING 6.4



$$y = \cos x$$

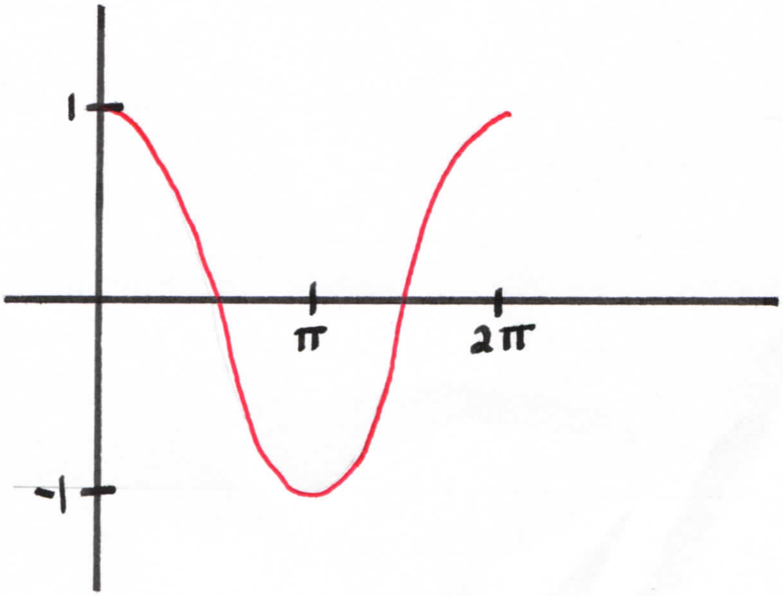
$$x = \cos^{-1} y$$

DRAWING 6.5



$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
$\frac{\pi}{4}$
$\frac{\pi}{4}$
$\frac{\pi}{4}$
1

DRAWING 6.6

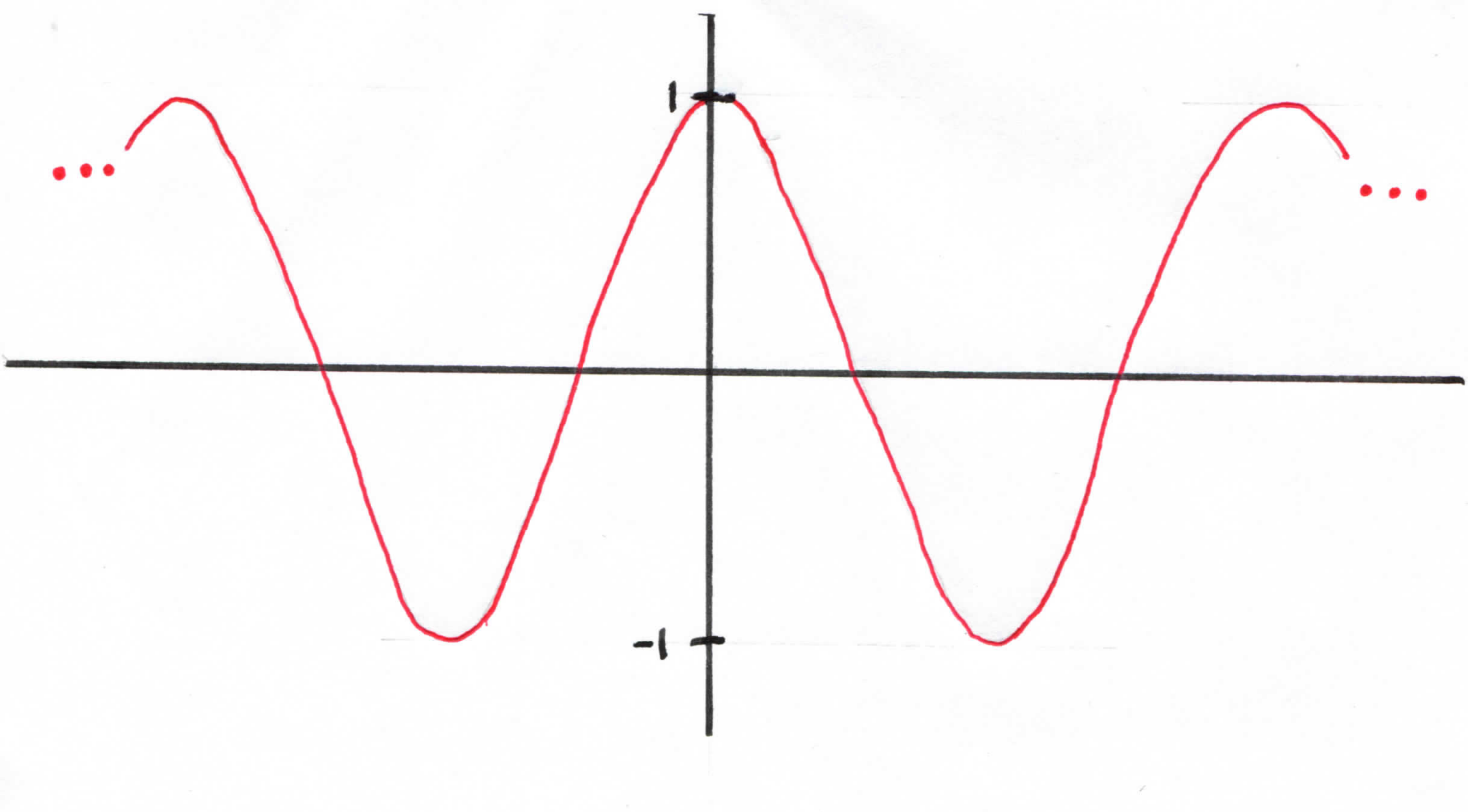


$y = \cos x$

$0 \leq x \leq 2\pi$

→ (6.3 viii)

$y = \cos x$



CHAPTER VII: Trig and Triangles, Law of Cosines, Law of Sines.

The previous chapter defined **trig** (short for **trigonometry or trigonometric**) functions cosine and sine as the real and imaginary parts of the exponential function restricted to the imaginary axis (Definition 6.1).

In this chapter we will relate the trig functions, at least for angles that measure between 0 and $\frac{\pi}{2}$, to certain triangles (Theorem 7.5). We will begin with the Law of Cosines (Theorem 7.1). This will give a quick proof of our triangle characterization of trig functions, which we will apply, after some examples, to the Law of Sines (Theorem 7.7). The Law of Cosines and the Law of Sines are powerful results about triangles; for an indication of their power, see, for example, Chapter X.

The Law of Cosines will be proven with the dot product (Definition 4.1). The Law of Sines will also follow from novel results about angles appearing in different ways in a circle (Theorem 7.14), which have numerous other applications.

We have already observed (Definitions 2.3 and DRAWING 2.8(a) at the end of Chapter II) that triangles may be described by vectors (two are sufficient, three are sometimes desirable); this will enable us to get the superpowers of the dot product into play.

Theorem 7.1: Law of Cosines. If a, b, c are the lengths of the sides of a triangle and θ is the measure of the angle opposite the side of length c, then

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

See DRAWING 7.1 at the end of this chapter.

Proof: Represent the triangle by vectors $\vec{a}, \vec{b}, \vec{c}$, with

$$a = \|\vec{a}\|, b = \|\vec{b}\|, c = \|\vec{c}\|,$$

$\vec{a}$ and $\vec{b}$ having the same initial point; denote, as usual, by θ the measure of the angle between $\vec{b}$ and $\vec{a}$ (see DRAWING 7.1 at the end of this chapter).

Then, by 4.5 and Theorem 6.9,

$$c^2 = \|\vec{a}-\vec{b}\|^2 = (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) = \vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{b}-2\vec{a}\cdot\vec{b} = \|\vec{a}\|^2+\|\vec{b}\|^2-2\|\vec{a}\|\|\vec{b}\|\cos\theta \equiv a^2+b^2-2ab\cos\theta.$$

$\square$

Examples 7.2. In each of the drawings in DRAWINGS 7.2 at the end of this chapter, fill in, if possible, the missing length of side. In (a), also get the missing angle measurements.

We will show later (Examples 7.11(e)), using triangles, that $\cos(\frac{\pi}{3}) = \frac{1}{2}$; see also Examples 6.5 and HWII.11(g).

Solutions. (a) Let c be the length of the side opposite the angle of measure $\frac{\pi}{3}$. The Law of Cosines says

$$c^2 = 3^2 + 4^2 - 2\cdot 3\cdot 4\cos(\frac{\pi}{3}) = 13,$$

thus the missing side has length $\sqrt{13}$.

For the missing measures, we can also use the Law of Cosines:

$$3^2 = c^2 + 4^2 - 2\cdot 4\cdot c\cos(\theta_1) \to 9 = 13 + 16 - 8\sqrt{13}\cos(\theta_1) \to \cos(\theta_1) = \frac{5}{2\sqrt{13}},$$

so that (applying $\cos^{-1}$)

$$\theta_1 \sim 46 \text{ degrees}.$$

We could get θ_2 similarly, but it's easier to use the fact that the angle measures add up to 180 degrees:

$$\theta_2 = 180 - (60 + 46) = 74 \text{ degrees}.$$

We will show, in Chapter X (Theorems 10.4 and 10.6), that, given the lengths of two sides and the measure of the angle between those sides (called **SAS: side-angle-side**), a unique triangle with those measurements always exists, with the remaining side calculated uniquely by the Law of Cosines.

(b) Let a be the unknown side length. By the Law of Cosines,

$$5^2 = a^2 + 6^2 - 2a \cdot 6\cos(\frac{\pi}{3}),$$

so that a satisfies the quadratic equation

$$a^2 - 6a + 11 = 0.$$

By the quadratic formula (Definitions 0.6),

$$a = \frac{1}{2}\left[6 \pm \sqrt{36-44}\right] = 3 \pm \sqrt{-2} = 3 \pm \sqrt{2}i;$$

that is, no real solution of the quadratic equation exists.

This means there is no such triangle as we drew in DRAWINGS 7.2(b) at the end of this chapter. Apparently drawings can lie.

The information in DRAWINGS 7.2(b) at the end of this chapter is an example of **SSA: side-side-angle**; it is more confused when the known angle is not between the two known sides. We have just seen that there might be *no* such triangle. It is also possible to have exactly one such triangle (see DRAWINGS 7.2(c) at the end of this chapter), and it is possible to have two such triangles (see DRAWINGS 7.2(d) at the end of this chapter). See Chapter X, especially Theorem 10.7.

Remarks 7.3. Choosing $\theta = \frac{\pi}{2}$ in the Law of Cosines yields the Pythagorean Theorem (Proposition 4.15). Choosing another angle measure equal to $\frac{\pi}{2}$ will also, for angle measures between 0 and $\frac{\pi}{2}$, produce an equivalent definition of sine and cosine in Theorem 7.5.

The Law of Cosines (Sines) relates cosines (sines) to lengths of sides in a triangle. For positive angle measures less than $\frac{\pi}{2}$, sine and cosine may be, equivalently to Definition 6.1, *defined* in terms of certain triangles.

Definitions 7.4 An angle of measure $\frac{\pi}{2}$ is sometimes called a **right angle**. A **right triangle** is a triangle that contains a right angle.

The **hypotenuse** of a right triangle is the side opposite the right angle. The **legs** are the other two sides of the right triangle.

Here is a convenient place to mention one more trig function, the **tangent**:

$$\tan\theta \equiv \text{ tangent of } \theta \equiv \frac{\sin\theta}{\cos\theta} \ (\theta \text{ real}).$$

See DRAWING 7.3 at the end of this chapter.

It is also convenient to mention here an angle that arises often in practice. The **angle of elevation** from one object to another object that is higher above the ground is the angle between a horizontal line through the lower object and the line from the lower object to the higher object. See DRAWING 7.4 at the end of this chapter.

Theorem 7.5. For any right triangle, denote by c the length of the hypotenuse, by a and b the lengths of the legs, and by θ the measure of the angle between the hypotenuse and the side of length a.

Then

$$\cos\theta = \frac{a}{c} \text{ ("adjacent over hypotenuse")} \qquad \sin\theta = \frac{b}{c} \text{ ("opposite over hypotenuse")}$$

and

$$\tan\theta = \frac{b}{a} \text{ ("opposite over adjacent").}$$

See DRAWING 7.5 at the end of this chapter.

Proof: By the Law of Cosines applied to DRAWING 7.5 at the end of this chapter,

$$b^2 = c^2 + a^2 - 2ac\cos\theta,$$

so that, by the Pythagorean theorem,

$$\cos\theta = \frac{c^2 + a^2 - b^2}{2ac} = \frac{a^2 + b^2 + a^2 - b^2}{2ac} = \frac{a}{c}.$$

Thus

$$\frac{b}{c} = \cos(\frac{\pi}{2} - \theta) = \sin\theta,$$

by 6.3(vii) and Proposition 3.9. □

Examples 7.6. (a) Get sines, cosines, and tangents, of the angle measures θ_1 and θ_2 in DRAWINGS 7.6(a) at the end of this chapter.

(b) Find the angle measure θ in the triangle in DRAWINGS 7.6(b) at the end of this chapter.

(c) Suppose the measure of the angle of elevation to the top of a tree is 85 degrees, when you stand 10 feet from the base of the tree (see DRAWINGS 7.6(c) at the end of this chapter). Find the height of the tree.

(d) Suppose now your tree (or the tower of a medieval castle, if you prefer) is surrounded by a moat of unknown width. You measure the angle of elevation to the top of the tree at the edge of the moat to be 80 degrees, and the measure of the angle of elevation 20 feet away from the moat to be 40 degrees. See DRAWINGS 7.6(d) at the end of this chapter.

Find the height of the tree and the width (labeled w in DRAWING 7.6(d)) of the moat.

See DRAWINGS 7.6 at the end of this chapter.

Solutions. (a) The Pythagorean theorem implies that the hypotenuse is of length 5, thus we may use Theorem 7.5. $\sin(\theta_1) = \cos(\theta_2) = \frac{4}{5}$; $\sin(\theta_2) = \cos(\theta_1) = \frac{3}{5}$. $\tan(\theta_1) = \frac{4}{3}, \tan(\theta_2) = \frac{3}{4}$.

(b) $\sin(\theta) = \frac{7}{14} = \frac{1}{2}$, thus, since $\sin(\frac{\pi}{6}) = \frac{1}{2}$, the answer in degrees is $\theta = 30$ degrees. Or we could have used a calculator, to get $\sin^{-1}(0.5))$ $\theta = 30$ degrees.

(c) $\frac{H}{10 \text{ feet}} = \tan(85 \text{ degrees})$, thus

$$H = 10\tan(85 \text{ degrees}) \text{ feet} \sim 114 \text{ feet}.$$

(d) As with (c),

$$H = (20 + w)\tan(40 \text{ degrees}) \sim (20 + w)0.84 \quad \text{and} \quad H = w\tan(80 \text{ degrees}) \sim 5.67w.$$

There are many ways to solve for H and w. Let's set the two expressions for H together:

$$(20 + w)0.84 = 5.67w \to 16.8 + 0.84w = 5.67w \to 16.8 = 4.83w \to w \sim 3.48,$$

so that $H \sim 5.67 \times 3.48 = 19.7$; height 19.7 feet, width of moat 3.48 feet.

See Examples 7.9 for an easier way (using the Law of Sines) to do (d).

Theorem 7.7: Law of Sines. If a triangle has sides of length c_1, c_2, c_3, and, for $j = 1, 2, 3, \theta_j$ is the measure of the angle opposite the side of length c_j, then

$$\frac{c_1}{\sin\theta_1} = \frac{c_2}{\sin\theta_2} = \frac{c_3}{\sin\theta_3}.$$

Proof: Label the sides of the triangle as vectors $\vec{c}_1, \vec{c}_2, \vec{c}_3$, with

$$\|\vec{c}_j\| = c_j, \;\; j = 1, 2, 3,$$

numbered so that $\vec{c}_1$ is the longest side:

$$c_1 \geq c_2 \;\text{ and }\; c_1 \geq c_3.$$

For $j = 1, 2, 3$, let P_j be the vertex opposite the side $\vec{c}_j$.

See DRAWINGS 7.8(a) at the end of this chapter.

Note first that $\text{proj}_{\vec{c}_1}(P_1)$ is on the line segment $\overrightarrow{P_2P_3}$; that is, in DRAWINGS 7.7(a)–(c) (see end of chapter), both (b) and (c) are impossible, for the following reason:

By the Pythagorean Theorem, in DRAWINGS 7.7(b) $c_3 > c_1$, while in DRAWINGS 7.7(c) $c_2 > c_1$; in either (b) or (c), $\vec{c}_1$ is not the longest side.

Finally, let

$$h \equiv \|P_1 - \text{proj}_{\vec{c}_1}(P_1)\|, \;\; d_1 \equiv \|P_2 - \text{proj}_{\vec{c}_1}(P_1)\|, \;\; d_2 \equiv \|P_3 - \text{proj}_{\vec{c}_1}(P_1)\|.$$

See DRAWINGS 7.8 at the end of this chapter. Note that $c_1 = (d_1 + d_2)$.

Theorem 7.5 implies that

$$\sin(\theta_2) = \frac{h}{c_3} \;\text{ and }\; \sin(\theta_3) = \frac{h}{c_2}.$$

Solving for h in the first equation, and substituting into the second equation gives us

$$\sin(\theta_3) = \frac{c_3\sin(\theta_2)}{c_2},$$

which implies that

$$\frac{c_2}{\sin\theta_2} = \frac{c_3}{\sin\theta_3}. \quad (*)$$

By (6.3), (6.4), Theorem 7.5, and Proposition 3.9,

$$\frac{c_1}{\sin\theta_1} = \frac{c_1}{\sin(\pi - (\theta_2 + \theta_3))} = \frac{(d_1 + d_2)}{\sin(\theta_2 + \theta_3)} = \frac{(d_1 + d_2)}{((\sin\theta_2)(\cos\theta_3) + (\sin\theta_3)(\cos\theta_2))}$$

$$= \frac{(d_1 + d_2)}{\left((\frac{h}{c_3})(\frac{d_2}{c_2}) + (\frac{h}{c_2})(\frac{d_1}{c_3})\right)} = \frac{c_2c_3(d_1 + d_2)}{h(d_2 + d_1)} = c_2(\frac{c_3}{h}) = \frac{c_2}{\sin\theta_2};$$

combined with (*), this concludes the proof. □

Remark 7.8. For a right triangle, the Law of Sines follows from Theorem 7.5, with $\frac{c_j}{\sin\theta_j}$ equal to the length of the hypotenuse, $j = 1, 2, 3$.

Examples 7.9. (a) Find all missing lengths of sides and measures of angles, up to two decimal places, in DRAWINGS 7.9(a) at the end of this chapter.

(b) Answer Examples 7.6(d) using Law of Sines.

(c) Without getting wet, you wish to know the width of a stream. From two places 100 feet apart on the edge of the stream, you measure the angle to a tree on the opposite side, as in DRAWINGS 7.9(c) at the end of this chapter.

Find the width of the stream (see red drawings in DRAWINGS 7.9(c) at the end of this chapter), and the distances from your two places to the tree on the opposite side.

Solutions. (a) Let c_1 be the length of the side opposite the angle of measure $\frac{\pi}{4}$, and let c_3 be the length of the remaining side.

The missing angle measure, call it θ_3, equals $\pi - (\frac{\pi}{4} + \frac{\pi}{6}) = \frac{7\pi}{12}$.

By the Law of Sines,

$$\frac{c_1}{\sin\frac{\pi}{4}} = \frac{50}{\sin\frac{\pi}{6}} = \frac{c_3}{\sin\frac{7\pi}{12}},$$

so that, making decimal approximations,

$$c_1 = 100 \times \frac{1}{\sqrt{2}} \sim 70.71, \quad c_3 = 100 \times \sin(\frac{7\pi}{12}) \sim 96.59.$$

(b) See DRAWINGS 7.9(b) at the end of this chapter, where we have labeled all side lengths and angle measures.

By the Law of Sines, applied to the triangle on the left with sides of length c_1, c_2, 20, we also have $c_2 = 20$. (More generally, see Theorem 7.10(a) $\iff$ (c), an application of the Law of Sines to isosceles triangles.)

Now we may focus on the right triangle on the right of the picture:

$$\sin(80 \text{ degrees}) = \frac{H}{20} \to H = 20\sin(80 \text{ degrees}) \sim 19.70; \text{ similarly, } w = 20\cos(80 \text{ degrees}) \sim 3.47.$$

(c) See DRAWINGS 7.9(c) at the end of this chapter, where the missing angle measure is

$$\pi - (\frac{2\pi}{3} + \frac{\pi}{4}) = \frac{\pi}{12}.$$

By the Law of Sines,

$$\frac{c_3}{\sin(\frac{2\pi}{3})} = \frac{c_2}{\sin(\frac{\pi}{4})} = \frac{100}{\sin(\frac{\pi}{12})} \sim 386,$$

thus

$$c_3 \sim 386\sin(\frac{2\pi}{3}) \sim 334 \quad \text{and} \quad c_2 \sim 386\sin(\frac{\pi}{4}) \sim 273.$$

Those are the distances (in feet) to the tree, from the two points where we made angle measurements.

The width of the stream (see the last drawing in DRAWINGS 7.9(c)at the end of this chapter) is now seen to be

$$273\sin(\frac{\pi}{3}) \sim 236 \text{ feet}.$$

Examples 7.2, 7.6, and 7.9 are excellent detective work. Examples 7.9(c), for example, gets extensive information about the river and its opposite side, all while staying on one side of the river high and dry.

Recall the definition of an *isosceles* triangle in Chapter V (Definition 5.9). The following theorem gives many characterizations of being isosceles. Note that the equivalence of (a) and (c) in Theorem 7.10 shows that we could have equivalently replaced "side" with "angle" in the definition of an isosceles triangle.

See Definitions 3.11 for the definition of "midpoint" and "bisect."

Theorem 7.10. Let P, Q, R be the three vertices of a triangle (see DRAWINGS 7.10 at the end of this chapter). Then the following are equivalent.

(a) $\|\overrightarrow{PR}\| = \|\overrightarrow{QR}\|$.

(b) The orthogonal projection of the vertex R onto the opposite side $\overrightarrow{PQ}$ is the midpoint of $\overrightarrow{PQ}$.

(c) The measure of the interior angle at P equals the measure of the interior angle at Q.

(d) The line segment between R and its orthogonal projection onto $\overrightarrow{PQ}$ bisects the interior angle at R.

Proof: The equivalence of (a) and (b) is Theorem 5.10.

Label angle measures $\theta_j, j = 1, 2, 3$, as in the second page of DRAWINGS 7.10, at the end of this chapter.

(c) $\to$ (a) follows from the Law of Sines. The converse (a) $\to$ (c) *almost* immediately follows from the Law of Sines: (a) implies that $\sin(\theta_1) = \sin(\theta_2)$, which implies (see 6.3(ii) and (iv) and DRAWING 6.1 at the end of Chapter VI) that

$$\theta_1 = \theta_2 \quad \text{OR} \quad \theta_2 = \pi - \theta_1.$$

But $\theta_2 = \pi - \theta_1$ implies that $\theta_3 = \pi - (\theta_2 + \theta_1) = 0$, not much of a triangle. Thus θ_1 must equal θ_2, the assertion of (c).

So far we've shown the equivalence of (a), (b), and (c).

For the equivalence of (c) and (d), let $S \equiv \text{proj}_{\vec{PQ}}(R)$ and label additional angle measures θ_4, θ_5, as in DRAWING 7.11 at the end of this chapter. Note that (b) $\iff$ (c) implies that DRAWING 7.11 at the end of this chapter is accurate.

By Proposition 3.9, looking at triangles PSR and SQR in DRAWING 7.11 at the end of this chapter,

$$\theta_1 + \theta_4 + \frac{\pi}{2} = \pi = \frac{\pi}{2} + \theta_2 + \theta_5,$$

so that $\theta_1 = \theta_2 \iff \theta_4 = \theta_5$; the latter equality is (d), while the former is (c), so that (c) and (d) are equivalent. $\square$

Examples 7.11. In (a)–(d) of DRAWINGS 7.12 at the end of this chapter, label, where possible, missing side lengths or angle measures.

(e) Use Theorem 7.10 and 7.5 to get sine and cosine of $\frac{\pi}{4}$, $\frac{\pi}{3}$, and $\frac{\pi}{6}$.

Solutions. (a) By Theorem 7.10(a) $\iff$ (c), $\theta_2 = 72$ degrees, thus $\theta_1 = 180 - 2 \cdot 72 = 36$ degrees. From the Law of Cosines,

$$s^2 = 5^2 + 5^2 - 2 \cdot 5 \cdot 5 \cos(36) \to s \sim 3.1.$$

(b) Again by Theorem 7.10(a) $\iff$ (c), $s_1 = 7$. The measure $\theta = 180 - 2 \cdot 40 = 100$, so, as with (a),

$$s_2^2 = 7^2 + 7^2 - 2 \cdot 7 \cdot 7 \cos(100) \to s_2 \sim 10.7.$$

We remark that (a) and (b) could also be done with the Law of Sines.

(c) By Theorem 7.10(a) $\iff$ (d), $\phi = 32$ degrees, thus $s_1 = 10\cos(32) \sim 8.5$, and both s_2 and s_3 equal $10\sin(32) \sim 5.3$.

(d) By Theorem 7.10(a) $\iff$ (b), $\theta_1 = \theta_2 = \frac{\pi}{2}$. By Theorem 7.10(a) $\iff$ (c), $\theta_5 = \theta_6$; by (a) $\iff$ (d), $\theta_4 = \theta_3$. Finally, focus on either half of the picture:

$$\cos\theta_5 = \frac{7}{16} \to \theta_5 \sim 64 \text{ degrees} \to \theta_4 = (90 - 64) = 26 \text{ degrees}.$$

(e) By Theorem 7.10(a) $\iff$ (c), $\cos(\frac{\pi}{4}) = \sin(\frac{\pi}{4}) \equiv x$, so (see DRAWINGS 7.12(e) at the end of this chapter), by the Pythagorean theorem, $x^2 + x^2 = 1 \to x = \frac{1}{\sqrt{2}}$.

For sine and cosine of $\frac{\pi}{3}$, denote $x \equiv \cos(\frac{\pi}{3}) = \sin(\frac{\pi}{6}), y \equiv \cos(\frac{\pi}{6}) = \sin(\frac{\pi}{3})$; see DRAWINGS 7.12(e) at the end of this chapter, where we have drawn a right triangle whose hypotenuse is length 1, base is x, height is y.

As drawn in DRAWINGS 7.12(e), paste a reflection of our original triangle on its right, to form a triangle all of whose angle measures are $\frac{\pi}{3}$. By Theorem 7.10(a) $\iff$ (c), all the side lengths are equal, hence equal to 1. By Theorem 7.10(a) $\iff$ (b), the base of the original triangle is $\frac{1}{2}$; thus $x = \frac{1}{2}$. Now apply the Pythagorean theorem to the original right triangle:

$$(\frac{1}{2})^2 + y^2 = 1 \quad \rightarrow \quad y = \frac{\sqrt{3}}{2}.$$

See also Examples 6.5 and HWII.11(g).

If we use both the Law of Cosines and the Law of Sines, we may improve Theorem 7.10(a) $\iff$ (c).

Corollary 7.12. In a triangle, one angle measure is greater than or equal to another angle measure if and only if the length of the side opposite to the first angle is greater than or equal to the length of the side opposite to the second angle. See DRAWING 7.13 at the end of this chapter.

Proof: Denote by θ_1, θ_2 the two angle measures being compared, and, for $j = 1, 2$, let c_j be the length of the side opposite θ_j (see DRAWING 7.13 at the end of this chapter).

Suppose $\theta_1 \geq \theta_2$.

If $\theta_1 \geq \frac{\pi}{2}$, then by the Law of Cosines, letting c_3 be the length of the third side,

$$c_1^2 = c_2^2 + c_3^2 - 2c_2c_3 \cos\theta_1 \geq c_2^2 + c_3^2 > c_2^2,$$

since $\cos\theta_1 \leq 0$, so that $c_1 \geq c_2$.

If $\theta_1 < \frac{\pi}{2}$, then $\theta_1 \geq \theta_2$ implies $\sin\theta_1 \geq \sin\theta_2$, thus by the Law of Sines,

$$c_1 = c_2 \left(\frac{\sin\theta_1}{\sin\theta_2} \right) \geq c_2.$$

Conversely, if $\theta_2 > \theta_1$, the same argument shows that $c_2 > c_1$. Thus $c_1 \geq c_2$ implies that $\theta_1 \geq \theta_2$. $\square$

Definitions 7.13. We have defined angles in terms of arclengths of circles; this means an angle is formed by the center of a circle and two points on the circle (Definitions 2.10). We could also form an angle by replacing the center with a third point on the circle. In this picture (DRAWING 7.14 at the end of this chapter), the angle formed at the center is called a **central angle**, while the angle formed at the third point is called an **inscribed angle**.

See DRAWING 7.14 at the end of this chapter, where the center of the circle is C, the initial points on the circle are Q and R and the third point on the circle is P; thus the inscribed angle ψ is between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ while the central angle θ is between $\overrightarrow{CQ}$ and $\overrightarrow{CR}$.

Theorem 7.14. Given any pair of points Q and R on a circle, the central angle formed by them measures twice the measure of the inscribed angle formed by them with a third point P on the circle; that is, $\theta = 2\psi$, in DRAWING 7.14 at the end of this chapter.

Proof: Throughout this proof, the points C, P, Q, R are as in DRAWING 7.14 at the end of this chapter.

CASE 1: $\overrightarrow{PQ}$ or $\overrightarrow{PR}$ contains C.

Without loss of generality, assume C is on $\overrightarrow{PQ}$; see DRAWING 7.15 at the end of this chapter.

Since $\|\overrightarrow{PC}\| = \|\overrightarrow{CR}\|$ (the radius of the circle), Theorem 7.10 implies that ϕ (from DRAWING 7.15 at the end of this chapter) equals ψ, thus, by Proposition 3.9,

$$\psi + \psi + (\pi - \theta) = \pi,$$

so that $\theta = 2\psi$.

CASE 2: C is enclosed by $\overrightarrow{PQ}, \overrightarrow{PR}$, and the counterclockwise arc from R to Q, as in DRAWING 7.14 at the end of this chapter and DRAWINGS 7.16 at the end of this chapter.

Add, to DRAWING 7.14 at the end of this chapter, the diameter consisting of the line segment PC, extended to the point on the circle opposite P; see DRAWINGS 7.16 at the end of this chapter, including labels for measures of angles that follow from Case 1.

By Case 1, $2\psi = 2\psi_1 + 2\psi_2 = \theta_1 + \theta_2 = \theta$. See DRAWING 7.16 at the end of this chapter.

CASE 3: C is outside the subset of $\mathbf{R}^2$ enclosed by $\overrightarrow{PQ}, \overrightarrow{PR}$, and the counterclockwise arc from R to Q, in DRAWING 7.14 at the end of this chapter; see DRAWINGS 7.17 at the end of this chapter.

Add on the same diameter that we did in Case 2; see DRAWINGS 7.17 at the end of this chapter, where we have labeled measures of angles that follow from Case 1.

By Case 1, $2(\psi + \psi_1) = \theta + 2\psi_1 \rightarrow 2\psi = \theta$. See DRAWING 7.17 at the end of this chapter. $\square$

We may generalize Theorem 7.14 considerably in Theorem 7.15, in each part by drawing an extra line. First let's reformulate DRAWING 7.14 at the end of this chapter and the conclusion of Theorem 7.14 in terms of arclength (see Definitions 2.10), on a circle of radius r: see DRAWING 7.18 at the end of this chapter.

Theorem 7.15. Suppose the circles in DRAWINGS 7.19 and 7.20 at the end of this chapter are both of radius r.

(a) In DRAWING 7.19, $\theta = \frac{1}{2}(\phi + \psi)$.

(b) In DRAWING 7.20, $\theta = \frac{1}{2}(\phi - \psi)$.

Proof: DRAWING 7.21 at the end of this chapter is DRAWING 7.19 with one additional line, and the angle measures that then follow from Theorem 7.14, added; the same relationship holds between DRAWING 7.22 at the end of this chapter and DRAWING 7.20.

As drawn in DRAWINGS 7.21 and 7.22, all that remains is to use the facts that the sum of the measures of angles in a triangle is π (Proposition 3.9) and the sum of supplementary (see Proposition 3.6) angle measures is π. $\square$

Examples 7.16. In each of the examples in DRAWINGS 7.23 at the end of this chapter, find the measure of the angle θ.

Solutions. (a) Since $r = 4$, the angle measure ϕ in DRAWING 7.19 at the end of this chapter is $\frac{5\pi}{4}$. Since ψ in DRAWING 17.19 is 0, $\theta = \frac{1}{2}\phi = \frac{5\pi}{8}$.

OR we could have used DRAWINGS 7.18 at the end of this chapter.

(b) Since $r = 8$, we now have $\phi = \frac{\pi}{4}, \psi = \frac{\pi}{8}$, so $\theta = \frac{1}{2}(\frac{\pi}{4} + \frac{\pi}{8}) = \frac{3\pi}{16}$.

(c) Now $\phi = \frac{5\pi}{12}$ and $\psi = \frac{\pi}{6}$, but θ is outside the disc, so $\theta = \frac{1}{2}(\frac{5\pi}{12} - \frac{\pi}{6}) = \frac{\pi}{8}$.

(d) We could use Theorem 7.15(a) with $\phi = \psi = \frac{3\pi}{4}$, or we could use the original definition of angle measure (2.10), to get $\theta = \frac{3\pi}{4}$.

We may also use Theorem 7.14 to embellish the Law of Sines (Theorem 7.7), as follows, specifying *what* the common ratio of side length to sine of opposite angle measure is.

Theorem 7.17. If there is a circle of radius r that contains the vertices of a triangle, then the common ratio from Theorem 7.7, of length of side to sine of measure of angle opposite side equals the diameter $2r$; that is (terminology from Theorem 7.7),

$$\frac{c_j}{\sin\theta_j} = 2r, \quad j = 1, 2, 3.$$

Proof: Let C be the center of the specified circle. Let $\vec{c}_j, c_j, \theta_j, P_j, j = 1, 2, 3$, be as in the proof of Theorem 7.7. See DRAWING 7.24 at the end of this chapter, where we have added on the central angle measuring $2\theta_1$ guaranteed by Theorem 7.14.

By the Law of Cosines (Theorem 7.1),

$c_1^2 \equiv \|\vec{c}_1\|^2 = \|\overrightarrow{CP_3}\|^2 + \|\overrightarrow{CP_2}\|^2 - 2\|\overrightarrow{CP_3}\|\|\overrightarrow{CP_2}\|\cos(2\theta_1) = 2r^2 - 2r^2\cos(2\theta_1) = 2r^2(1-\cos(2\theta_1)) = 4r^2\sin^2(\theta_1)$,
by 6.4(vii); thus

$$\frac{c_1}{\sin\theta_1} = 2r.$$

The same argument applies to $\frac{c_2}{\sin\theta_2}$ and $\frac{c_3}{\sin\theta_3}$. $\square$

HOMEWORK

HWVII.1. Get the sines, cosines, and tangents, of the angles of measure θ_1 and θ_2, in DRAWING 7.25 at the end of this chapter.

HWVII.2. Where possible in DRAWINGS 7.26 at the end of this chapter, find missing lengths of sides and measures of angles.

HWVII.3. Suppose that, when you are 50 meters from a skyscraper, the angle of elevation from your feet to the top of the skyscraper has measure 60 degrees.

Get the height of the skyscraper. See DRAWING 7.27 at the end of this chapter for a pictorial hint.

HWVII.4. Suppose now that the skyscraper has a rectangular front and has a rectangle of broken glass perpendicular to the front of the skyscraper. At the edge of the broken glass, the angle of elevation from your feet to the top of the skyscraper is 88 degrees. 100 meters away from the edge of the broken glass, the angle of elevation from your feet to the top of the skyscraper is 70 degrees.

Get the height of the skyscraper and the width of the broken glass in front of it. See DRAWING 7.28 at the end of this chapter for a pictorial hint.

HWVII.5. A triangle is **equilateral** if each of its sides has the same length. Show that a triangle is equilateral if and only if each of its vertices have interior angles of the same measure.

HWVII.6. In DRAWINGS 7.29 at the end of this chapter, find all missing lengths of sides and measures of angles, where possible.

HWVII.7. In each part of DRAWINGS 7.30 at the end of this chapter, find the angle measure θ.

HWVII.8. Suppose a quadrilateral has all its vertices on a circle. Show that opposite angle measures add up to π.

HWVII.9. Suppose a parallelogram has all its vertices on a circle. Show that it must be a rectangle.

HWVII.10. Suppose a triangle has all its vertices on a circle, with two of its vertices on a diameter of the circle. Show that the other vertex has a right angle as its interior angle.

HOMEWORK ANSWERS

HWVII.1. The Pythagorean theorem tells us the length of the vertical leg is 12.
$\sin(\theta_1) = \cos(\theta_2) = \frac{12}{13}; \sin(\theta_2) = \cos(\theta_1) = \frac{5}{13}; \tan(\theta_1) = \frac{12}{5}; \tan(\theta_2) = \frac{5}{12}$.

HWVII.2. See DRAWINGS 7.31 at the end of this chapter.

HWVII.3. $50\tan(60 \text{ degrees}) = 50\sqrt{3}$ meters, ~ 86.60 meters. See DRAWING 7.32 at the end of this chapter.

HWVII.4. The height of the skyscraper is $\sin(88 \text{ degrees})\left(\frac{100\sin(70 \text{ degrees})}{\sin(18 \text{ degrees})}\right) \sim 303.91$ meters.

The width of the broken glass is $\cos(88 \text{ degrees})\left(\frac{100\sin(70 \text{ degrees})}{\sin(18 \text{ degrees})}\right) \sim 10.61$ meters.
See DRAWING 7.33 at the end of this chapter.

HWVII.5. Theorem 7.10 (a) $\iff$ (c), applied to each pair of sides.

HWVII.6. (a) $s_1 = 7\sin(20 \text{ degrees}) = s_2, s_3 = 7\cos(20 \text{ degrees}), \theta_1 = (180 - 20 - 90)$ degrees $=$ 70 degrees $= \theta_2, \theta_3 = 20$ degrees, by Theorem 7.10 (a) $\iff$ (d).

(b) $s_1 = 6, \theta = 120$ degrees, $s_2 = 6\sqrt{3}$ (could be done by Law of Sines, Law of Cosines, or by chopping up the triangle into two right triangles.

(c) $\theta_4 = \theta_5 = 90$ degrees; $s = 8, \theta_3 = \cos^{-1}(\frac{3}{5}) = \theta_6; \theta_1 = \cos^{-1}(\frac{4}{5}) = \theta_2$.

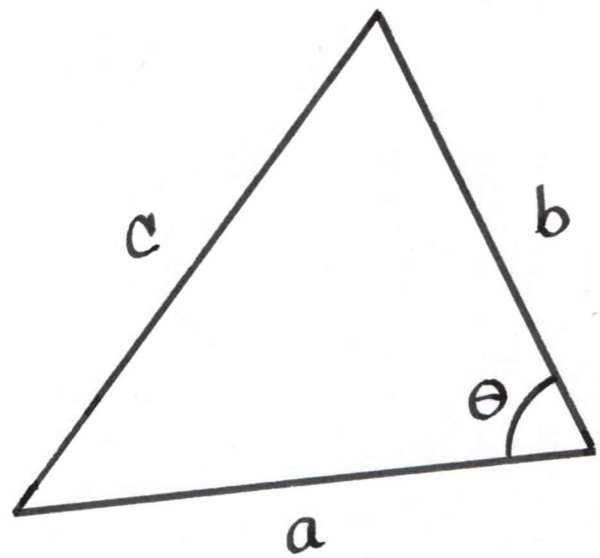
HWVII.7. (a) $\frac{4\pi}{3}$ (b) $\frac{3\pi}{20}$ (c) $\frac{3\pi}{20}$ (d) $\frac{\pi}{4}$.

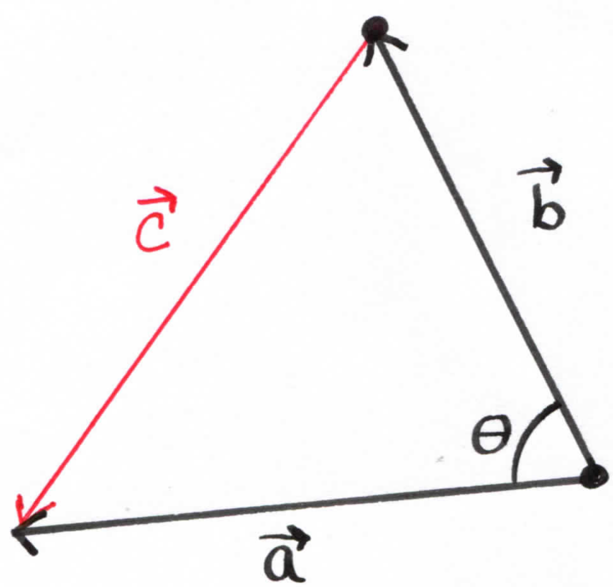
HWVII.8. See DRAWING 7.34 at the end of this chapter.

HWVII.9. See DRAWING 7.35 at the end of this chapter.

HWVII.10. See DRAWING 7.36 at the end of this chapter.

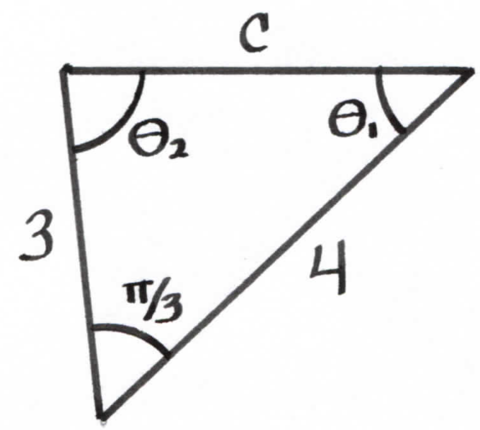
DRAWING 7.1



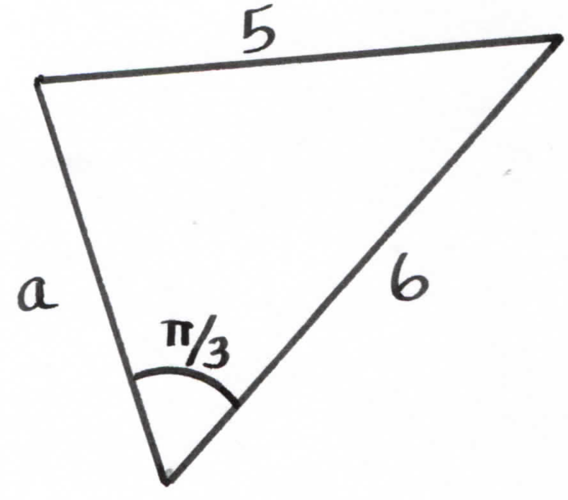


DRAWINGS 7.2

(a)



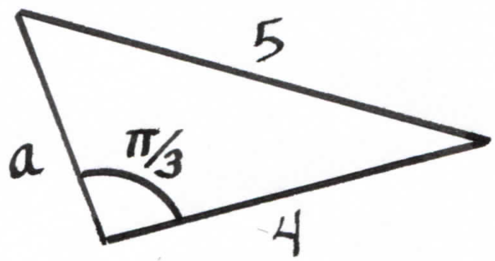
(b)



NO SUCH number a exists
(use Law of Cosines)

DRAWINGS 7.2 (continued)

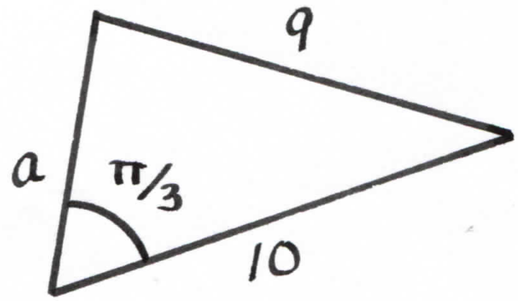
(c)



a must equal $2+\sqrt{13}$

(use Law of Cosines)

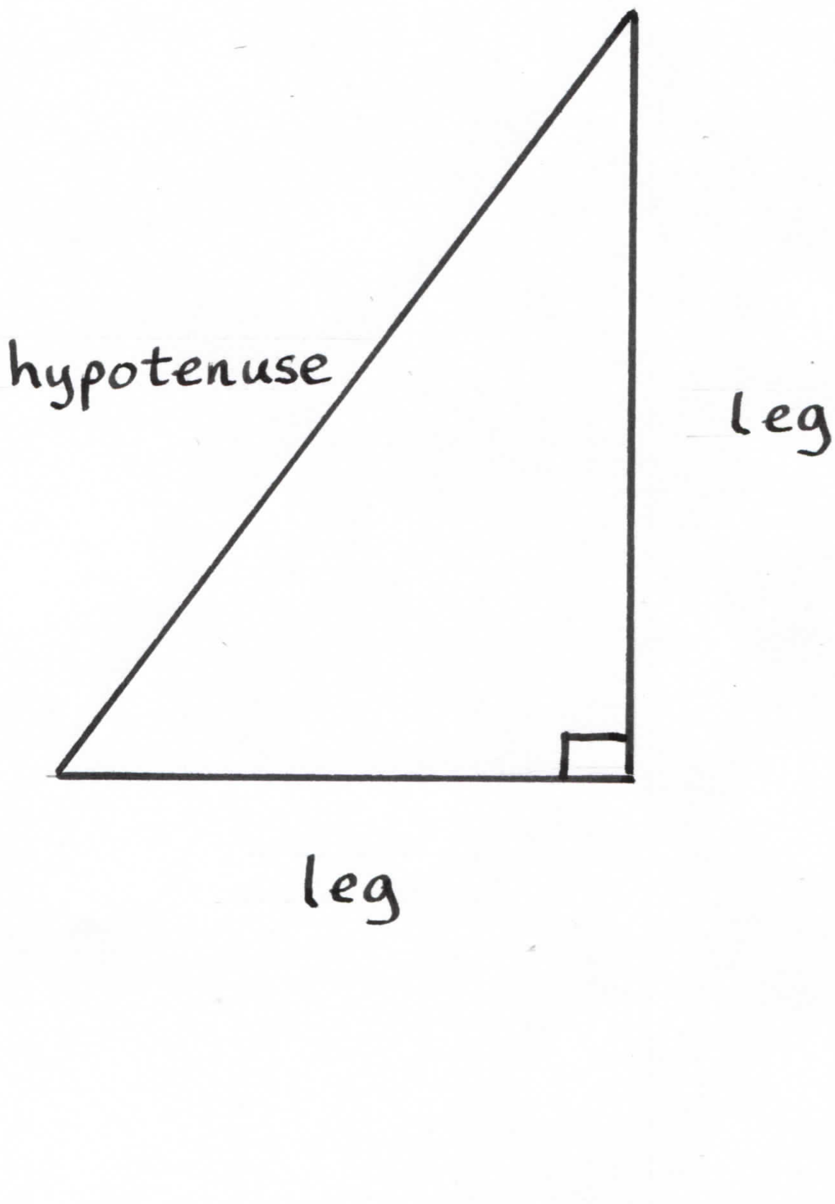
(d)



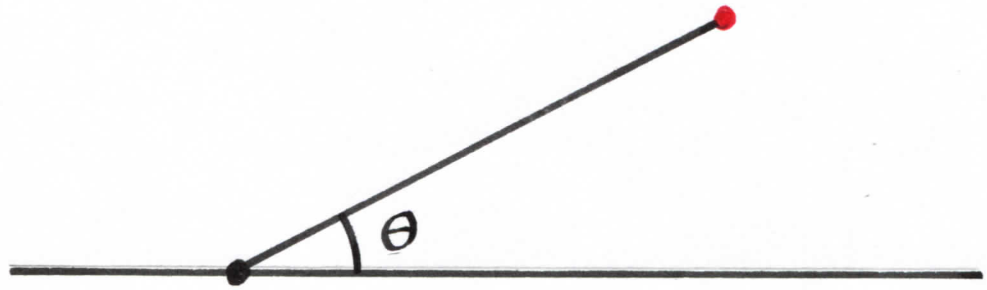
a could equal $(5+\sqrt{6})$ or $(5-\sqrt{6})$

(use Law of Cosines)

DRAWING 7.3

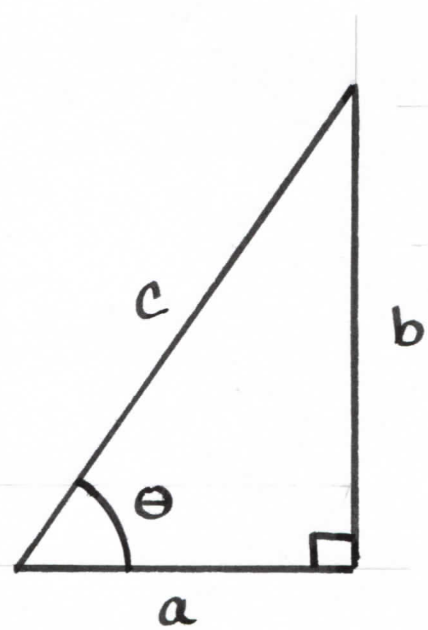


DRAWING 7.4



θ = measure of angle of elevation to

DRAWING 7.5



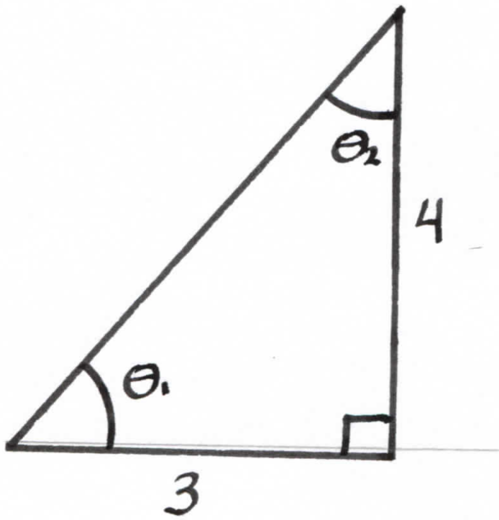
$$\cos \theta = \frac{a}{c}$$

$$\sin \theta = \frac{b}{c}$$

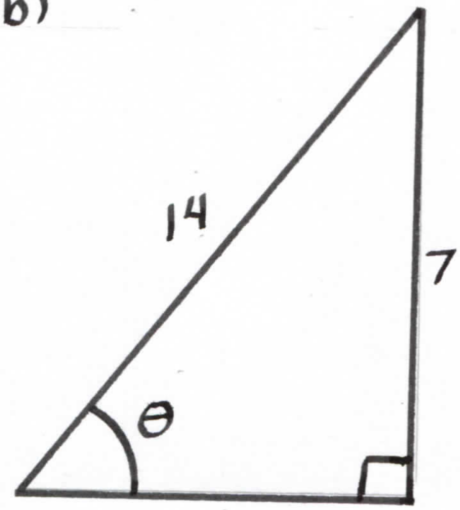
$$\tan \theta = \frac{b}{a}$$

DRAWINGS 7.6

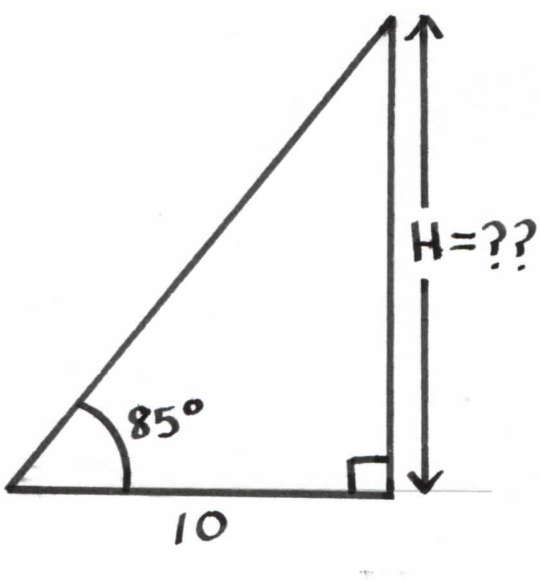
(a)



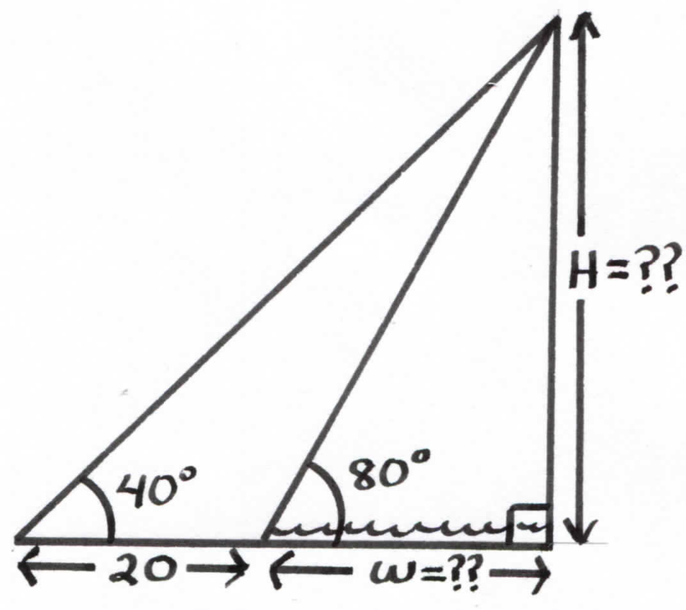
(b)



(c)

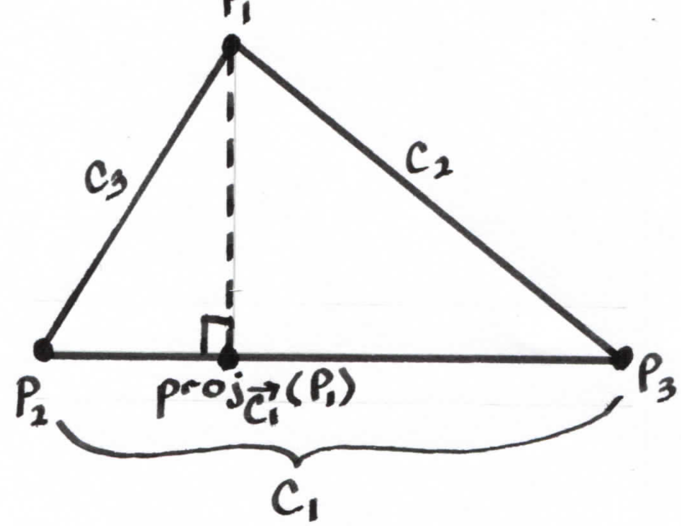


(d)

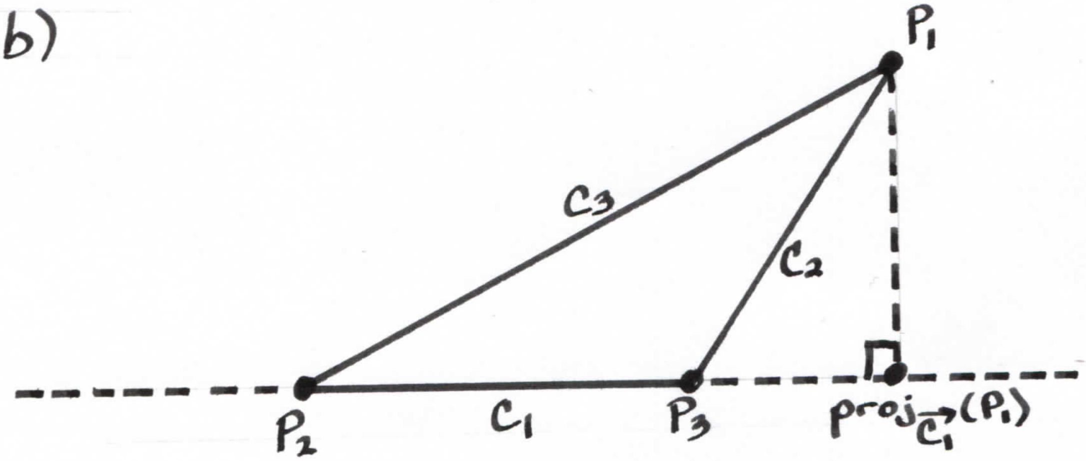


DRAWINGS 7.7

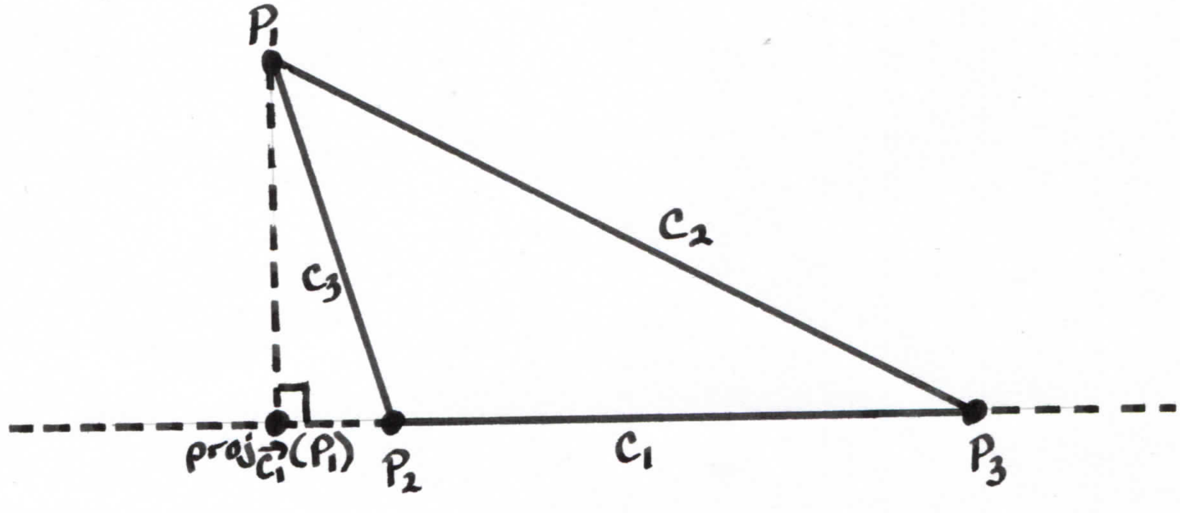
(a)



(b)

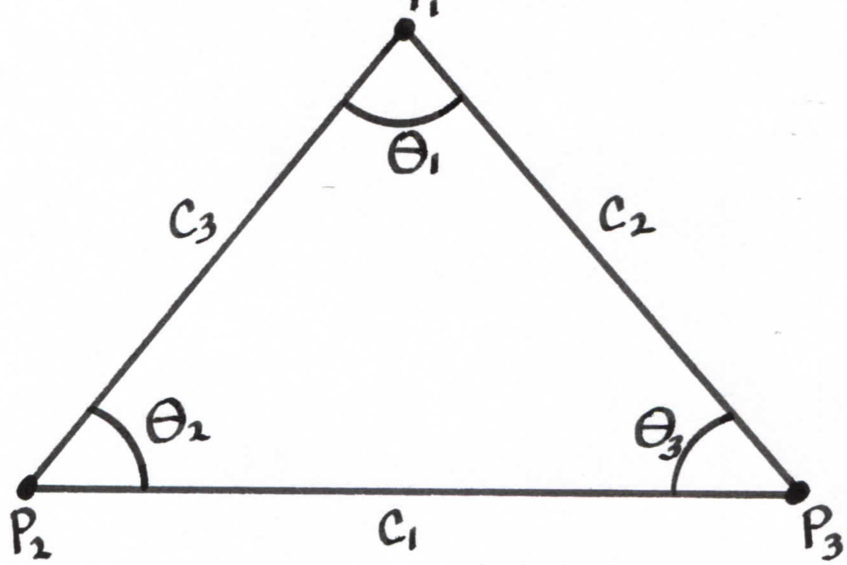


(c)

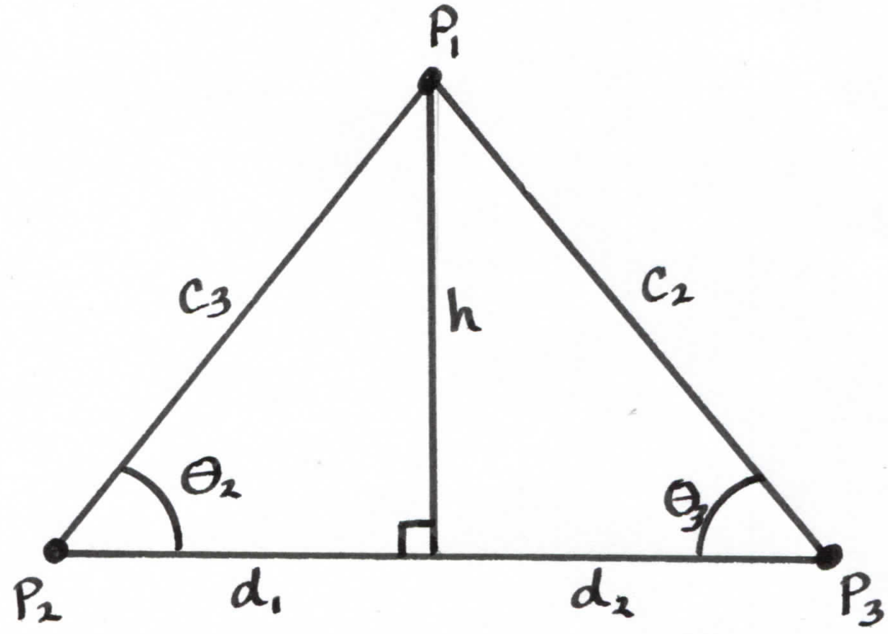


DRAWINGS 7.8

(a)

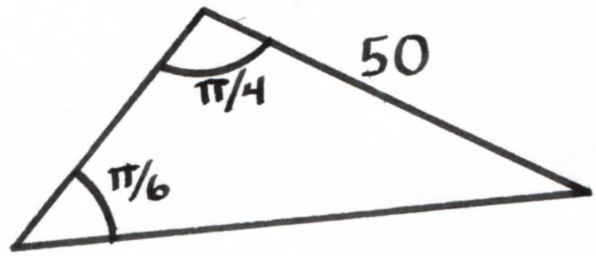


(b)

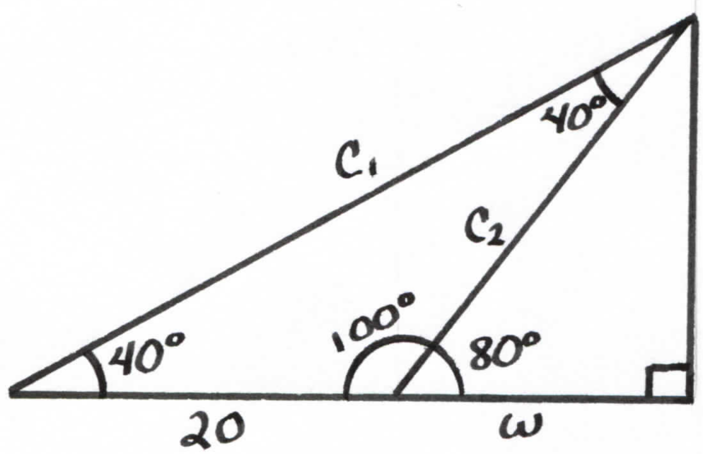


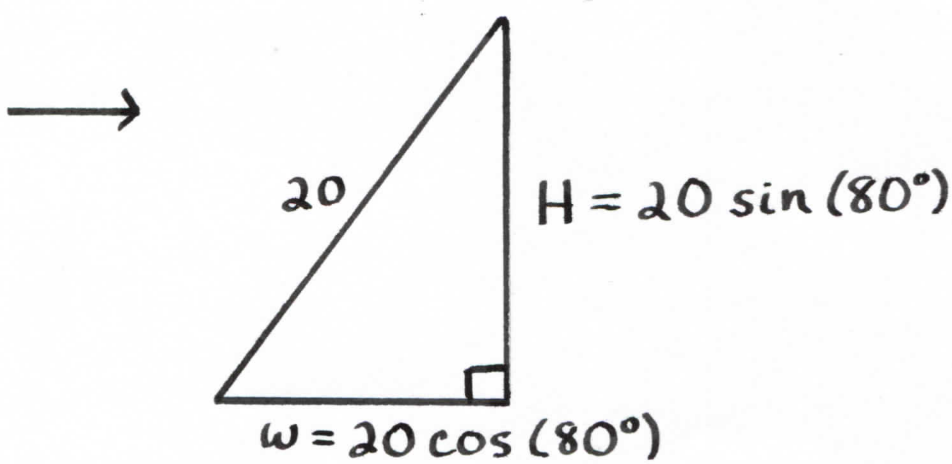
DRAWINGS 7.9

(a)



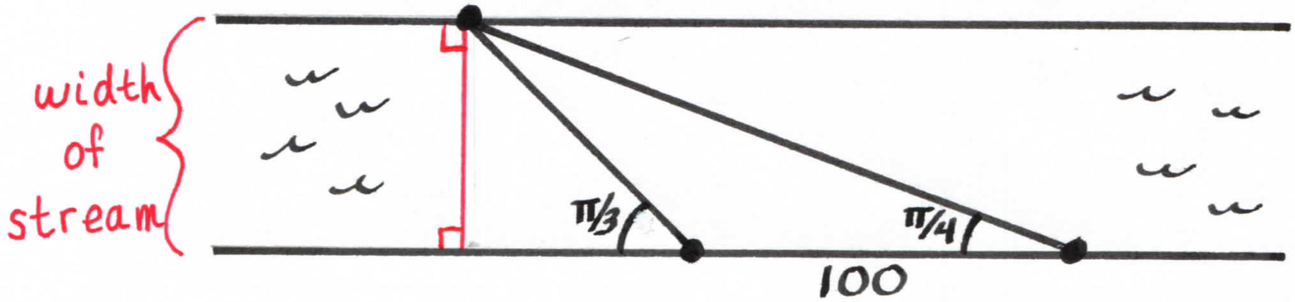
(b)

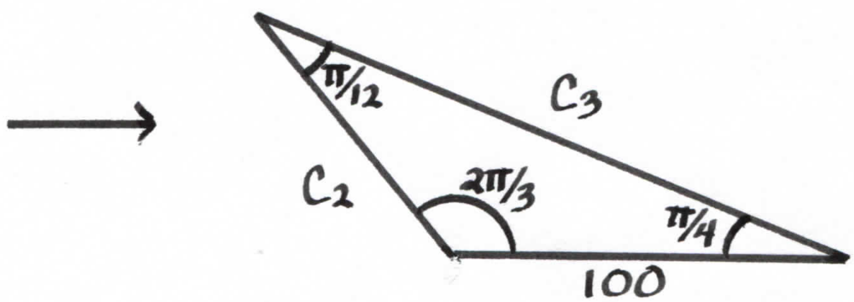




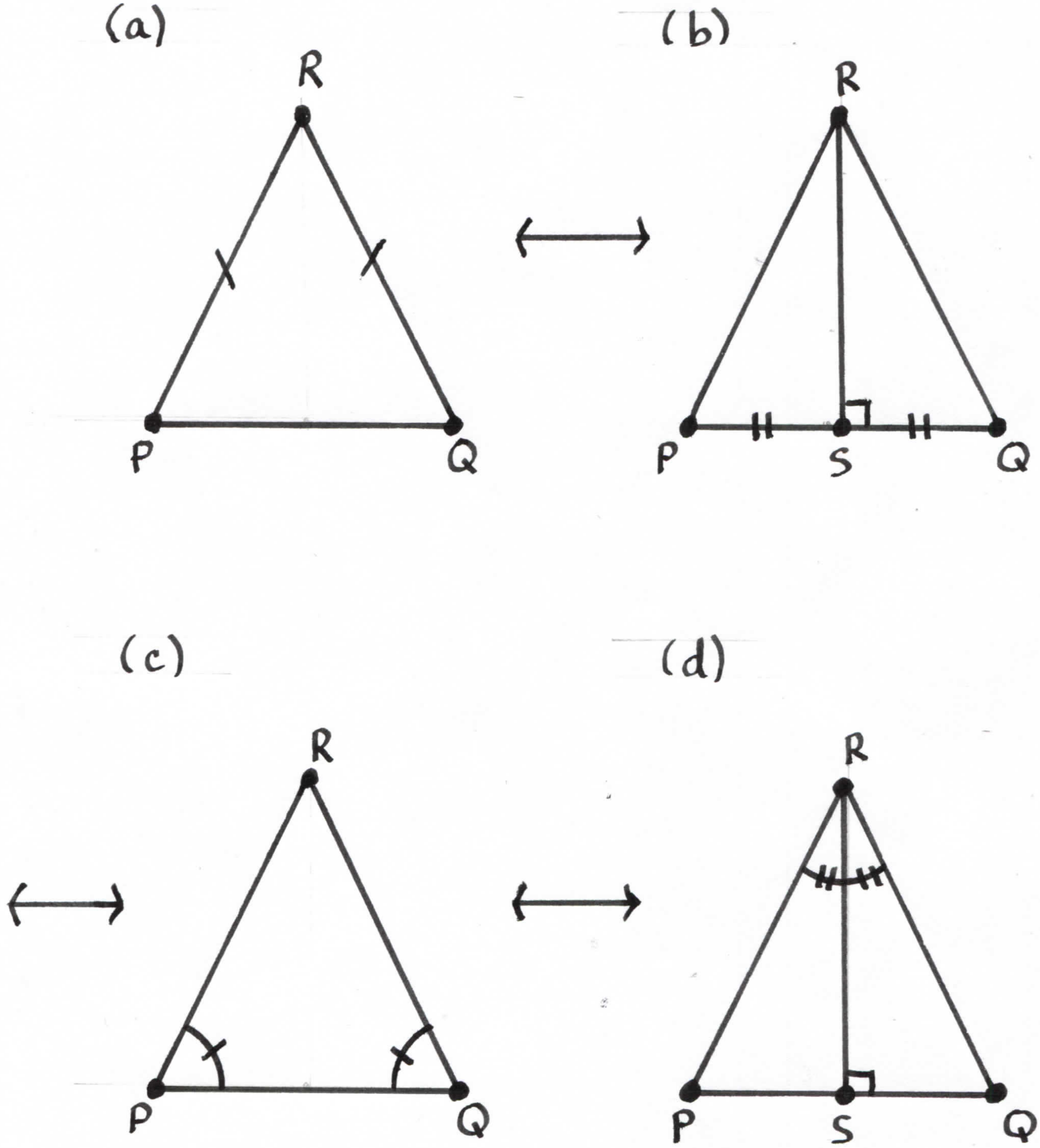
DRAWINGS 7.9 (continued)

(c)

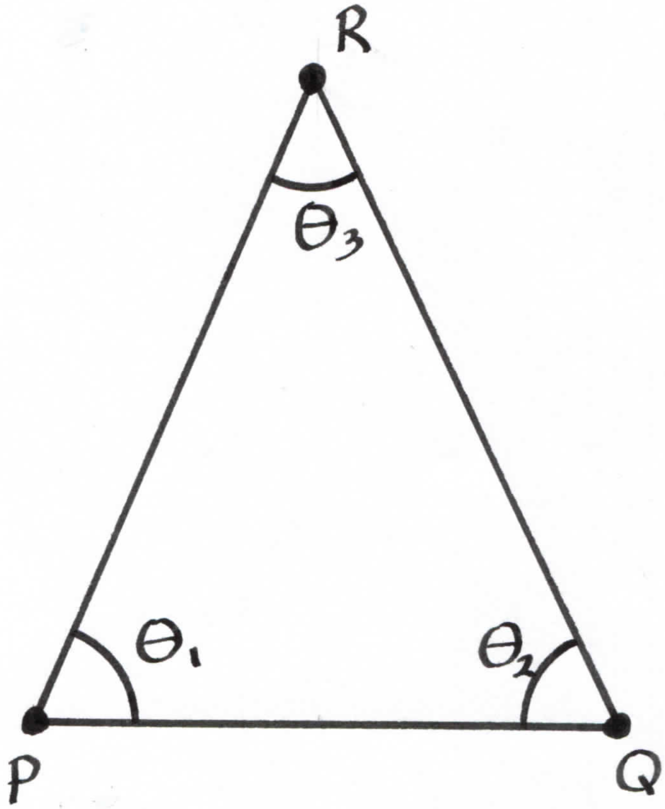




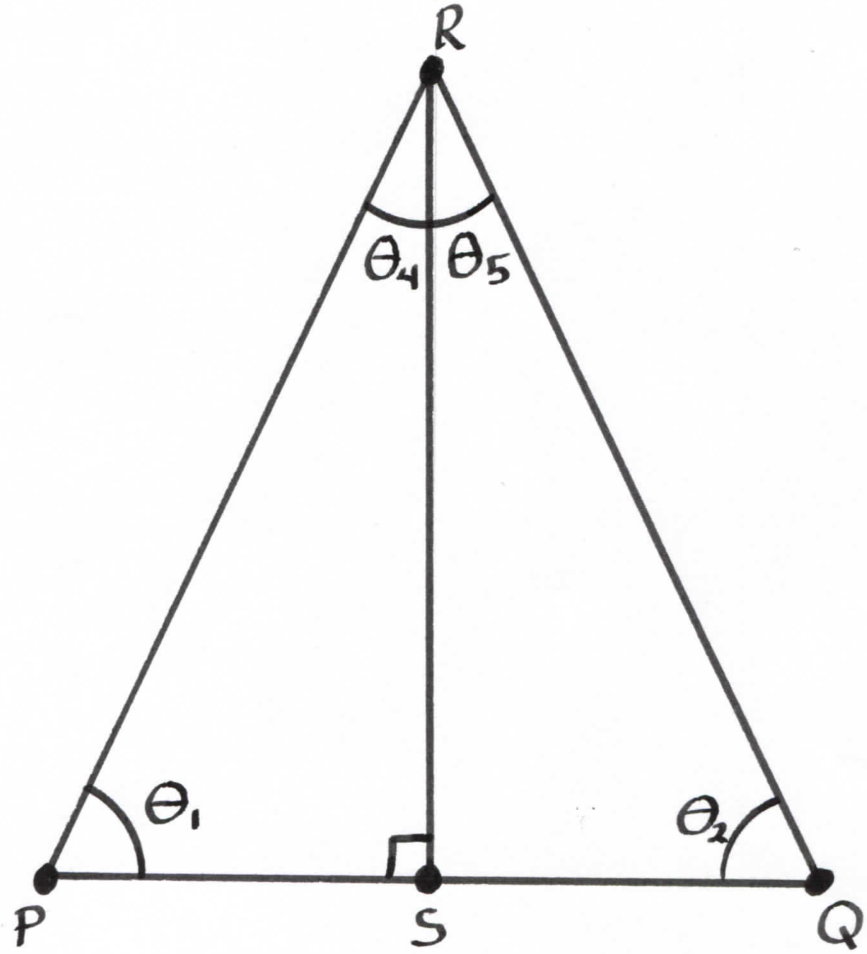
DRAWINGS 7.10



DRAWINGS 7.10 (continued)

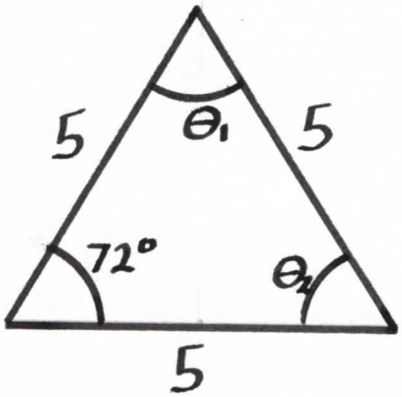


DRAWING 7.11

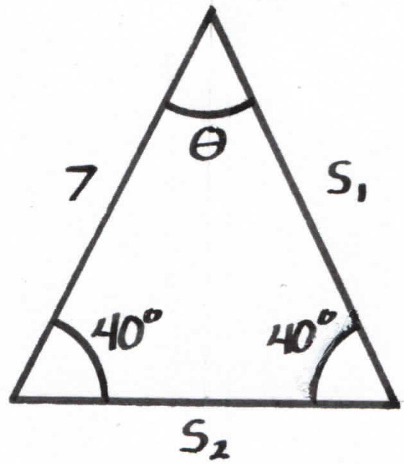


DRAWINGS 7.12

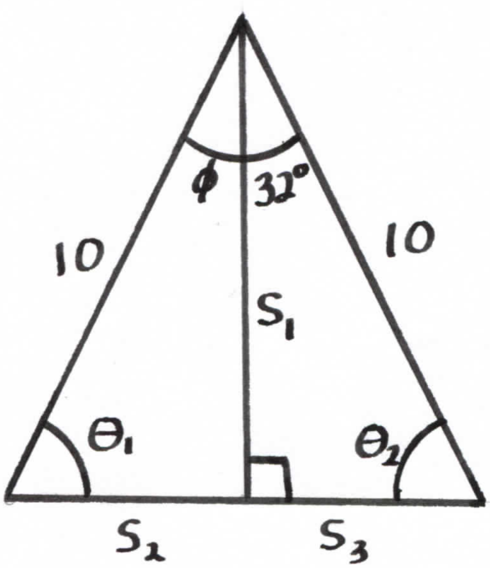
(a)



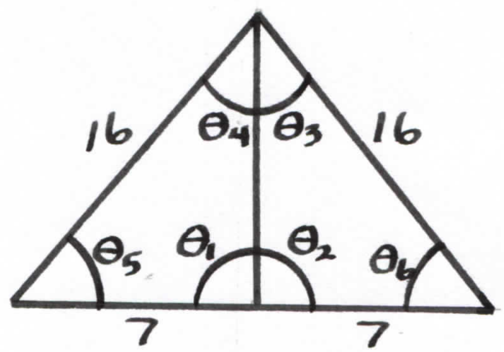
(b)

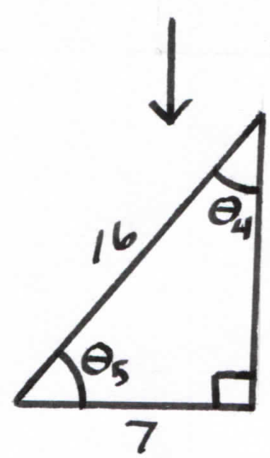


(c)



(d)



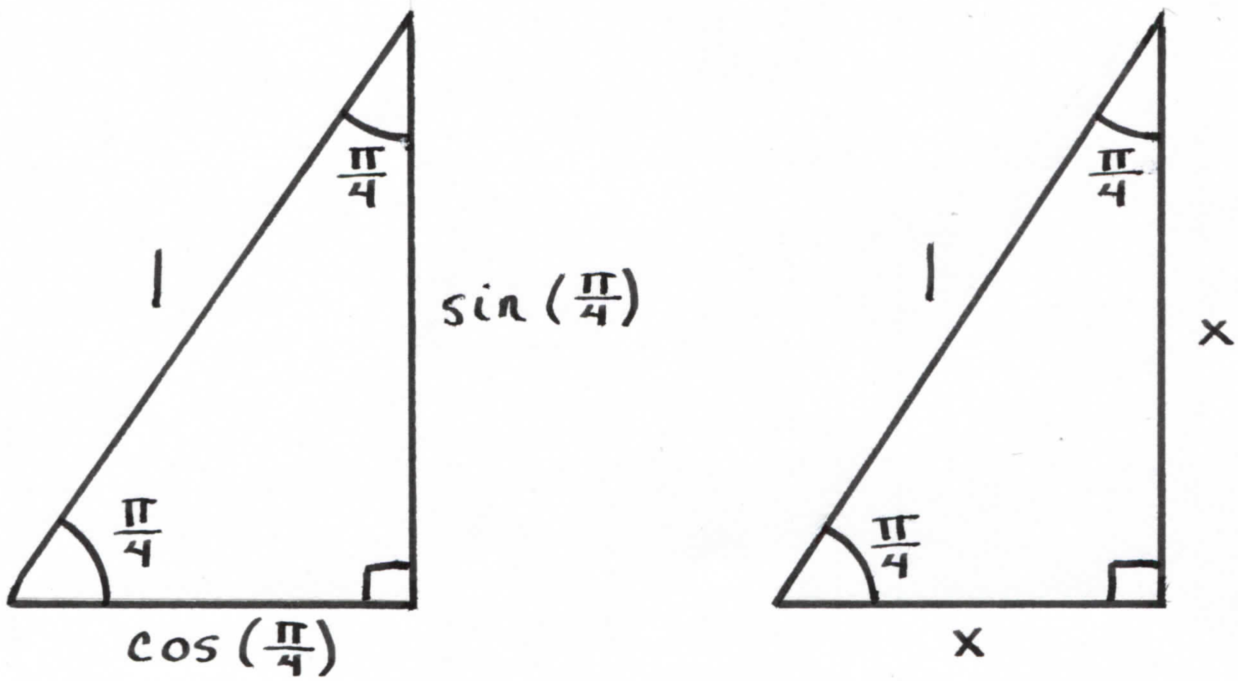


$$\cos\theta_5 = \frac{7}{16}$$

$$= \sin\theta_4$$

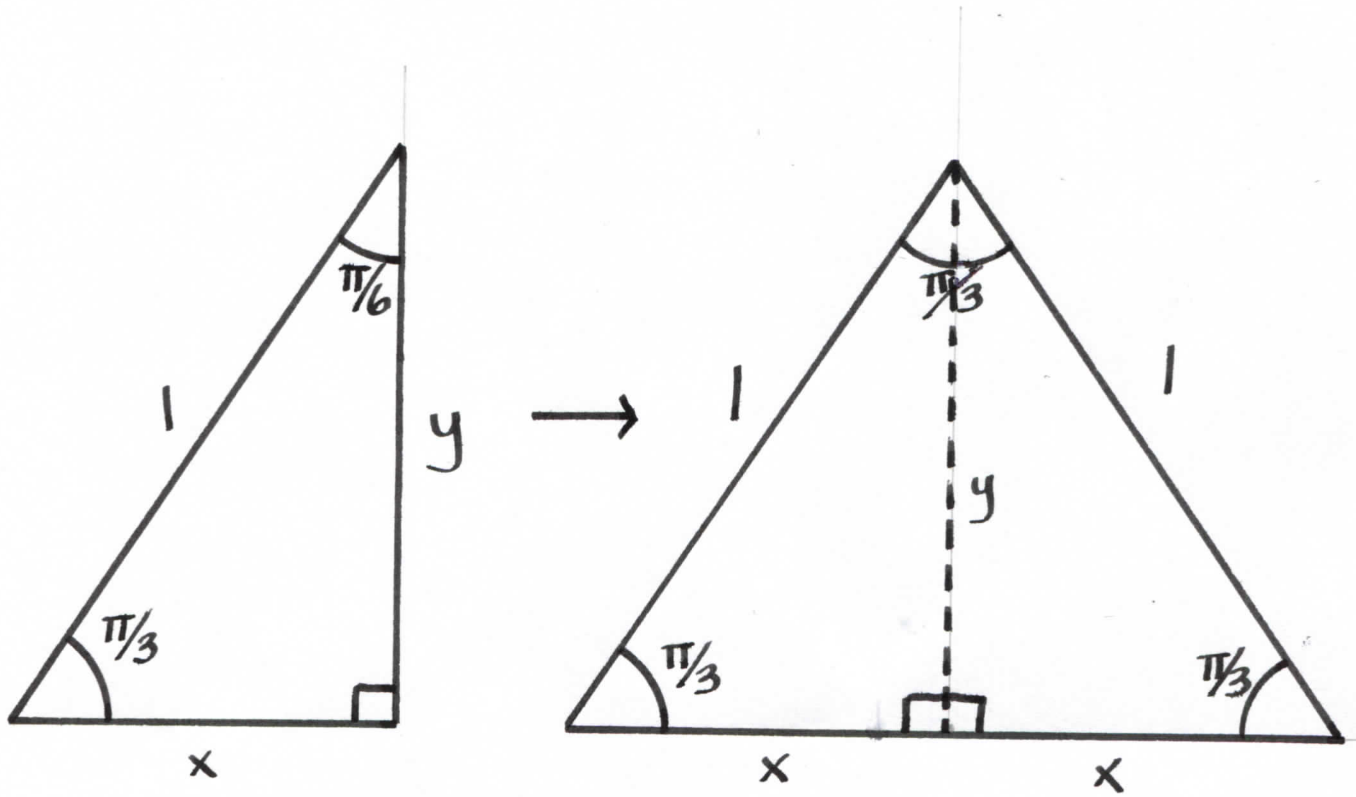
DRAWINGS 7.12 (continued)

(e)



$$\longrightarrow \quad x^2 + x^2 = 1 \quad \longrightarrow \quad x = \frac{1}{\sqrt{2}}$$

DRAWINGS 7.12(e) (continued)

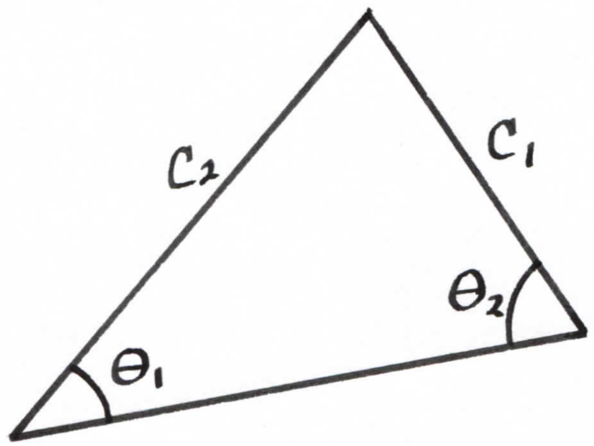


$\longrightarrow \quad x = \frac{1}{2}; \qquad x^2 + y^2 = 1 \rightarrow y = \frac{\sqrt{3}}{2}$

$\cos\frac{\pi}{3} = \sin\frac{\pi}{6} = x = \frac{1}{2}$

$\cos\frac{\pi}{6} = \sin\frac{\pi}{3} = y = \frac{\sqrt{3}}{2}$

DRAWING 7.13

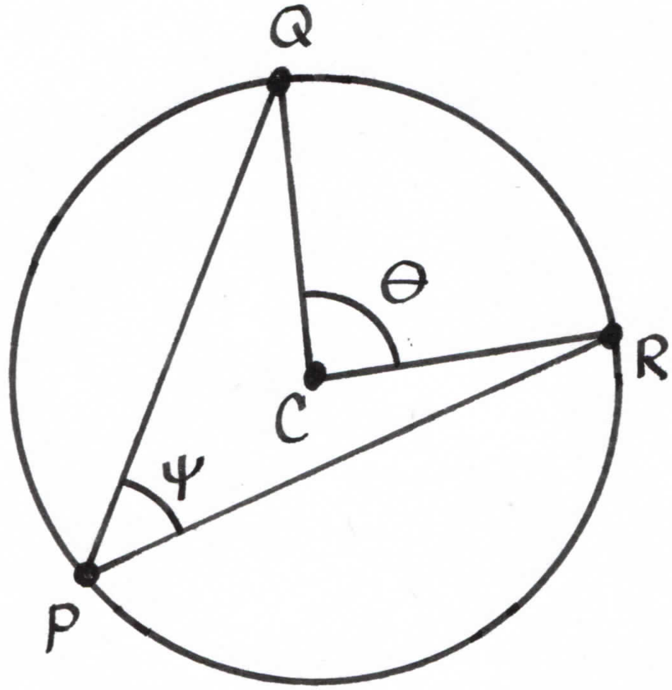


$c_2 \geq c_1$

$\longleftrightarrow$

$\theta_2 \geq \theta_1$

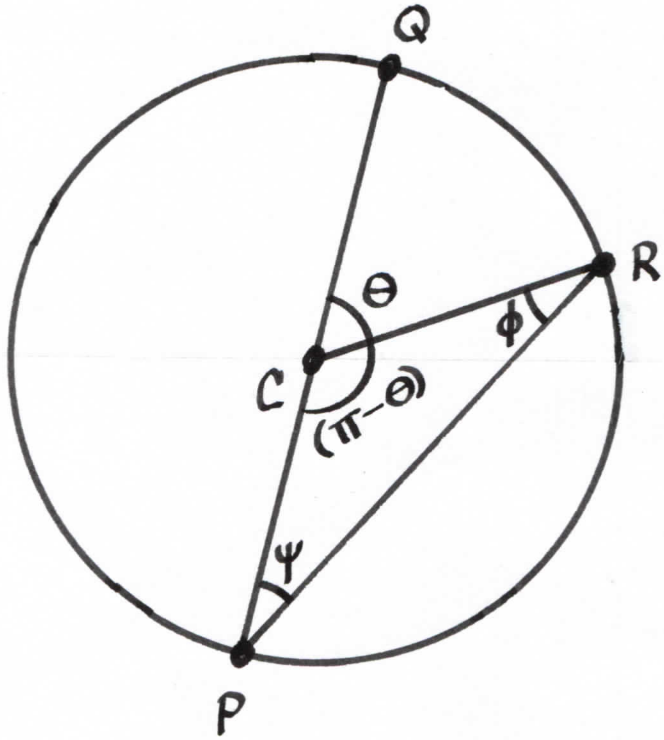
DRAWING 7.14



θ = central angle

ψ = inscribed angle

DRAWING 7.15 (CASE 1)

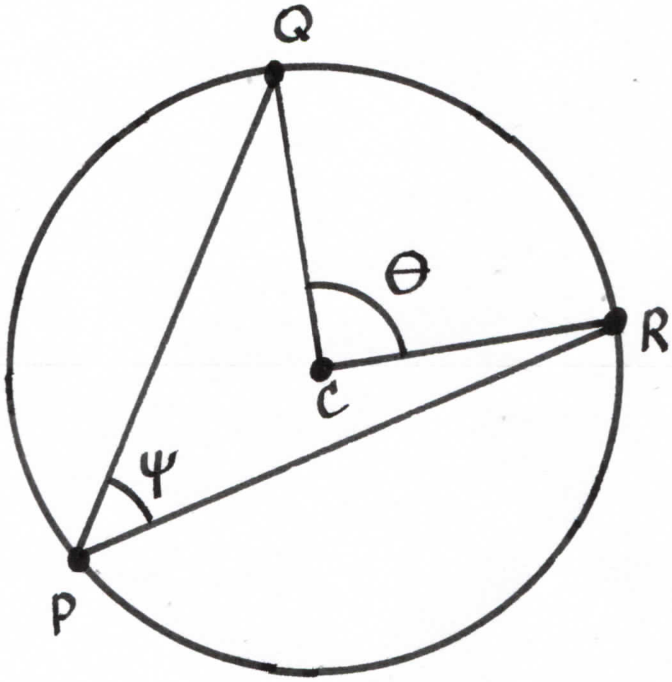


$$\phi = \psi$$

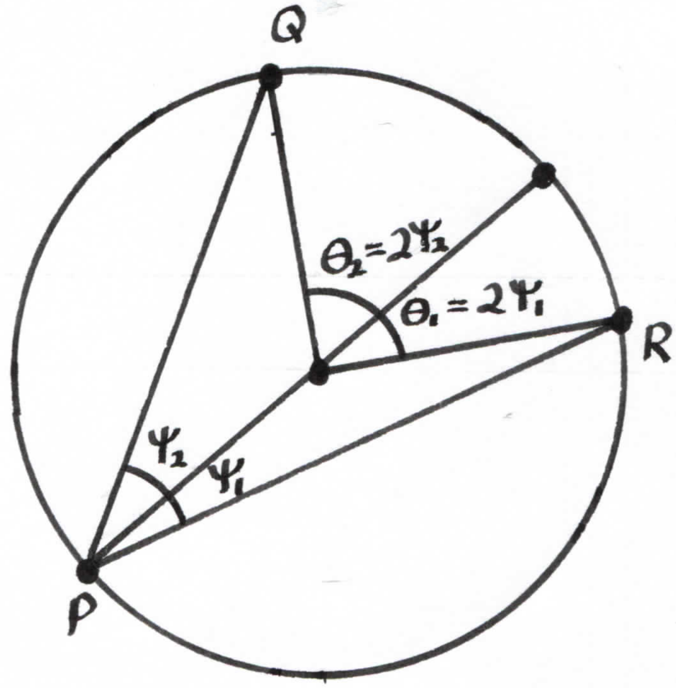
$$\longrightarrow \quad \pi = (\pi - \theta) + \psi + \psi$$

$$\longrightarrow \quad \theta = 2\psi$$

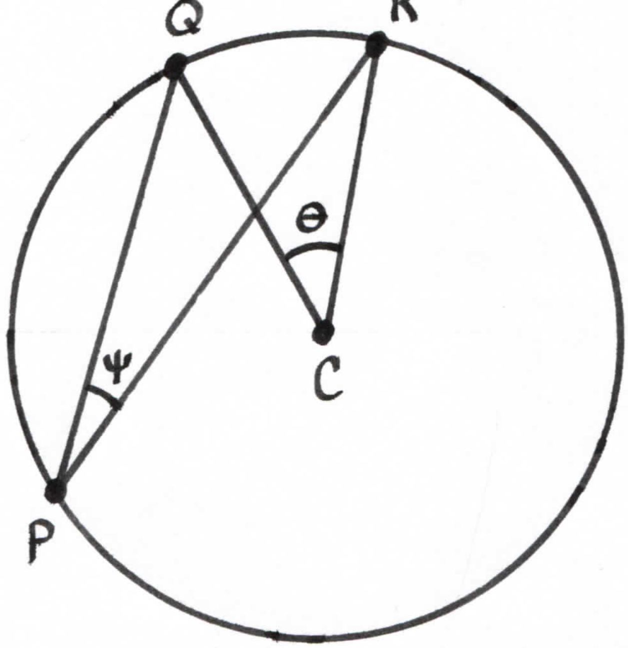
DRAWINGS 7.16 (CASE 2)

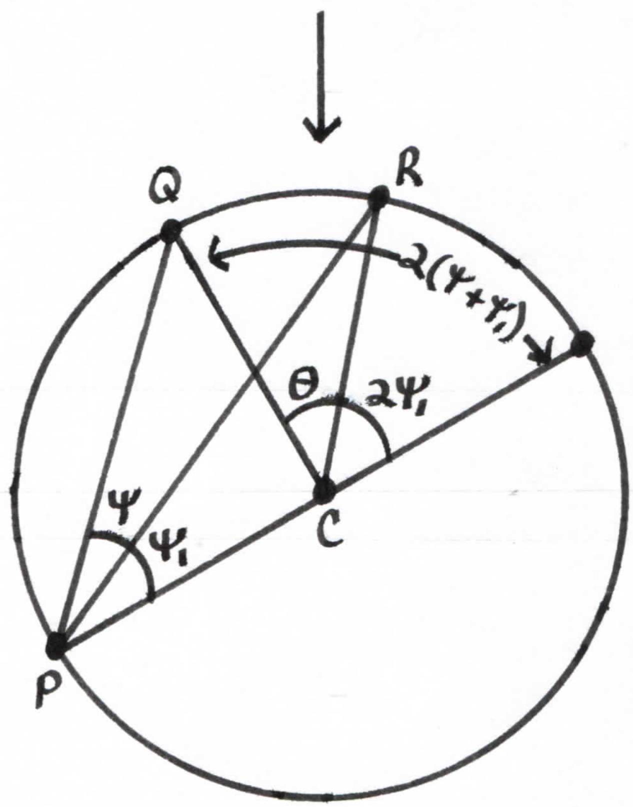


↓



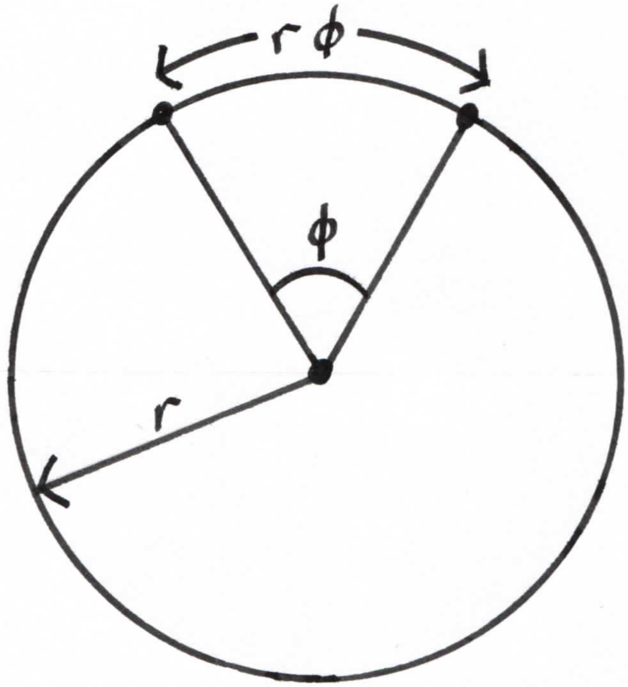
DRAWINGS 7.17 (CASE 3)

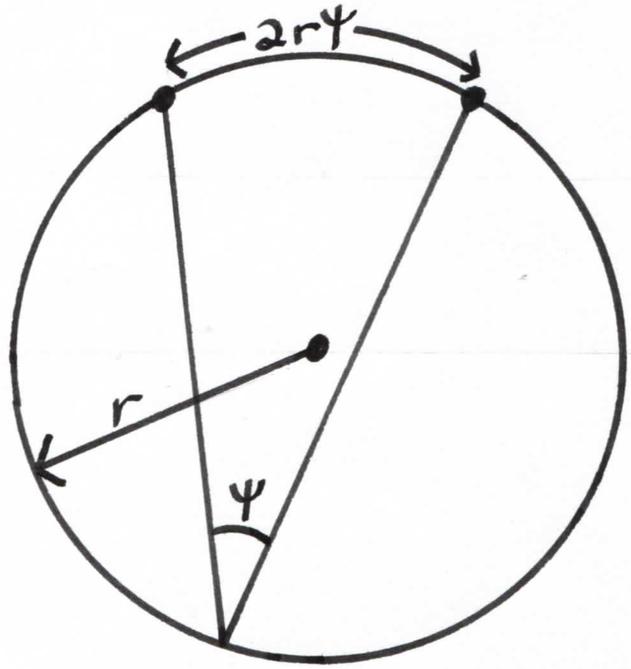




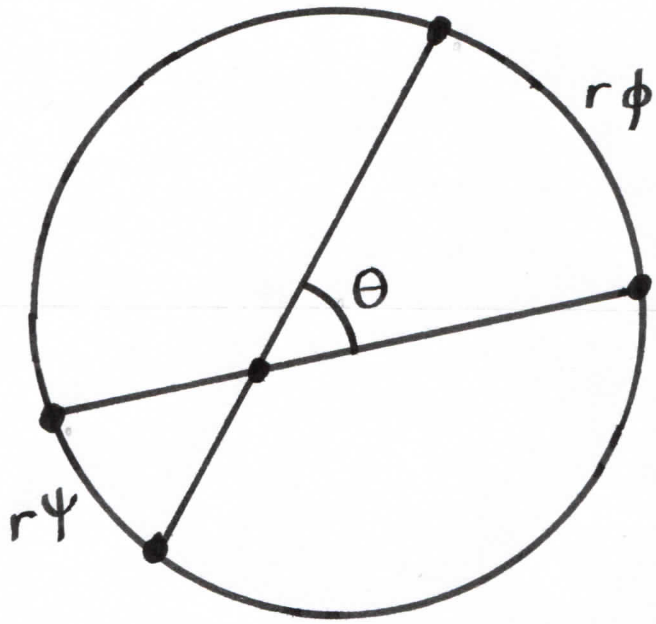
$$2(\Psi + \Psi_1) = \Theta + 2\Psi_1 \longrightarrow \Theta + 2\Psi$$

DRAWINGS 7.18

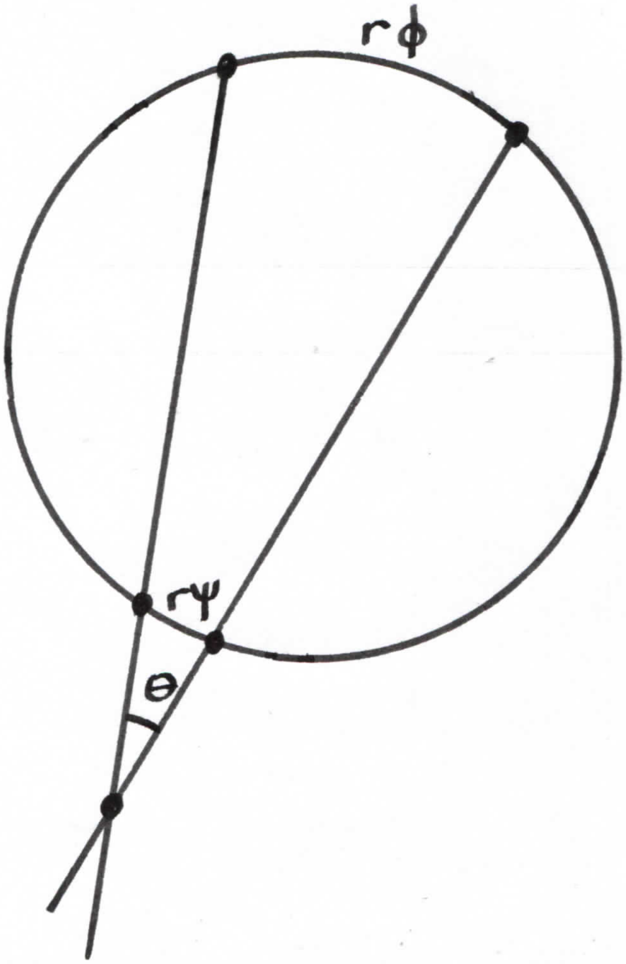




DRAWING 7.19

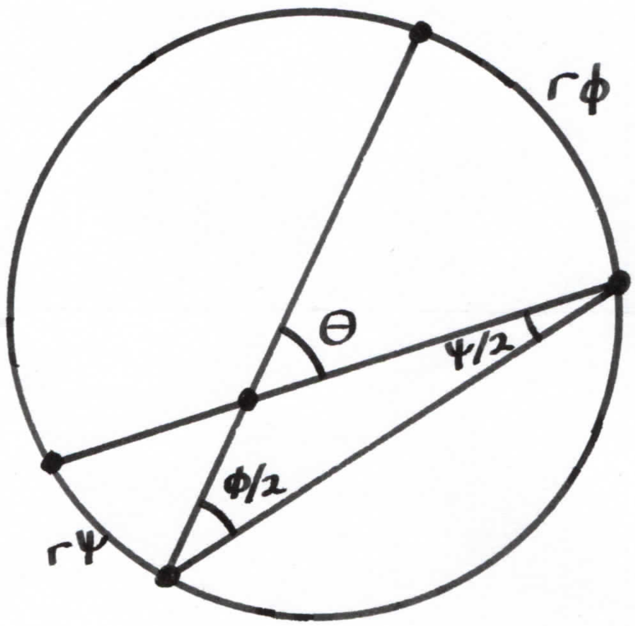


DRAWING 7.20



DRAWING 7.21

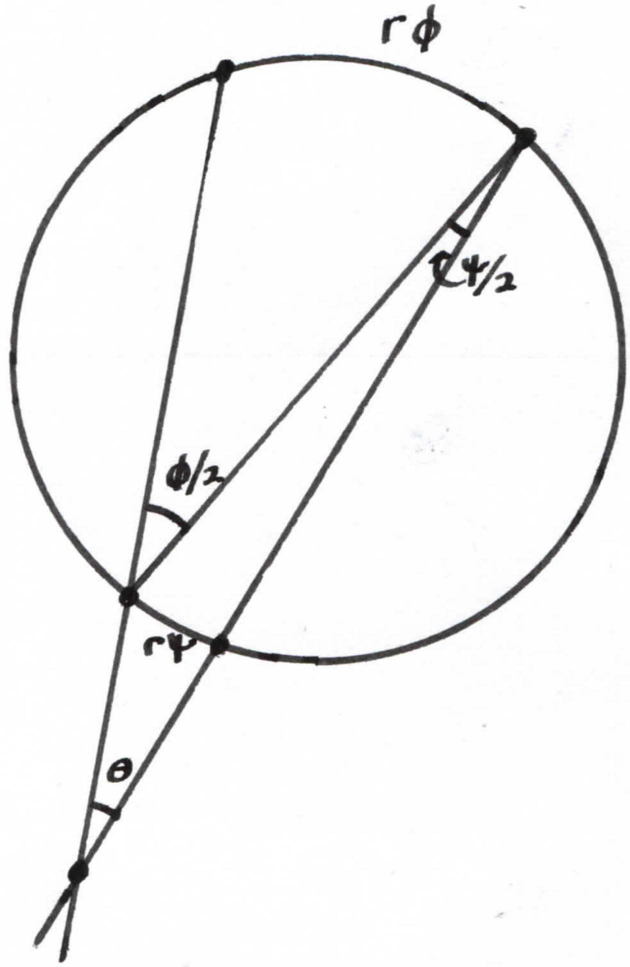
DRAWING 7.19 ⟶



$$\pi = \theta + \left(\pi - \frac{\psi}{2} - \frac{\phi}{2}\right) \longrightarrow \theta = \frac{1}{2}(\phi + \psi)$$

DRAWING 7.22

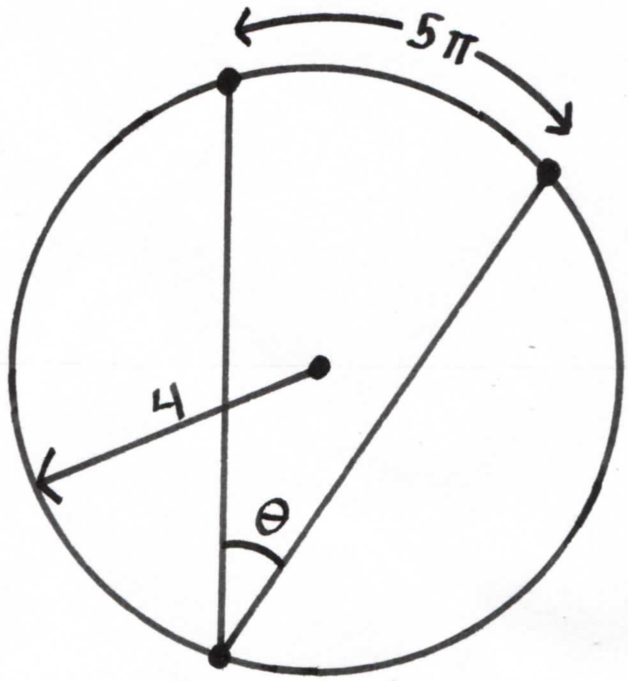
DRAWING 7.20 $\longrightarrow$



$$\pi = \left(\pi - \frac{\phi}{2}\right) + \frac{\psi}{2} + \theta \longrightarrow \theta = \frac{1}{2}(\phi - \psi)$$

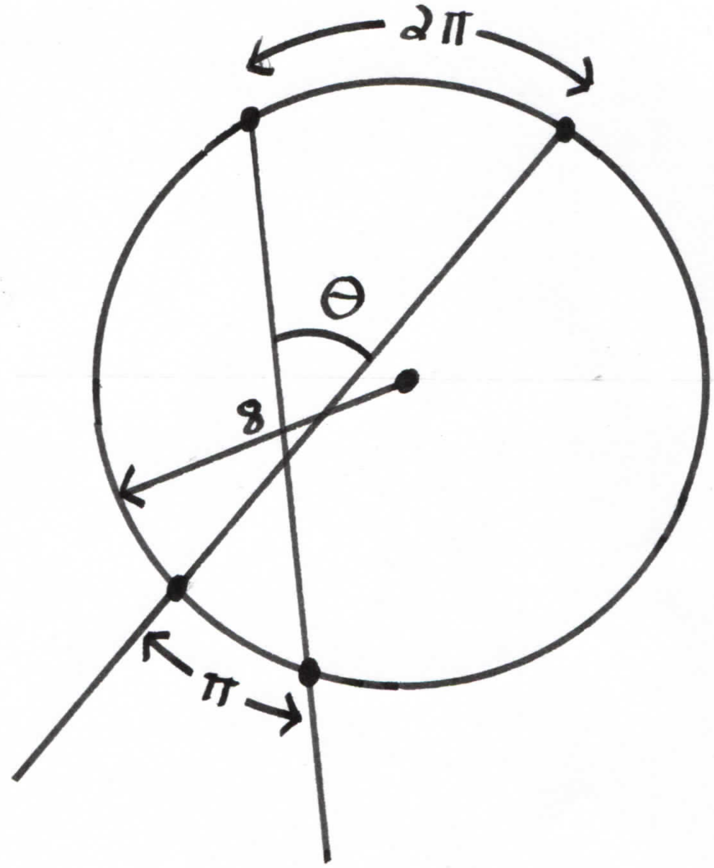
DRAWINGS 7.23

(a)



(radius=4)

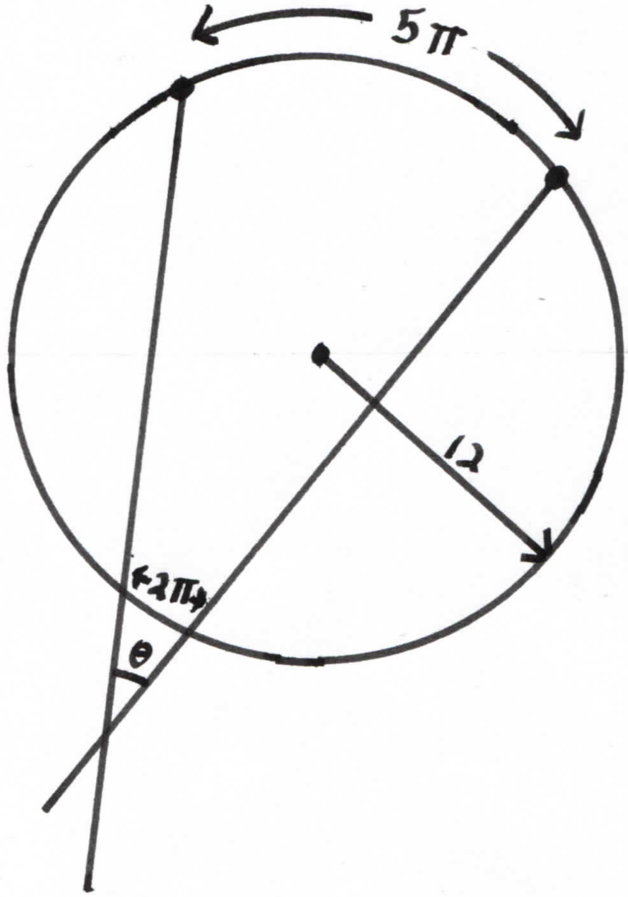
(b)



(radius=8)

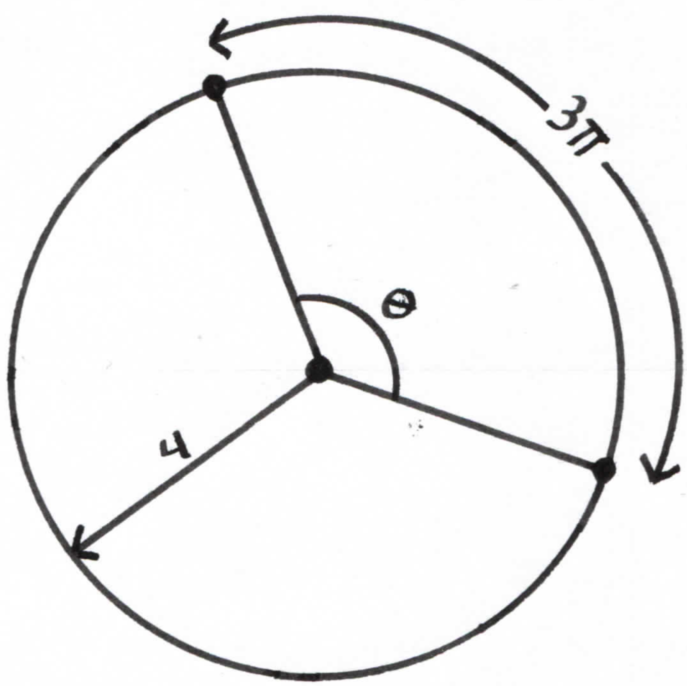
DRAWINGS 7.23 (continued)

(c)



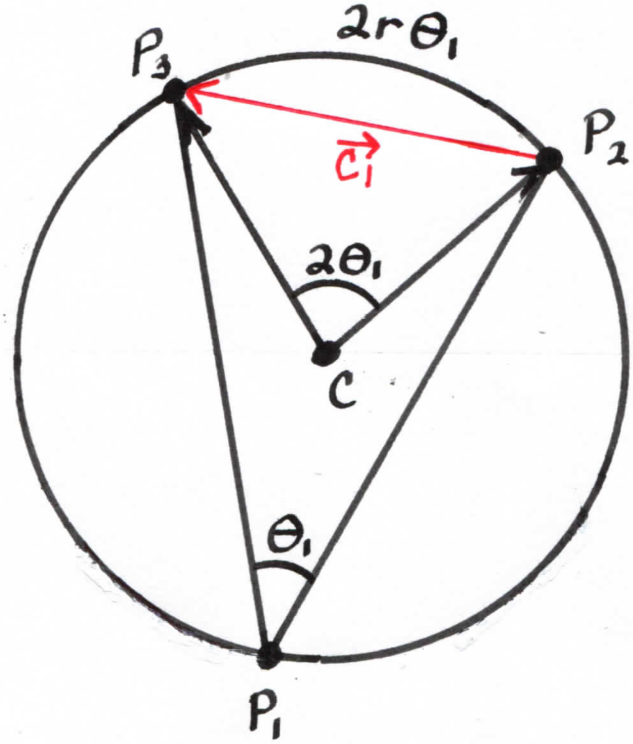
(radius = 12)

(d)

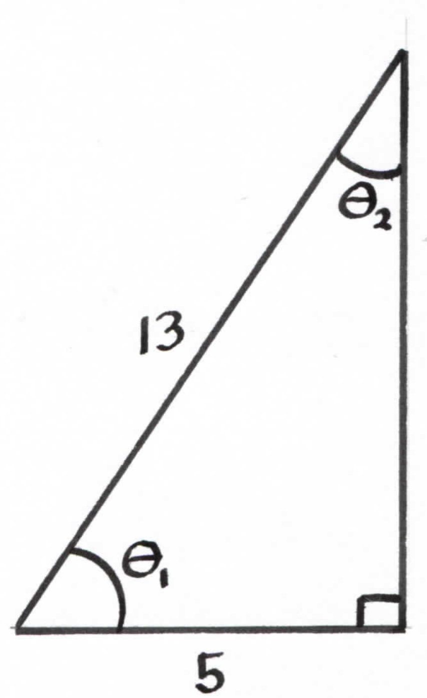


(radius = 4)

DRAWING 7.24



DRAWING 7.25

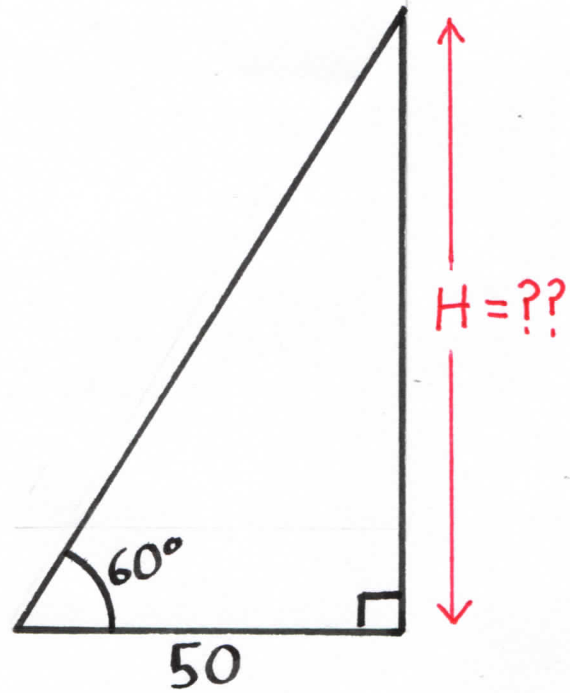


DRAWINGS 7.26

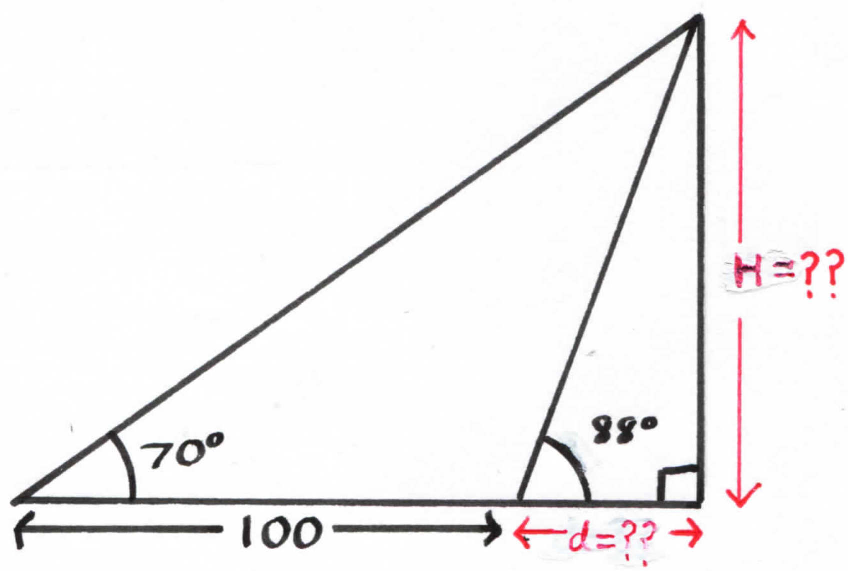
(a)

(b)

DRAWING 7.27

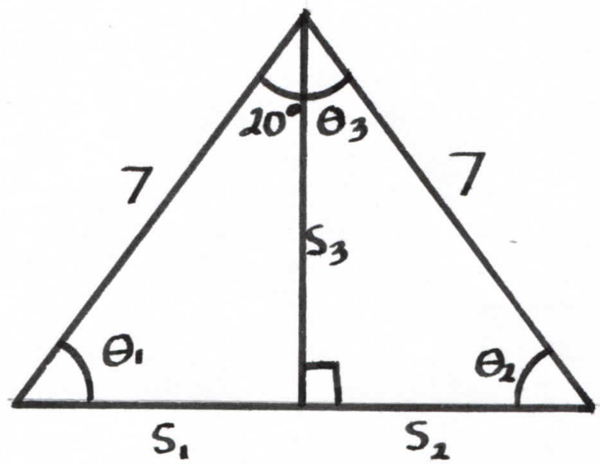


DRAWING 7.28

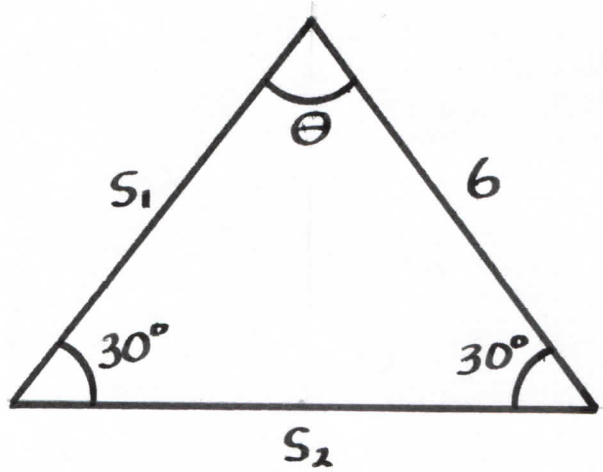


DRAWINGS 7.29

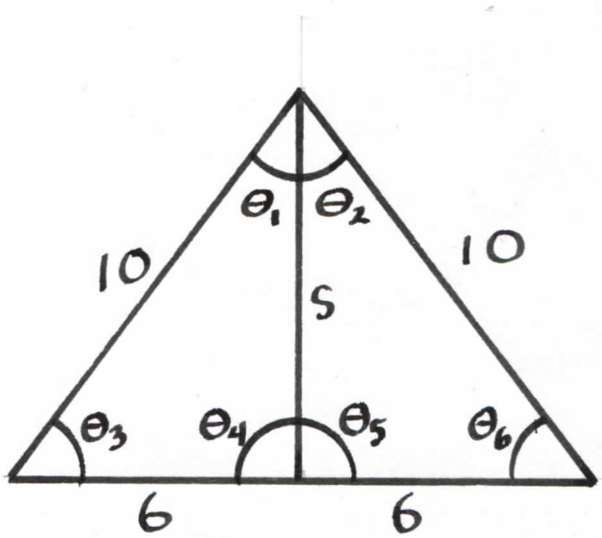
(a)



(b)

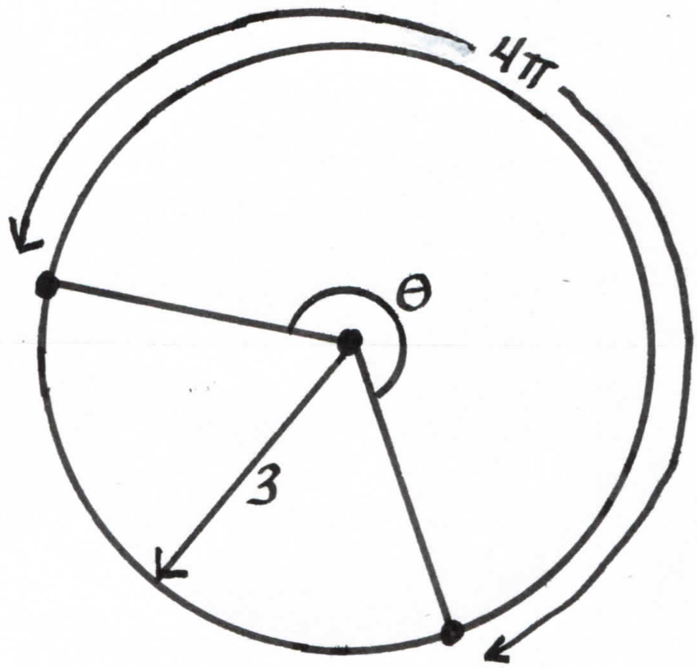


(c)



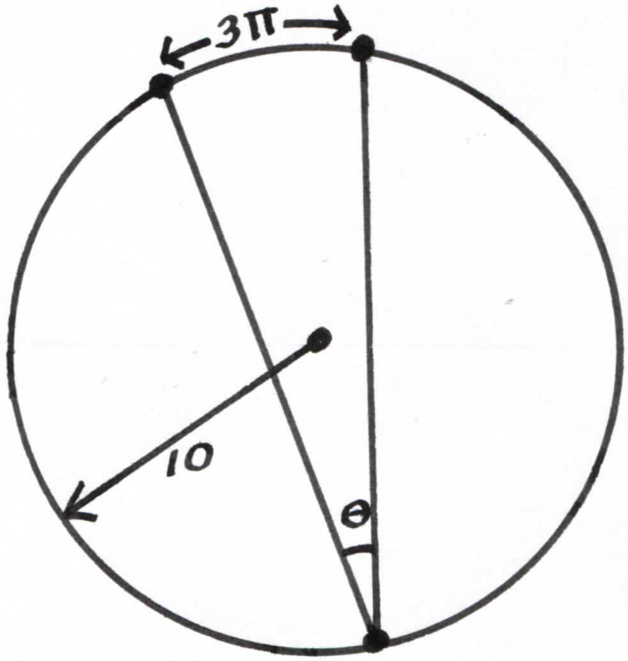
DRAWINGS 7.30

(a)



(radius = 3)

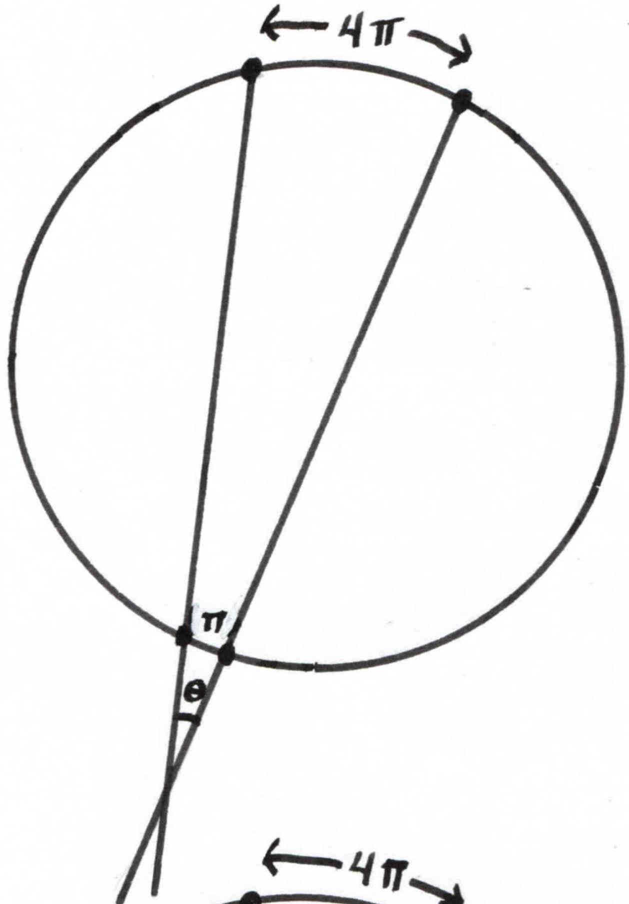
(b)



(radius = 10)

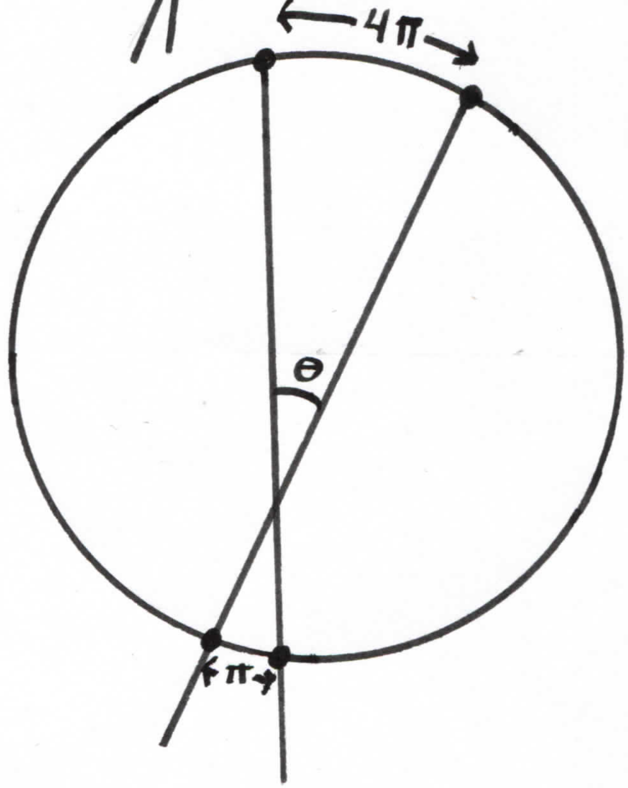
DRAWINGS 7.30 (continued)

(c)



(radius = 10)

(d)

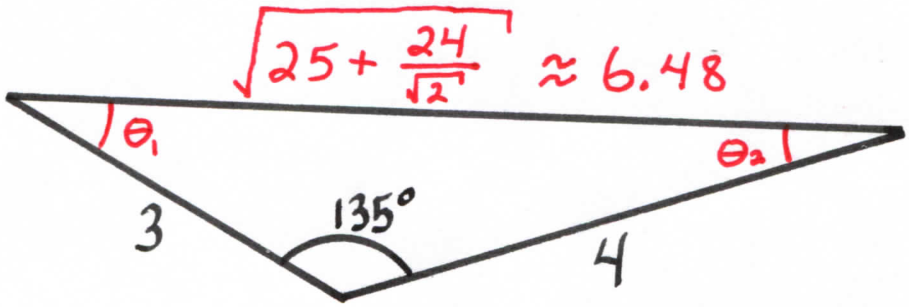


(radius = 10)

DRAWINGS 7.31

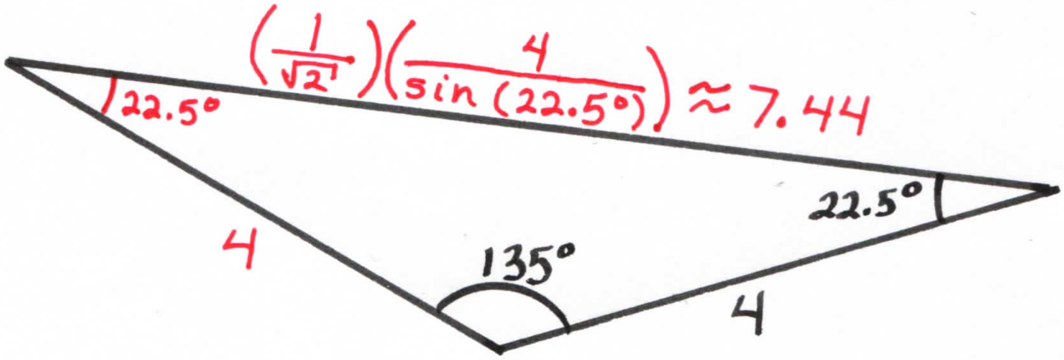
(a)

$\theta_1 \approx \cos^{-1}(0.9) \approx 25.84°$



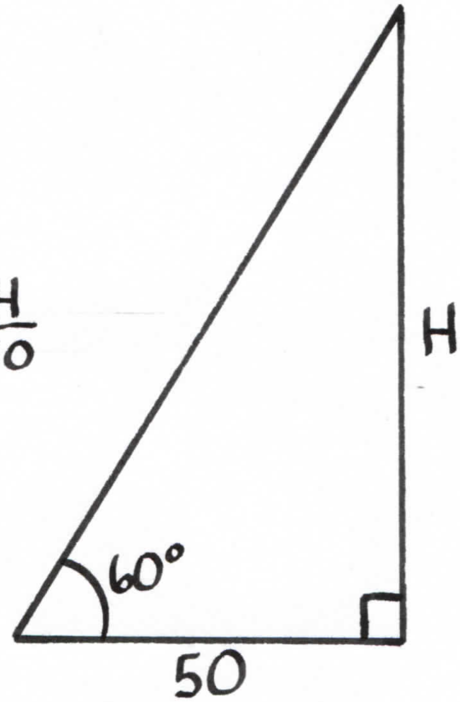
$\theta_2 \approx 180° - (25.84° + 135°)$
$= 19.16°$

(b)



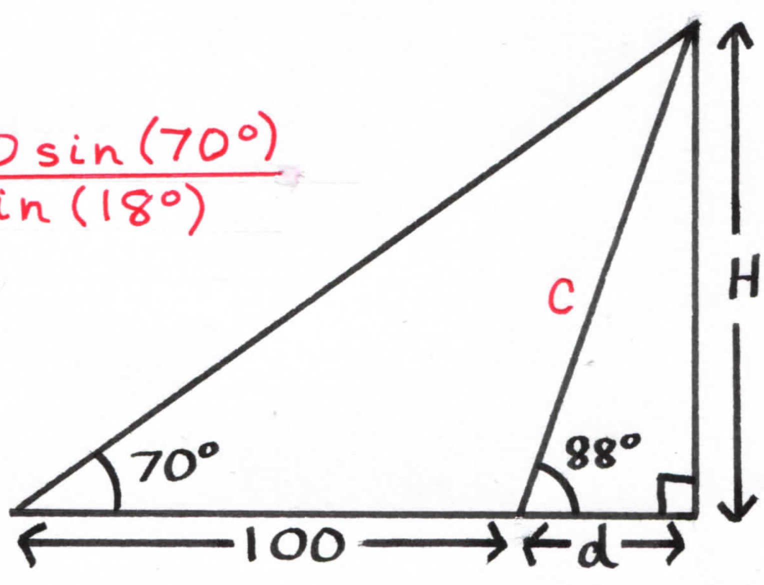
DRAWING 7.32

$$\tan(60^\circ) = \frac{H}{50}$$



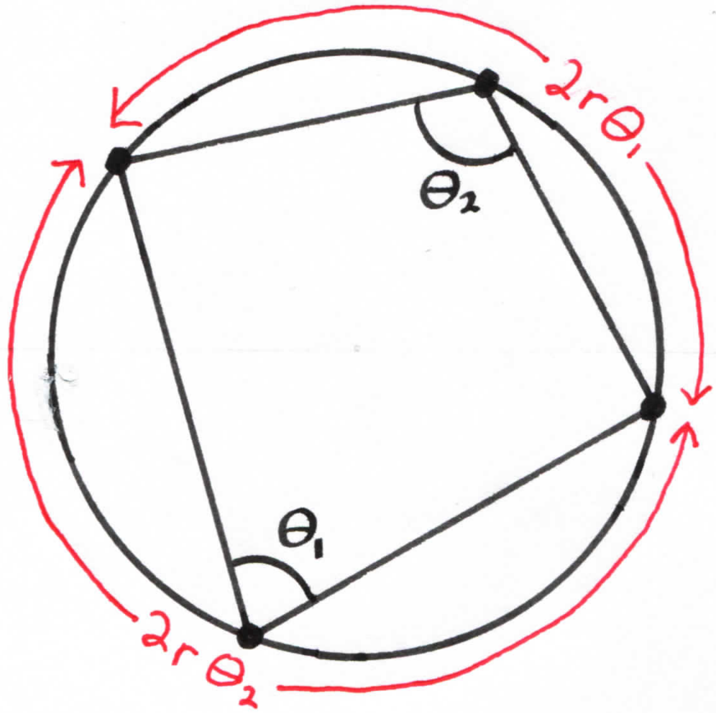
DRAWING 7.33

$$c = \frac{100 \sin(70^\circ)}{\sin(18^\circ)}$$



DRAWING 7.34

$r \equiv$ radius of circle

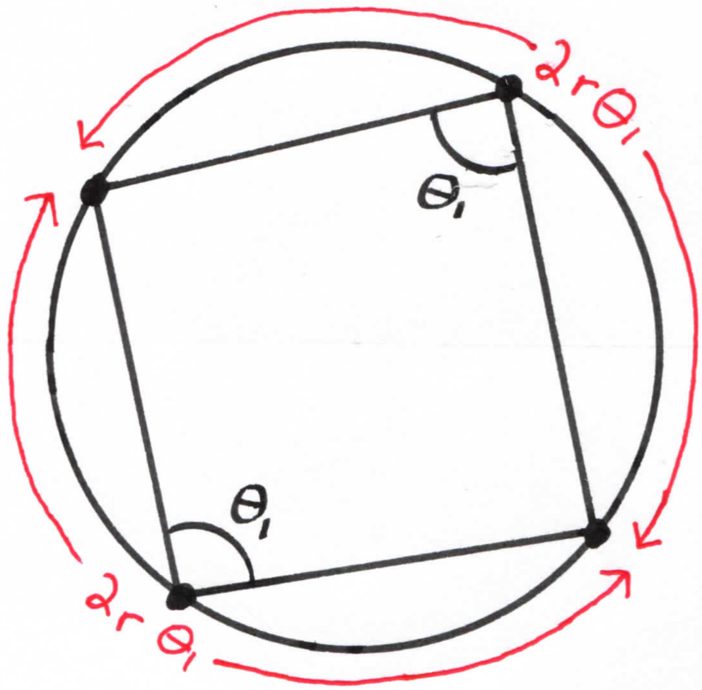


$$2\pi r = (2r\theta_1 + 2r\theta_2) \longrightarrow (\theta_1 + \theta_2) = \pi$$

DRAWING 7.35

$r \equiv$ radius of circle

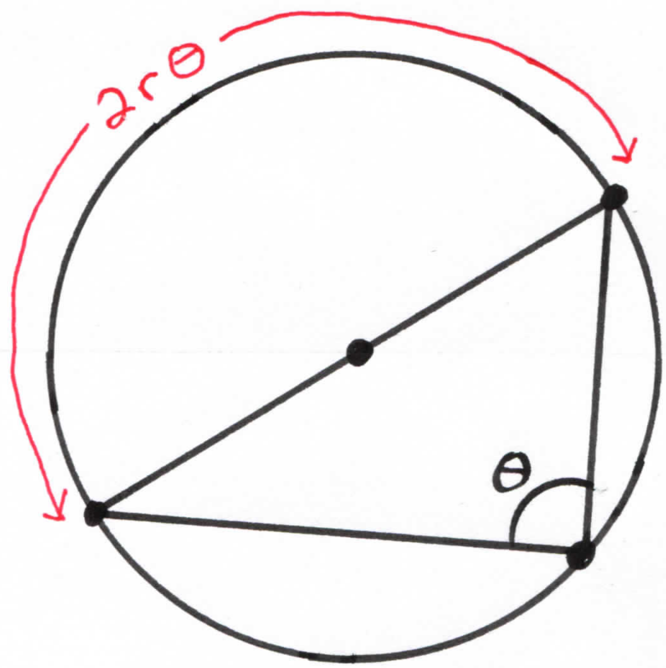
Proposition 3.8 $\longrightarrow$



$2(2r\theta_1) = 2\pi r \longrightarrow \theta_1 = \frac{\pi}{2} \longrightarrow$ rectangle (see Definitions 5.4)

DRAWING 7.36

$r \equiv$ radius of circle



$$2r\theta = r\pi \longrightarrow \theta = \frac{\pi}{2}$$

CHAPTER VIII: Congruence and Similarity.

An object might get moved around without changing its size or shape; that is, without changing length, angle measure, or area. Even though the moved object might consist of different points in the plane than the original object, we would like to think of it as being the same, in some sense (see Theorem 8.3).

For a physical model, imagine the object of interest as cut out of cardboard. We can move the object around on top of the plane, or even (in the case of reflection) in three dimensions before returning to the plane, so that it occupies different points, yet we have the same piece of cardboard.

Theorem 8.3(a) is a precise statement of the intuition of the last two paragraphs, preceded by a precise description of the motions we are applying to the desired objects (Definitions 8.1(a), (b), and (c); see also DRAWING 8.5(a)–(c) at the end of this chapter).

We also discuss a motion (Definition 8.1(d); see also DRAWING 8.5(d) at the end of this chapter) that preserves shape, but not necessarily size, of an object, in Theorem 8.3(b). Physically, we could get this effect, at least in our perception of said object, by zooming in or out on that piece of cardboard from paragraph two representing said object.

The proof of Theorem 8.3 is put off until Chapter IX, which begins with a discussion of *matrices*, the fundamental idea in the subject known as *linear algebra*. The reader who is willing to take Theorem 8.3 as an intuitive axiom may skip Chapter IX, without jeopardizing geometry. In fact, if a student prefers to avoid, in addition, the language of functions (Appendix 0), DRAWING 8.5(a)–(d) at the end of this chapter, along with Proposition 8.5 and the one-line statements of preservation in Theorem 8.3(a) and (b), provide sufficiently believable postulates to adopt.

Definitions 8.4 contain the definitions of *congruent* (informally, same size and shape) and *similar* (informally, same shape) objects.

Definitions 8.1. Here are the motions (also called *actions*) of interest: informally, sliding (translation), rotating, flipping (reflection), and stretching/shrinking (magnification). These can be thought of as functions (see Appendix 0): writing f for the function representing a motion, if I is a point in the plane, then the **image** of I, denoted $f(I)$, is the point after the motion has been applied; that is, where I is moved to.

We will write I' for $f(I)$.

Note that these functions may also be thought of as acting on vectors:

$$f(< v_1, v_2 >) \equiv < v_1', v_2' > \quad \text{if} \quad f(v_1, v_2) \equiv (v_1', v_2');$$

see Definition 9.3 for the consistency of these definitions.

In each of (a)–(d), I is an arbitrary point in the plane and I' is the image of I.

(a) Given a vector $\vec{v}$, the **translation** by $\vec{v}$ means $I' \equiv I + \vec{v}$. See DRAWING 8.1 at the end of this chapter.

(b) Given an angle measure θ and a point P the (counterclockwise) **rotation** means I' is on the same circle centered at P as I, with the measure of the counterclockwise angle from $\overrightarrow{PI}$ to $\overrightarrow{PI'}$ equal to θ.

Equivalently, $\|\overrightarrow{PI}\| = \|\overrightarrow{PI'}\|$ and the measure of the counterclockwise angle from $\overrightarrow{PI}$ to $\overrightarrow{PI'}$ is θ. See DRAWING 8.2 at the end of this chapter.

In the language of complex numbers,

$$I' = P + (I - P)e^{i\theta}.$$

See DRAWING 8.2 at the end of this chapter.

(c) Given a line ℓ, the **reflection** thru ℓ means that ℓ is the perpendicular bisector of the line segment between I and I' (see Definition 5.8). See DRAWING 8.3 at the end of this chapter.

(d) Given a positive number R, **magnification** by R moves (a, b) to (Ra, Rb); that is, if $I \equiv (a, b)$, then $I' = (Ra, Rb)$. See DRAWING 8.4 at the end of this chapter.

For Ω a subset of $\mathbf{R}^2$, f a composition of any of the functions of Definitions 8.1(a)–(d),

$$f(\Omega) \equiv \{f(J) \mid J \text{ is in } \Omega\}.$$

Definition 8.2. The actions of translation, rotation and reflection in Definitions 8.1(a)–(c) and DRAWING 8.5(a)–(c) at the end of this chapter are called **rigid motions**.

Theorem 8.3. (a) Rigid motions preserve length, angle measure, and area.

By this, we mean that, if f is a function representing (a), (b), or (c) of Definitions 8.1, then, for any vectors $\vec{a}, \vec{b}$, polygon Ω,

(i) the length of $f(\vec{a})$ equals the length of $\vec{a}$; and

(ii) the angle between $f(\vec{a})$ and $f(\vec{b})$ and the angle between $\vec{a}$ and $\vec{b}$ have equal measures; and

(iii) the area of $f(\Omega)$ equals the area of Ω.

(b) Magnification by R preserves angle, multiplies length by R, and multiplies area by R^2.

By this, we mean that, if f is the function representing (d) of Definitions 8.1, then, for any vectors $\vec{a}, \vec{b}$, polygon Ω,

(i) the length of $f(\vec{a})$ equals R times the length of $\vec{a}$; and

(ii) the angle between $f(\vec{a})$ and $f(\vec{b})$ and the angle between $\vec{a}$ and $\vec{b}$ have equal measure; and

(iii) the area of $f(\Omega)$ equals R^2 times the area of Ω.

See DRAWING 8.5 at the end of this chapter.

Definitions 8.4. Two sets are said to be **congruent** if one set may be obtained from the other by applying rigid motions.

Two sets are said to be **similar** if one set may be obtained from the other by applying any of the motions (a)–(d) of Definition 8.1 or DRAWING 8.5(a)–(d) at the end of this chapter.

If f is a composition of the motions (a)–(d) in Definitions 8.1 and Ω_1 and Ω_2 are subsets of $\mathbf{R}^2$ such that $f(\Omega_1) = \Omega_2$ (so that Ω_1 and Ω_2 are similar) and ω is a subset of Ω_1 then $f(\omega)$ is the subset of Ω_2 **corresponding** to ω and the measure of $f(\omega)$ **corresponds** to the measure of ω.

By Theorem 8.3, congruent sets have corresponding lengths, angle measures, and areas in common, while similar objects have corresponding angle measures in common; see Theorem 10.1(b) for a converse of the latter assertion. Here is a useful consequence of similarity for triangles.

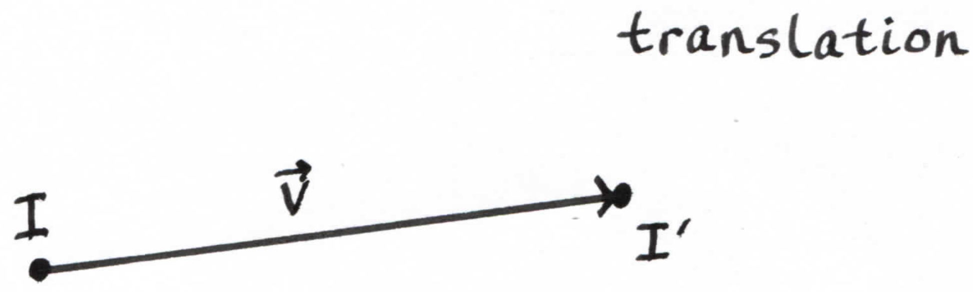
Proposition 8.5. If two triangles are similar, then ratios of corresponding lengths of sides are equal. That is, suppose T_1 and T_2 are triangles and f is a composition of (a)–(d) in Definitions 8.1, with $T_2 = f(T_1)$. Further suppose that $\vec{S}_1$ and $\vec{S}_2$ are two sides of T_1. Then

$$\frac{\|\vec{S}_1\|}{\|\vec{S}_2\|} = \frac{\|f(\vec{S}_1)\|}{\|f(\vec{S}_2)\|}.$$

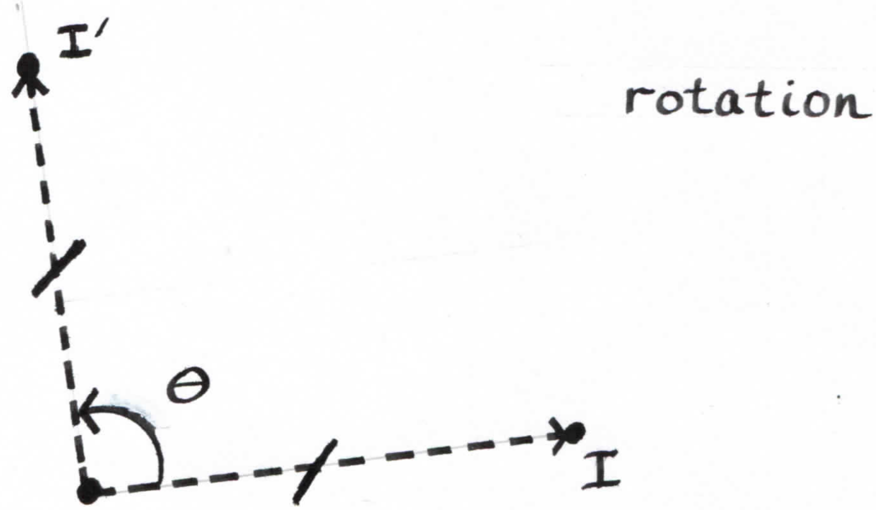
It can be shown that rigid motions are the only functions from the plane into itself that preserve length, angle measure, and area. See Remark 9.14, which shows this under the assumption that the function is *linear*, which is equivalent to being represented as matrix multiplication (see Definition 9.3).

Thus congruence is *equivalent* to having corresponding lengths, angle measures, and areas in common. Congruent sets are precisely the ones we think of as being the same, as described in the first two paragraphs of this chapter. Simple identification of congruence and similarity will be the theme of Chapter X.

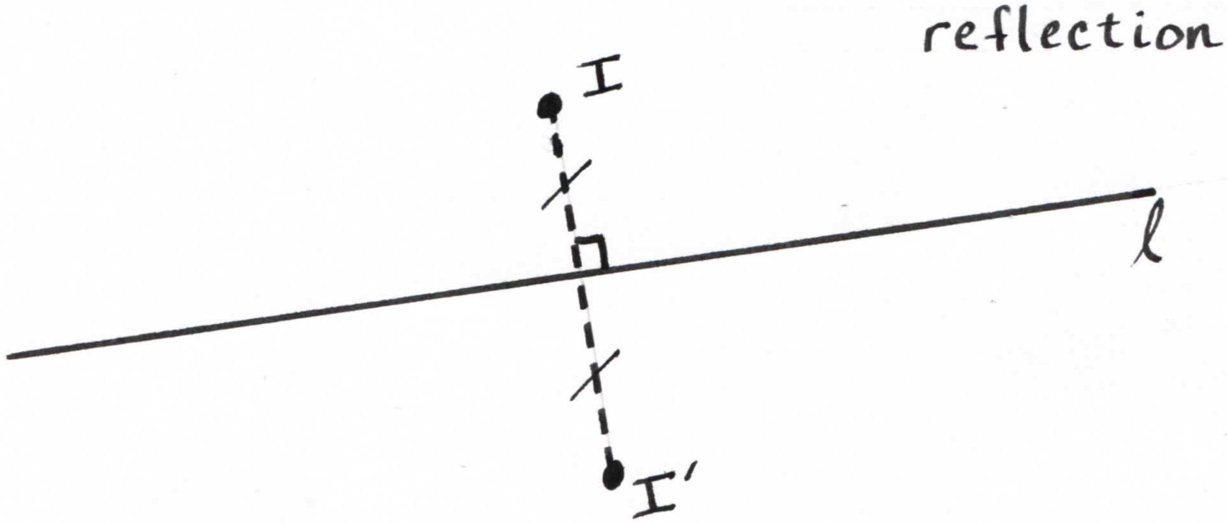
DRAWING 8.1



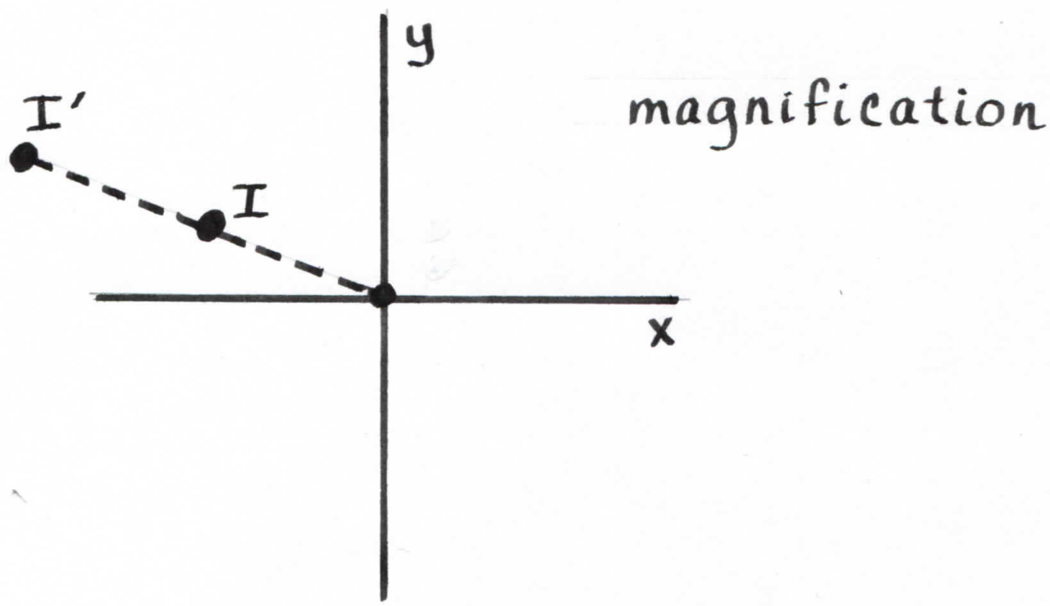
DRAWING 8.2



DRAWING 8.3

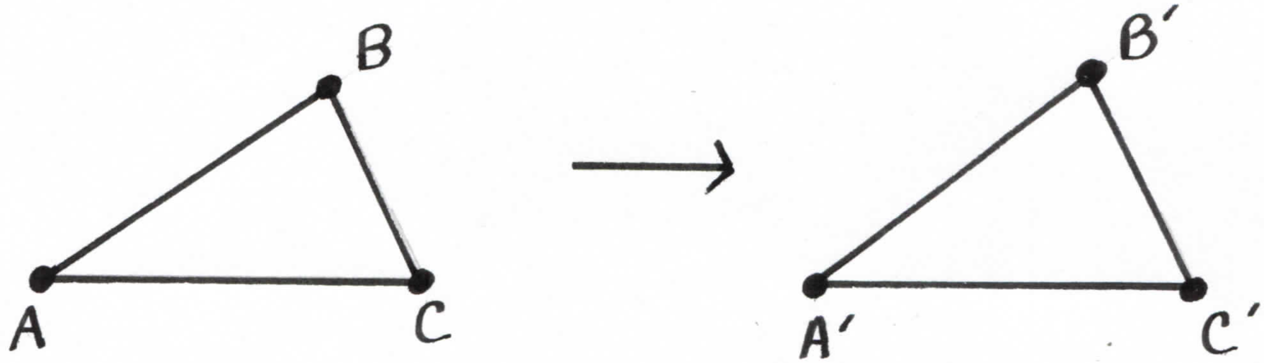


DRAWING 8.4



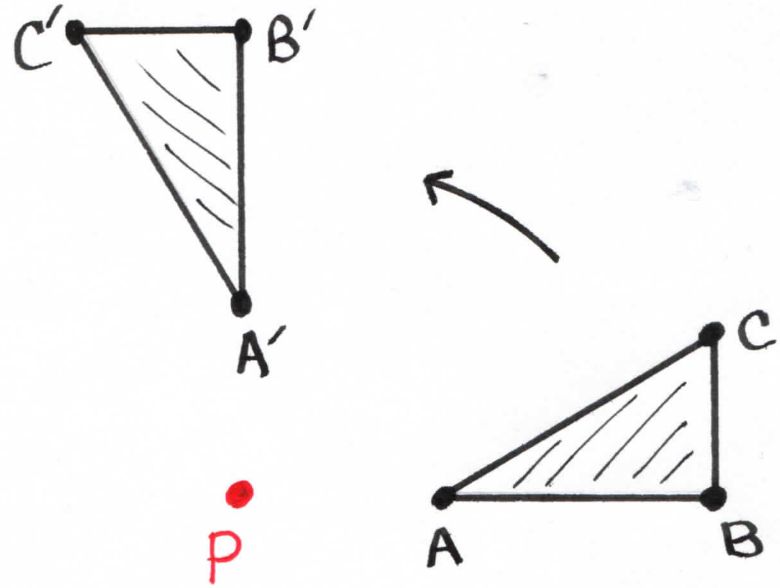
DRAWING 8.5(a)

translation



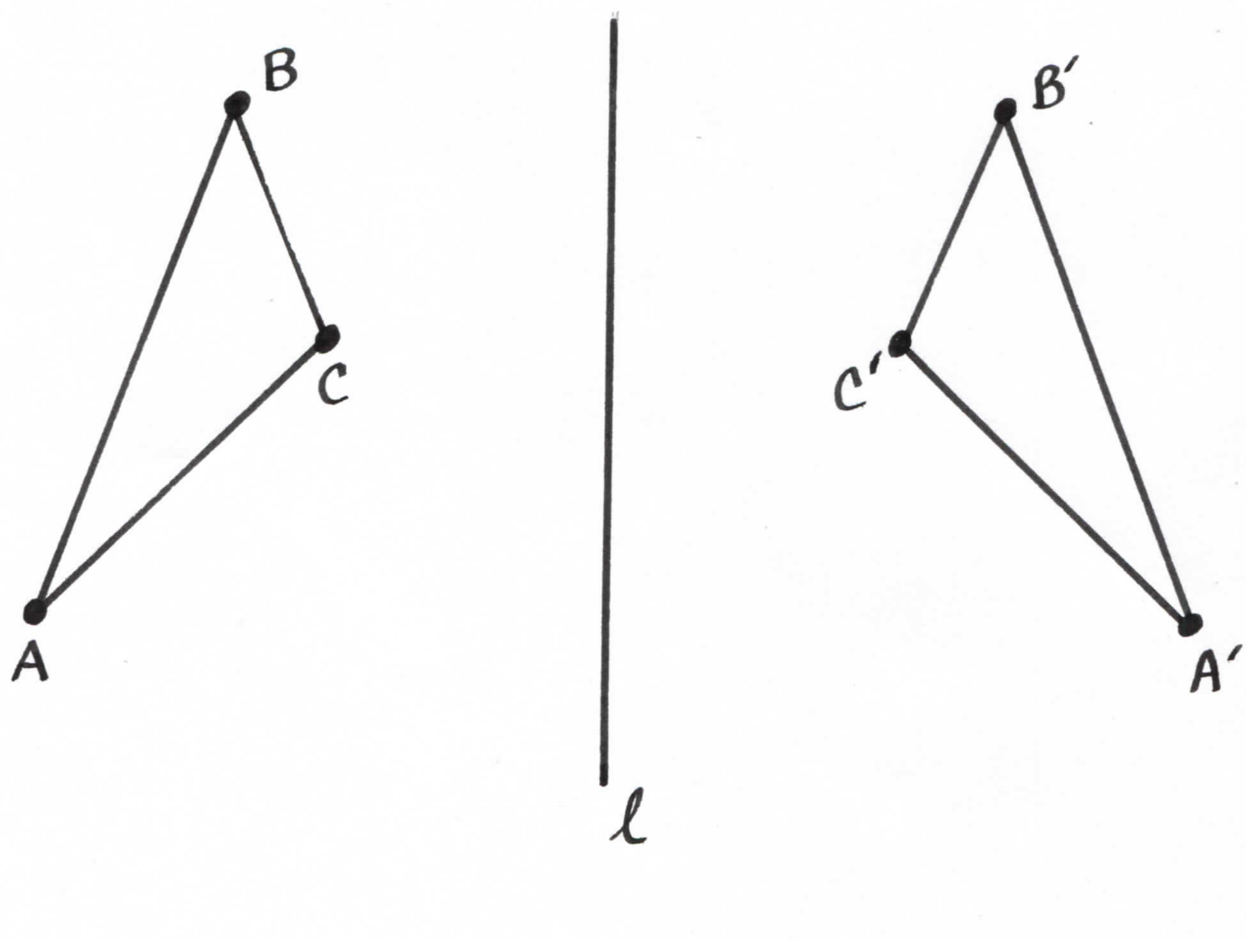
DRAWING 8.5(b)

rotation (by 90 degrees)



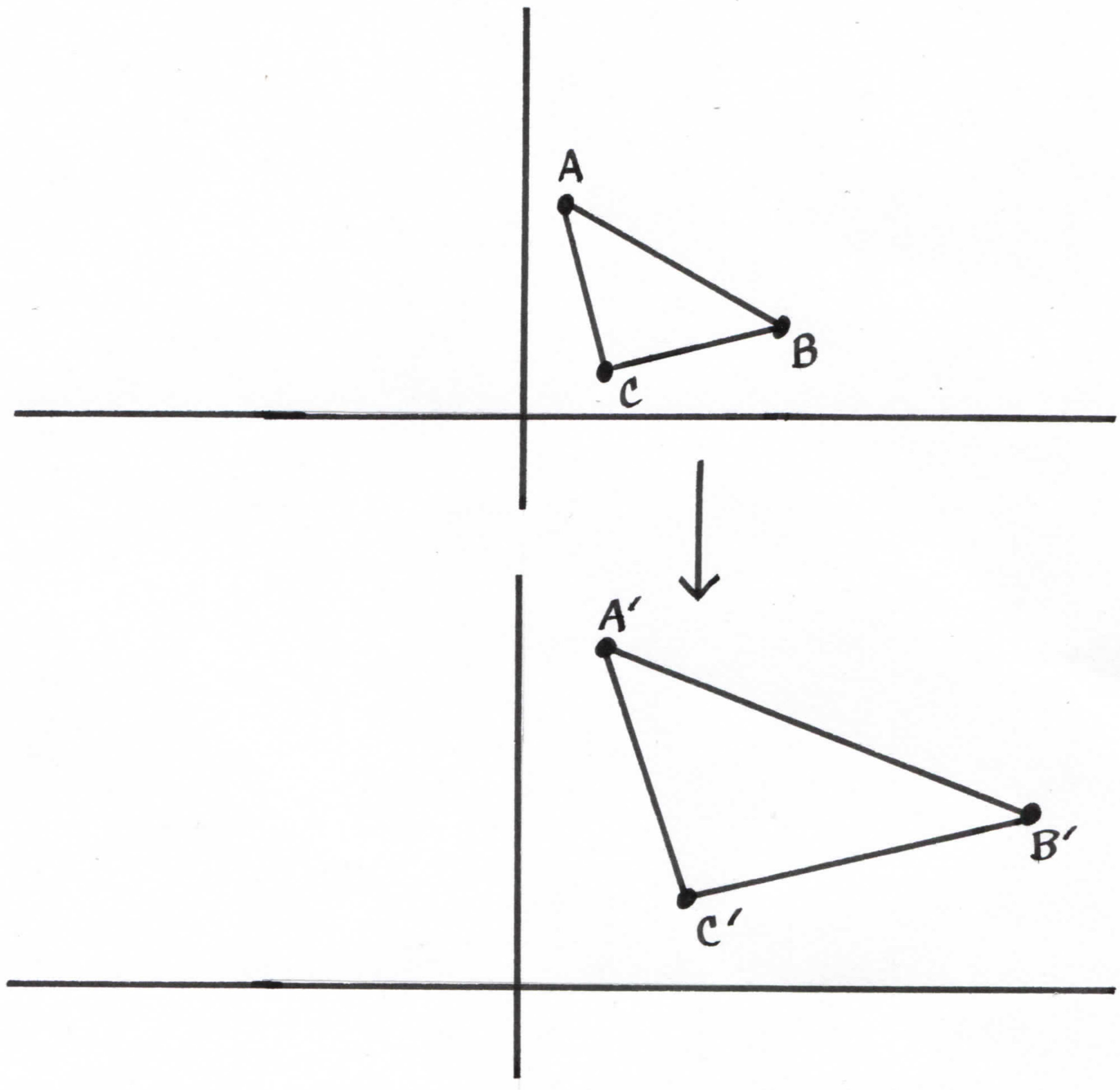
DRAWING 8.5(c)

reflection



DRAWING 8.5(d)

magnification



CHAPTER IX: Proof of Theorem 8.3, via Matrices.

We will study rigid motions by representing them (except for translation) as multiplication of vectors by certain *matrices* (see Definitions 9.1 and 9.2).

Definitions 9.1. A 2×2 ("two by two") **matrix** is a rectangular array of two rows and two columns $A \equiv \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, where $a_{11}, a_{12}, a_{21}, a_{22}$ are numbers.

The **rows** of A are $\begin{bmatrix} a_{11} & a_{12} \end{bmatrix}$ and $\begin{bmatrix} a_{21} & a_{22} \end{bmatrix}$; the **columns** are $\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$ and $\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$.

A 2×1 matrix is a column of two numbers $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$.

We will write $\begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix} \equiv \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ and $\begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix} \equiv \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$. Either of those matrices can be thought of as the pair of vectors $\vec{a} \equiv < a_1, a_2 >, \vec{b} \equiv < b_1, b_2 >$.

Definition 9.2. We may multiply some matrices. If $A \equiv \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $B \equiv \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ and x and y are numbers, then (see Definition 4.1)

$$AB \equiv \begin{bmatrix} (< a_{11}, a_{12} > \cdot < b_{11}, b_{21} >) & (< a_{11}, a_{12} > \cdot < b_{12}, b_{22} >) \\ (< a_{21}, a_{22} > \cdot < b_{11}, b_{21} >) & (< a_{21}, a_{22} > \cdot < b_{12}, b_{22} >) \end{bmatrix} \text{ and } A \begin{bmatrix} x \\ y \end{bmatrix} \equiv \begin{bmatrix} (< a_{11}, a_{12} > \cdot < x, y >) \\ (< a_{21}, a_{22} > \cdot < x, y >) \end{bmatrix}.$$

In the slicker language from Definition 9.1,

$$\begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \end{bmatrix} \begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix} \equiv \begin{bmatrix} (\vec{a}_1 \cdot \vec{b}_1) & (\vec{a}_1 \cdot \vec{b}_2) \\ (\vec{a}_2 \cdot \vec{b}_1) & (\vec{a}_2 \cdot \vec{b}_2) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \equiv \begin{bmatrix} \vec{a}_1 \cdot < x, y > \\ \vec{a}_2 \cdot < x, y > \end{bmatrix}.$$

Definition 9.3. Our interest in matrices is using them to define functions from $\mathbf{R}^2$ to $\mathbf{R}^2$. If A is a 2×2 matrix, we want the function f_A to represent multiplication by A. Complicated appearances arise from the fact that we now have *three* ways (not including complex numbers) to arrange an ordered pair of numbers x, y :

$$(x, y), \quad < x, y >, \quad \text{and} \quad \begin{bmatrix} x \\ y \end{bmatrix}.$$

Define $f_A : \mathbf{R}^2 \to \mathbf{R}^2$ by

$$f_A(x, y) = (x', y'), \quad \text{where} \quad \begin{bmatrix} x' \\ y' \end{bmatrix} \equiv A \begin{bmatrix} x \\ y \end{bmatrix}.$$

We may also apply f_A to vectors:

$$f_A(< v_1, v_2 >) =< v_1', v_2' >, \quad \text{where} \quad \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} \equiv A \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

In either case, A is called the **standard matrix** for f_A.

We should check that f_A applied to points is consistent with f_A applied to vectors. The definition of f_A on $\mathbf{R}^2$ necessitates the following definition of f_A applied to directed line segments: if $\overrightarrow{IT}$ is a directed line segment representing the vector $\vec{v}$, define

$$f_A(\overrightarrow{IT}) \equiv \overrightarrow{I'T'}, \text{ where } I' \equiv f_A(I), T' \equiv f_A(T).$$

Since $T = I + \vec{v}$, it is not hard to show that

$$T' \equiv f_A(T) = f_A(I + \vec{v}) = f_A(I) + f_A(\vec{v}) \equiv I' + f_A(\vec{v}),$$

so that the components of $\overrightarrow{I'T'}$ are $f_A(\vec{v})$; thus, when two directed line segments represent the same vector $\vec{v}$, their images both represent the vector $f_A(\vec{v})$, producing an unambiguous definition of f_A applied to vectors, as desired.

With the standard matrices for rigid motions (to be determined as in Remark 9.13) in mind, our goal now is the relationships between A and f_A.

Definition 9.4. det is short for **determinant**:

$$\det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \equiv a_1b_2 - a_2b_1.$$

Theorem 9.5. If A and B are 2×2 matrices, then det(AB) = (det A)(det B).

Proof: Denote $A \equiv \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B \equiv \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$. Then

$$\det(AB) = \det \begin{bmatrix} (< a_{11}, a_{12} > \cdot < b_{11}, b_{21} >) & (< a_{11}, a_{12} > \cdot < b_{12}, b_{22} >) \\ (< a_{21}, a_{22} > \cdot < b_{11}, b_{21} >) & (< a_{21}, a_{22} > \cdot < b_{12}, b_{22} >) \end{bmatrix}$$

$$= (a_{11}b_{11} + a_{12}b_{21})(a_{21}b_{12} + a_{22}b_{22}) - (a_{21}b_{11} + a_{22}b_{21})(a_{11}b_{12} + a_{12}b_{22}) =$$

$$(a_{11}b_{11}a_{21}b_{12}+a_{11}b_{11}a_{22}b_{22}+a_{12}b_{21}a_{21}b_{12}+a_{12}b_{21}a_{22}b_{22})-(a_{21}b_{11}a_{11}b_{12}+a_{21}b_{11}a_{12}b_{22}+a_{22}b_{21}a_{11}b_{12}+a_{22}b_{21}a_{12}b_{22})$$

$$= (a_{11}b_{11}a_{22}b_{22} + a_{12}b_{21}a_{21}b_{12}) - (a_{21}b_{11}a_{12}b_{22} + a_{22}b_{21}a_{11}b_{12}),$$

while

$$(\det A)(\det B) = (a_{11}a_{22}-a_{21}a_{12})(b_{11}b_{22}-b_{21}b_{12}) = a_{11}a_{22}b_{11}b_{22}-a_{11}a_{22}b_{21}b_{12}-a_{21}a_{12}b_{11}b_{22}+a_{21}a_{12}b_{21}b_{12}.$$

Amazingly, they appear to be the same. □

Proposition 9.6. If I is a point and $\vec{a}$ and $\vec{b}$ are vectors, then the area of the triangle with vertices $I, I + \vec{a}, I + \vec{b}$ is $\frac{1}{2}|\det \begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix}|$.

Proof: If $\vec{a}$ and $\vec{b}$ are parallel, we leave it to the reader to show that $\det \begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix}$ is zero. For the remainder of the proof, assume $\vec{a}$ and $\vec{b}$ are not parallel.

By Proposition APP1.3 and Proposition APP1.5, combined with Theorem 9.5 and the fact that the standard matrices for reflection thru an axis (see f and g in Definitions APP1.2) both have determinants (-1), we may assume that I is the origin and $\vec{a} \equiv < a_1, a_2 >$ is in the first quadrant; that is, both a_1 and a_2 are nonnegative.

First, we will show this proposition when $\vec{b}$ is horizontal; that is, $\vec{b} = < b_1, 0 >$, for some nonzero real b_1.

Note that here $|\det \begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix}| = |b_1a_2| =$ (base)(height).

There are three cases: (1) $b_1 > a_1$; (2) $0 < b_1 \leq a_1$; (3) $b_1 < 0$. See DRAWINGS 9.1 at the end of this chapter.

Using Corollary APP1.7:

in case (1), the area of our triangle is $\frac{1}{2}a_2a_1 + \frac{1}{2}a_2(b_1 - a_1) = \frac{1}{2}a_2b_1 = \frac{1}{2}|a_2b_1|$;

in case (2), the area of our triangle is $\frac{1}{2}a_2a_1 - \frac{1}{2}a_2(a_1 - b_1) = \frac{1}{2}a_2b_1 = \frac{1}{2}|a_2b_1|$;

in case (3), the area of our triangle is $\frac{1}{2}a_2(a_1 - b_1) - \frac{1}{2}a_2a_1 = -\frac{1}{2}a_2b_1 = \frac{1}{2}|a_2b_1|$.

Now we remove all restrictions on $\vec{b}$, except that we will separately consider $b_2 = a_2$ and $b_2 \neq a_2$.

If $b_2 = a_2$, then, breaking into the same three cases presented just before DRAWINGS 9.1 at the end of this chapter, and using Corollary APP1.7 again and Definitions 2.4(ii) (see DRAWINGS 9.2 at the end of this chapter):

in case (1), the area of our triangle is

$$a_2b_1 - \frac{1}{2}a_1a_2 - \frac{1}{2}a_2b_1 = \frac{1}{2}a_2(b_1 - a_1) = |\frac{1}{2}\det\begin{bmatrix} a_1 & b_1 \\ a_2 & a_2 \end{bmatrix}| = \frac{1}{2}|\det\begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix}|;$$

in case (2), the area of our triangle is

$$a_2a_1 - \frac{1}{2}a_2b_1 - \frac{1}{2}a_2a_1 = \frac{1}{2}a_2(a_1 - b_1) = |\frac{1}{2}\det\begin{bmatrix} a_1 & b_1 \\ a_2 & a_2 \end{bmatrix}| = \frac{1}{2}|\det\begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix}|;$$

in case (3), the area of our triangle is

$$a_2(a_1 - b_1) - \frac{1}{2}|b_1|a_2 - \frac{1}{2}a_1a_2 = a_2a_1 - a_2b_1 + \frac{1}{2}b_1a_2 - \frac{1}{2}a_1a_2 = |\frac{1}{2}\det\begin{bmatrix} a_1 & b_1 \\ a_2 & a_2 \end{bmatrix}| = \frac{1}{2}|\det\begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix}|.$$

See DRAWING 9.2 at the end of this chapter.

If $b_2 \neq a_2$, let c be the x intercept of the line thru (a_1, a_2) and (b_1, b_2) and let $\vec{c} \equiv < c, 0 >$.

By the first part of our proof, after a reflection thru the x axis if necessary (see Proposition APP1.5) the area of the triangle formed by $\vec{a}$ and $\vec{c}$, call it $A_{\vec{a}}$, is $\frac{1}{2}|c|a_2$ and the area of the triangle formed by $\vec{b}$ and $\vec{c}$, call it $A_{\vec{b}}$, is $\frac{1}{2}|cb_2|$. See DRAWINGS 9.3 at the end of this chapter.

As drawn in DRAWINGS 9.4 at the end of this chapter, we may write the area of our triangle as either a sum (if $b_2 < 0$) of $A_{\vec{a}}$ and $A_{\vec{b}}$, or (if $b_2 > 0$) the absolute value of the difference between $A_{\vec{a}}$ and $A_{\vec{b}}$.

We claim the area of the triangle formed by $\vec{a}$ and $\vec{b}$ is $\frac{1}{2}|c(b_2 - a_2)|$. We will show this claim first for b_2 positive, then for b_2 negative ($b_2 = 0$ was covered at the beginning of the proof).

If $b_2 > 0$, then our area equals

$$\frac{1}{2}\left[|cb_2| - |ca_2|\right] = \frac{1}{2}\left[|c|(b_2 - a_2)\right] = \frac{1}{2}|c(b_2 - a_2)| \quad \text{if} \quad b_2 > a_2$$

or

$$\frac{1}{2}\left[|ca_2| - |cb_2|\right] = \frac{1}{2}\left[|c|(a_2 - b_2)\right] = \frac{1}{2}|c(b_2 - a_2)| \quad \text{if} \quad a_2 > b_2.$$

If $b_2 < 0$, then our area is

$$\frac{1}{2}\left(|ca_2| + |cb_2|\right) = \frac{1}{2}\left(|c|a_2 - |c|b_2\right) = \frac{1}{2}|c|(a_2 - b_2) = \frac{1}{2}|c(b_2 - a_2)|.$$

See DRAWING 9.4 at the end of this chapter.

This proves our claim about the area of the triangle formed by $\vec{a}$ and $\vec{b}$, in terms of c. To see that it is the area given in the statement of the proposition, we must solve for c:

The line thru (a_1, a_2) and (b_1, b_2) is

$$y = a_2 + \left(\frac{(b_2 - a_2)}{(b_1 - a_1)}(x - a_1)\right),$$

so

$$0 = a_2 + \left(\frac{(b_2 - a_2)}{(b_1 - a_1)}(c - a_1)\right),$$

hence

$$c = a_1 - a_2\left(\frac{(b_1 - a_1)}{(b_2 - a_2)}\right),$$

hence

$$c(b_2 - a_2) = a_1(b_2 - a_2) - a_2(b_1 - a_1) = a_1 b_2 - a_2 b_1 = \det \begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix},$$

concluding the proof. □

Proposition 9.7. If I is a point and $\vec{a}$ and $\vec{b}$ are vectors, then the area of the parallelogram with vertices $I, I + \vec{a}, I + \vec{b}, I + \vec{a} + \vec{b}$ is $|\det \begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix}|$.

Proof: This follows from Proposition 9.6 and Corollary APP1.6. □

Theorem 9.8. For Ω (the interior of) a polygon, the area of $f_A(\Omega) \equiv \{f_A(J) \,|\, J \text{ is in } \Omega\}$ equals

$$|\det(A)| \,(\text{area of } \Omega)\,.$$

Proof: By Propositions 9.6 and 9.7, since any polygon is a union of triangles (asserted in Definitions 2.3), we may assume Ω is (the interior of) a parallelogram. Let $I, \vec{a}, \vec{b}$ be as in Corollary 3.4; that is,

$$\Omega = \{I + s\vec{a} + t\vec{b}\,|, 0 \leq s, t \leq 1\}.$$

Since $f_A(I + s\vec{a} + t\vec{b}) = f_A(I) + s f_A(\vec{a}) + t f_A(\vec{b})$, $f_A(\Omega)$ is a parallelogram formed by $f_A(\vec{a})$ and $f_A(\vec{b})$, thus, by Proposition 9.7 and Theorem 9.5, the area of $f_A(\Omega)$ is

$$|\det \begin{bmatrix} f_A(\vec{a}) & f_A(\vec{b}) \end{bmatrix}| = |\det \left(A \begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix}\right)| = |\,(\det(A)) \left(\det \left(\begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix}\right)\right)|$$

$$= |\det(A)| \; | \left(\det \left(\begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix}\right)\right) | = |\det(A)| \,(\text{area of } \Omega)\,,$$

by Proposition 9.7. □

This gives a geometric picture of determinant: $|\det(A)|$ is a **magnification factor** for multiplication by the matrix A.

Of particular interest is to have the magnification factor equal to one; this is equivalent to preservation of area.

Corollary 9.9. If A is a 2×2 matrix, then $|\det(A)| = 1$ if and only if multiplication by A preserves area; that is, for Ω a polygon, $f_A(\Omega) \equiv \{f_A(J) \,|\, J \text{ is in } \Omega\}$ has the same area as Ω.

Definition 9.10. A matrix is **orthogonal** if its columns are orthogonal unit vectors (see Definitions 2.4 and Definition 4.3). That is, $A \equiv \begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix}$ is orthogonal if $\vec{a}$ and $\vec{b}$ are orthogonal unit vectors.

Theorem 9.11. If A is a 2×2 matrix, then A is orthogonal if and only if multiplication by A preserves length and angle measure; that is, for any vectors $\vec{x}, \vec{y}$, the measure of the angle between $f_A(\vec{x})$ and $f_A(\vec{y})$ equals the measure of the angle between $\vec{x}$ and $\vec{y}$ and the length of $f_A(\vec{x})$ equals the length of $\vec{x}$.

Proof: By 4.5(e) and Theorem 6.9, preservation of length and angle measure is equivalent to

$$(f_A(\vec{x})) \cdot (f_A(\vec{y})) = \vec{x} \cdot \vec{y},$$

for all $\vec{x}, \vec{y}$ in $\mathbf{R}^2$.

Denote $A \equiv \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \equiv \begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix}$. Then, for any $\vec{x}, \vec{y}$ in $\mathbf{R}^2$,

$$f_A(\vec{x}) =< a_1 x_1 + b_1 x_2, a_2 x_1 + b_2 x_2 >, \;\; f_A(\vec{y}) =< a_1 y_1 + b_1 y_2, a_2 y_1 + b_2 y_2 >,$$

thus

$$\begin{aligned} (f_A(\vec{x})) \cdot (f_A(\vec{y})) &= (a_1 x_1 + b_1 x_2)\,(a_1 y_1 + b_1 y_2) + (a_2 x_1 + b_2 x_2)\,(a_2 y_1 + b_2 y_2) \\ &= a_1^2 x_1 y_1 + a_1 b_1 x_1 y_2 + b_1 a_1 x_2 y_1 + b_1^2 x_2 y_2 + a_2^2 x_1 y_1 + a_2 b_2 x_1 y_2 + b_2 a_2 x_2 y_1 + b_2^2 x_2 y_2 \\ &= (a_1^2 + a_2^2) x_1 y_1 + (b_1^2 + b_2^2) x_2 y_2 + (a_1 b_1 + a_2 b_2) x_1 y_2 + (b_1 a_1 + b_2 a_2) x_2 y_1 \end{aligned}$$

$$= \|\vec{a}\|^2 x_1 y_1 + \|\vec{b}\|^2 x_2 y_2 + (\vec{a} \cdot \vec{b}) x_1 y_2 + (\vec{b} \cdot \vec{a}) x_2 y_1,$$

which equals $x_1 y_1 + x_2 y_2 \equiv \vec{x} \cdot \vec{y}$ for all $\vec{x}, \vec{y}$ in $\mathbf{R}^2$ if and only if

$$\|\vec{a}\| = 1 = \|\vec{b}\| \text{ and } \vec{a} \cdot \vec{b} = 0.$$

This is precisely the definition of A being orthogonal (see Definitions 9.10 and 4.3). $\square$

Corollary 9.12. If multiplication by a matrix A preserves length and angle measure, then it also preserves area.

Proof: By Theorem 9.11, the first column of A is a unit vector, thus (see Definitions 6.1) it equals $\begin{bmatrix} \cos\psi \\ \sin\psi \end{bmatrix}$, for some ψ between 0 and 2π. Also by Theorem 9.11, the second column is a unit vector orthogonal to the first column, thus it equals $\begin{bmatrix} -\sin\psi \\ \cos\psi \end{bmatrix}$, or $\begin{bmatrix} \sin\psi \\ -\cos\psi \end{bmatrix}$ (see DRAWING 9.5 at the end of this chapter);

that is, A equals either

$$\begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix} \text{ or } \begin{bmatrix} \cos\psi & \sin\psi \\ \sin\psi & -\cos\psi \end{bmatrix};$$

in either case, $|\det A| = 1$, so by Corollary 9.9, multiplication by A preserves area. $\square$

Remark 9.13. Some comments here about choosing a standard matrix (see Definition 9.3) for a function will be helpful for the proof of Theorem 8.3. Note that

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} \text{ and } \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}.$$

This tells us that, given a function $f : \mathbf{R}^2 \to \mathbf{R}^2$, a matrix A such that $f = f_A$ must have its first column equal to $f((1,0))$ and its second column equal to $f((0,1))$, where $f((1,0))$ and $f((0,1))$ are arranged as columns.

9.14. Proof of Theorem 8.3: The preservation of length, angle measure, and area under translation may be found in Propositions APP1.3 and APP2.3(a); recall (Definitions 2.10) that angle measure is a certain length

This means we may assume, for the remainder of the proof, that

(i) all circles involved in rotation (Definition 8.1(b)) are centered at the origin; and

(ii) all lines involved in reflection (Definition 8.1(c)) go thru the origin.

This allows us to represent Definitions 8.1(b), (c), and (d) with matrices, as in Definition 9.3; that is, each of (b), (c), and (d) will be written as f_A, with standard matrix A chosen, following Remark 9.13, in the following ways.

CLAIM 1. The motion in (b) is f_A, for $A \equiv \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

CLAIM 2. There's a real number ϕ so that the motion in (c) is f_A, for $A \equiv \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix}$; $< \cos\phi, \sin\phi >$ is then a direction vector for the line ℓ in (c).

CLAIM 3. The motion in (d) is f_A, with $A \equiv R \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Most of the proof of Theorem 8.3 involves proving these three claims about Definitions 8.1(b), (c), and (d).

Proof of CLAIM 1. Write an arbitrary point (x, y) in $\mathbf{R}^2$, via Definitions 6.1 or Definition 6.2, in the vector equivalent of polar form (Definition 1.15; see Lemma 2.8 and Glib Equivalences 1.16) $(r\cos\theta_0, r\sin\theta_0)$, for some real θ_0, with $r \equiv \| < x, y > \|$. Multiply

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r\cos\theta_0 \\ r\sin\theta_0 \end{bmatrix} = \begin{bmatrix} r(\cos\theta\cos\theta_0 - \sin\theta\sin\theta_0) \\ r(\sin\theta\cos\theta_0 + \cos\theta\sin\theta_0) \end{bmatrix} = \begin{bmatrix} r\cos(\theta+\theta_0) \\ r\sin(\theta+\theta_0) \end{bmatrix},$$

by Proposition 6.4(i) and (ii); this is $(r\cos\theta_0, r\sin\theta_0)$ rotated θ radians counterclockwise (see the complex form of Definitions 8.1(b)), as desired.

Proof of CLAIM 2. A "motion," in the sense of Definitions 8.1, that does *not* preserve length or area is the *projection* of Definition 4.6.

Our first step will be to describe reflection in terms of projection. Let ℓ be the line of Definitions 8.1(c), assumed (see beginning of proof of Theorem 8.3) to be going thru the origin.

Define $P : \mathbf{R}^2 \to \mathbf{R}^2$ by, for any I in $\mathbf{R}^2$,

$$I'' \equiv P(I) \equiv \mathrm{proj}_\ell(I)$$

(see Definition 4.6 and DRAWING 9.6 at the end of this chapter), the projection of I onto the line ℓ.

Comparing DRAWING 9.6 at the end of this chapter with DRAWING 8.3 at the end of Chapter VIII, we see that

$$R(I) - P(I) = \overrightarrow{II''} = \overrightarrow{I''I'} = P(I) - I;$$

that is,

$$R - P = P - Id,$$

where Id is the **identity** map $Id(I) \equiv I$, so that

$$R = 2P - Id \quad (*).$$

Since Id has standard matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the standard matrix for P will lead to the standard matrix for R.

Let $\vec{v}$ be a direction vector for ℓ of norm one. By Lemma 2.8 and Definitions 6.1, this means there is a real number ϕ so that

$$\vec{v} = < \cos\phi, \sin\phi > .$$

By Corollary 4.12, for any real x, y,

$$P((x, y)) = (< x, y > \cdot < \cos\phi, \sin\phi >) < \cos\phi, \sin\phi > = < (\cos\phi)^2 x + (\cos\phi\sin\phi)y, (\cos\phi)(\sin\phi)x + (\sin\phi)^2 y >$$

$$= \begin{bmatrix} (\cos\phi)^2 & (\cos\phi)(\sin\phi) \\ (\cos\phi)(\sin\phi) & (\sin\phi)^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix};$$

that is,

$$\begin{bmatrix} (\cos\phi)^2 & (\cos\phi)(\sin\phi) \\ (\cos\phi)(\sin\phi) & (\sin\phi)^2 \end{bmatrix}$$

is the standard matrix for P.

By (*) above, 6.3(v), and 6.4(i) and (ii), this implies that the standard matrix for R is

$$A = 2\begin{bmatrix} (\cos\phi)^2 & (\cos\phi)(\sin\phi) \\ (\cos\phi)(\sin\phi) & (\sin\phi)^2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2(\cos\phi)^2 - 1 & 2(\cos\phi)(\sin\phi) \\ 2(\cos\phi)(\sin\phi) & 2(\sin\phi)^2 - 1 \end{bmatrix}$$

$$= \begin{bmatrix} (\cos\phi)^2 - (\sin\phi)^2 & 2(\cos\phi)(\sin\phi) \\ 2(\cos\phi)(\sin\phi) & (\sin\phi)^2 - (\cos\phi)^2 \end{bmatrix} = \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix}$$

Proof of CLAIM 3. This is immediate:

$$R\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Rx \\ Ry \end{bmatrix},$$

for any real x, y.

The proof of Theorem 8.3(a) now follows from Corollary 9.12 and Theorem 9.11, applied to

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix}$$

while the proof of Theorem 8.3(b) follows from Theorem 9.8 (for area) and the fact that

$$< Rx_1, Ry_1 > \cdot < Rx_2, Ry_2 >= R^2 \left(< x_1, y_1 > \cdot < x_2, y_2 >\right),$$

for any real x_1, x_2, y_1, y_2 (see 4.5(e) and Theorem 6.9). □

Remark 9.15. The standard matrices for reflection and rotation are surprisingly similar. The standard matrix for reflection is the same as the standard matrix for counterclockwise rotation by 2ϕ, except that the second column is multiplied by (-1).

As we mentioned in the proof of Corollary 9.12, if A is the standard matrix for a function from $\mathbf{R}^2$ to $\mathbf{R}^2$ that preserves length and angle measure, then A equals, for some ψ, either

$$\begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix},$$

representing counterclockwise rotation by ψ, or

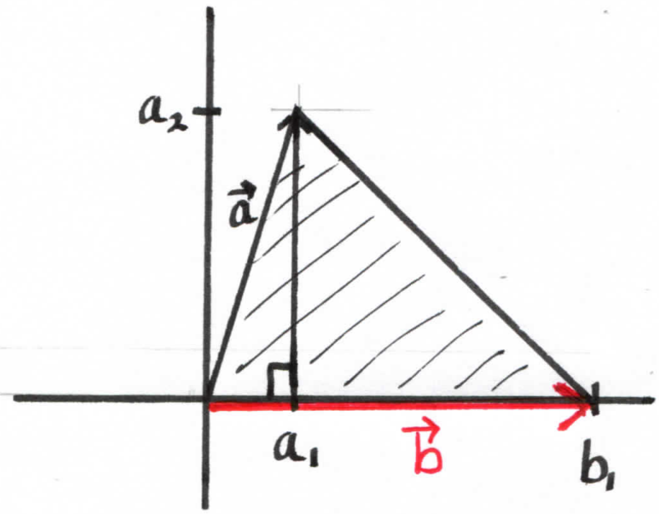
$$\begin{bmatrix} \cos\psi & \sin\psi \\ \sin\psi & -\cos\psi \end{bmatrix},$$

representing reflection thru the line thru the origin that makes an angle of measure $\frac{1}{2}\psi$ with the x axis.

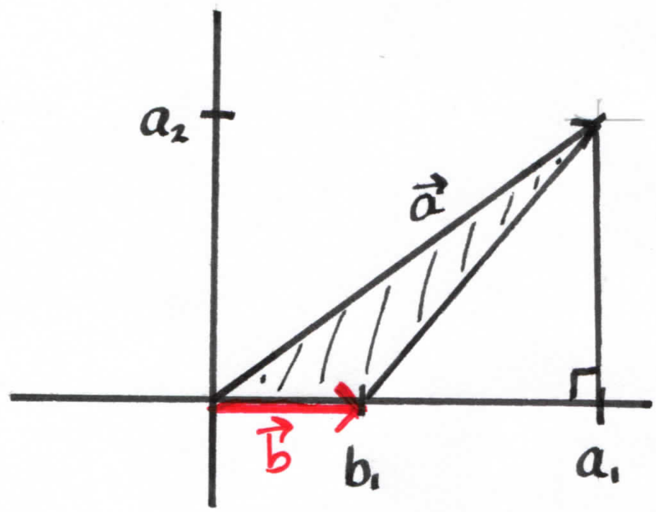
It is surprising that such a trivial-appearing algebraic change (multiplying the second column by (-1)) makes such a profound geometric change (rotation to reflection).

DRAWINGS 9.1

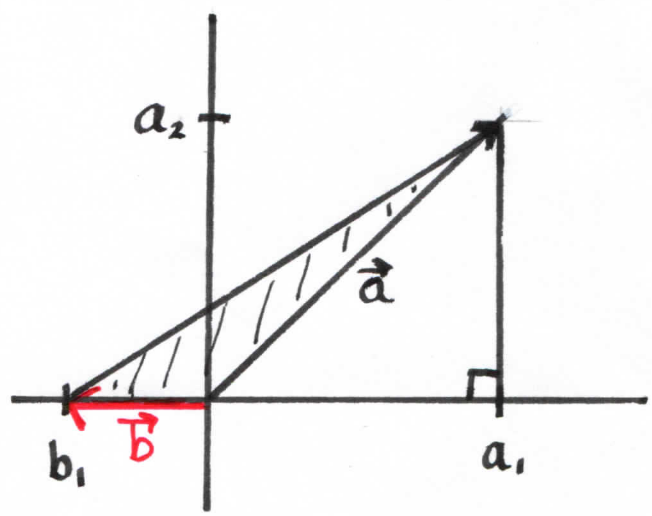
(1)



(2)

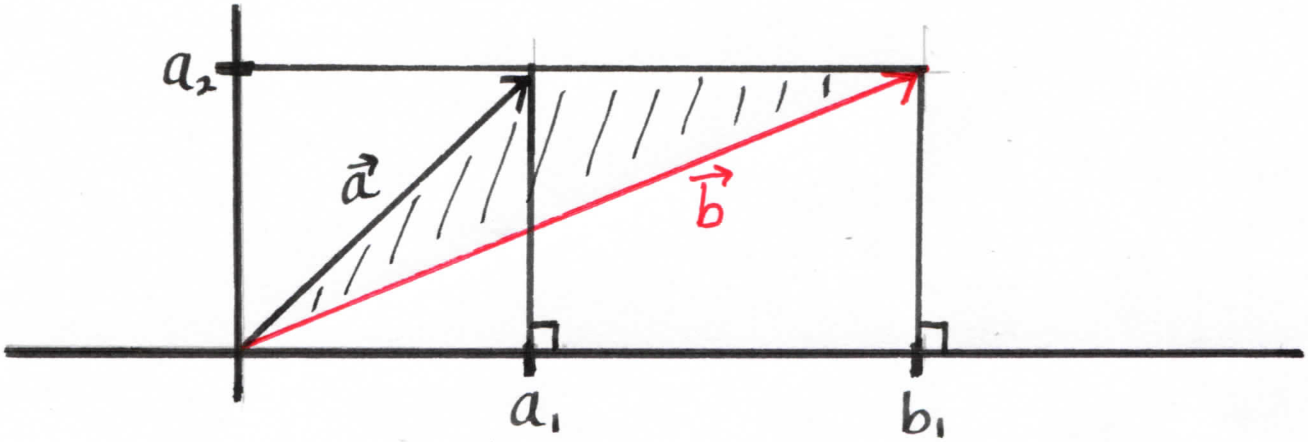


(3)

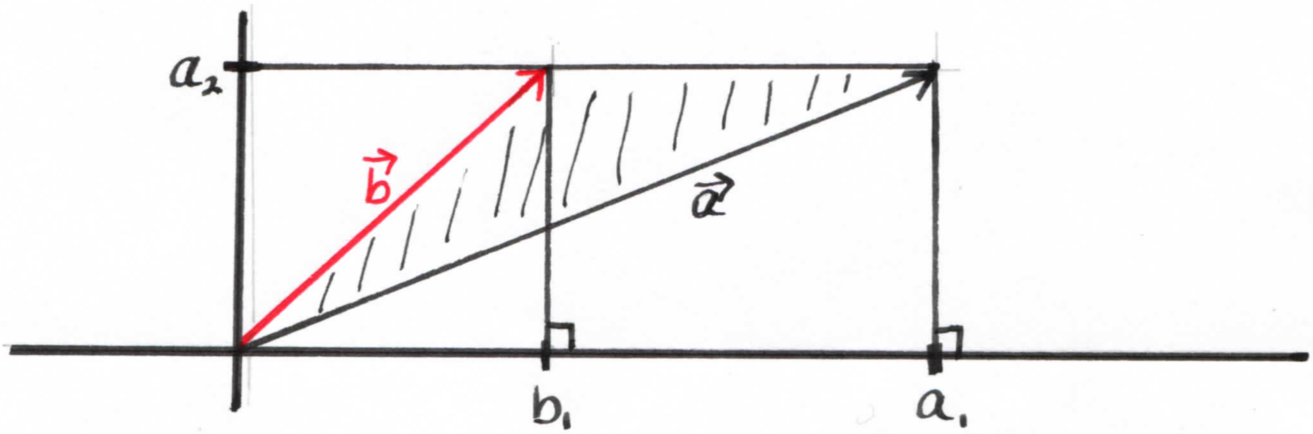


DRAWINGS 9.2

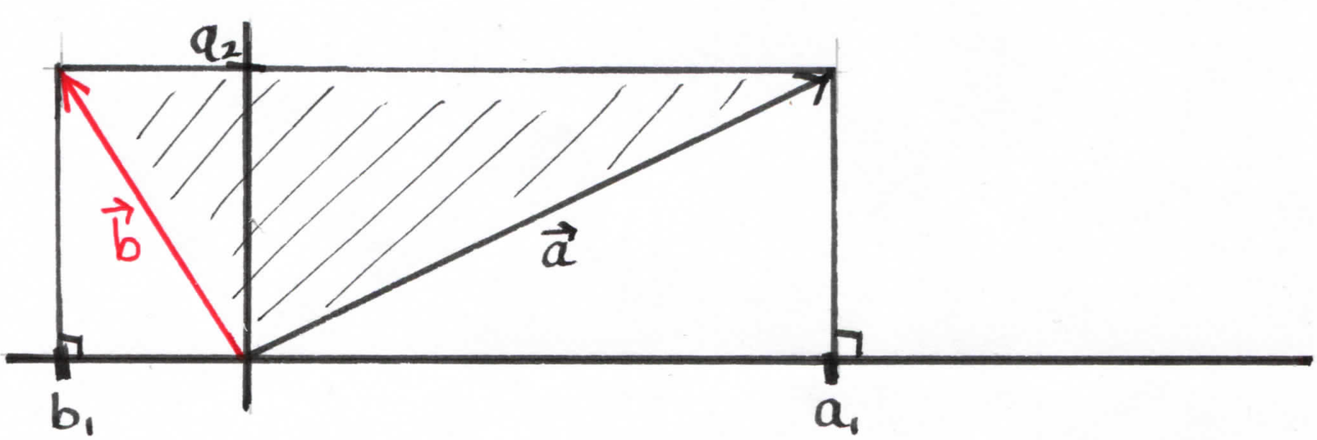
(1)



(2)

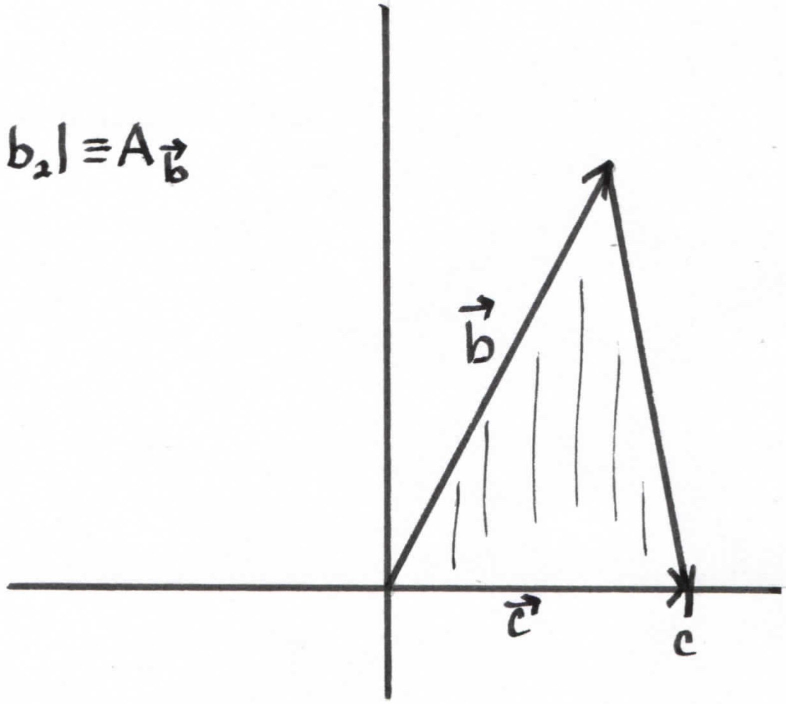


(3)

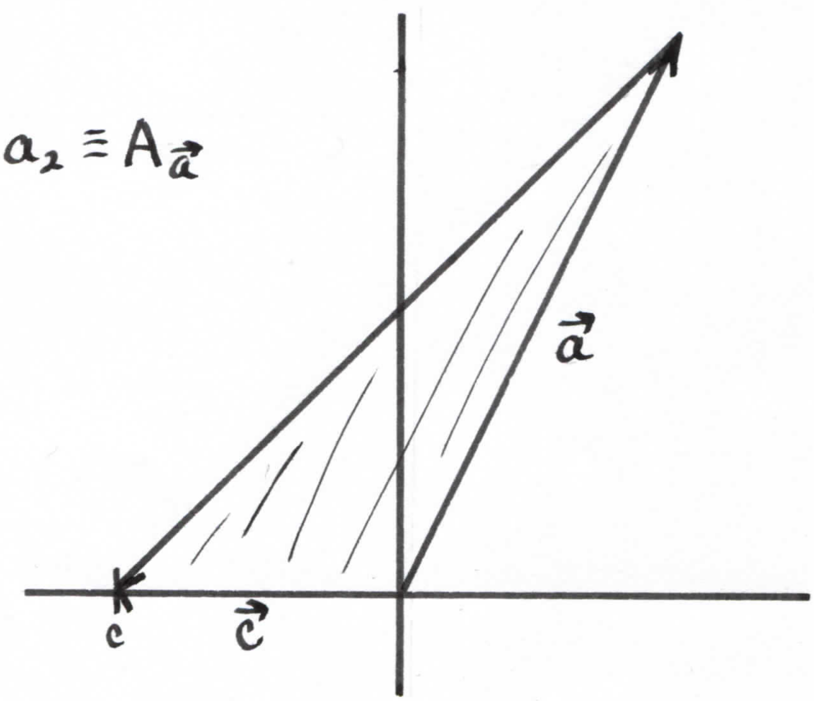


DRAWINGS 9.3

area $= \frac{1}{2}|c\,b_2| \equiv A_{\vec{b}}$

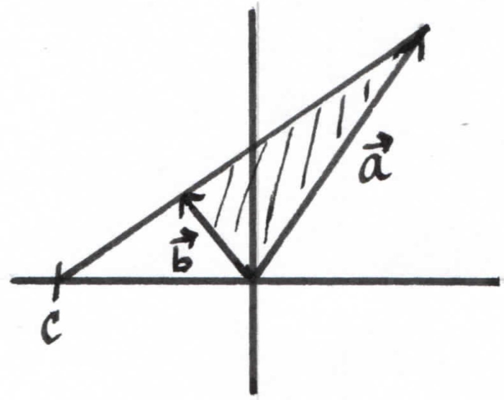


area $= \frac{1}{2}|c|a_2 \equiv A_{\vec{a}}$

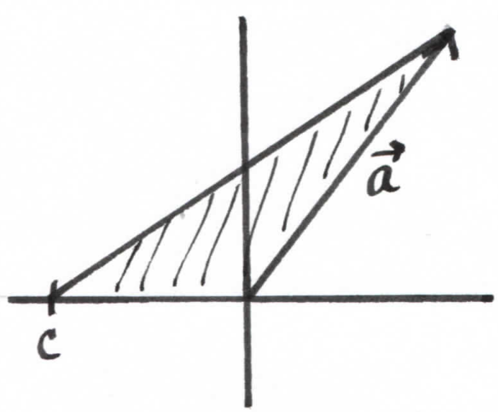


DRAWINGS 9.4

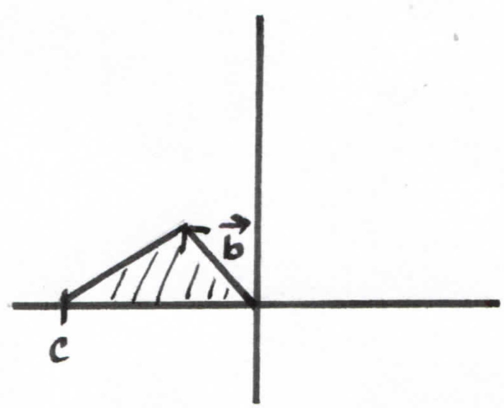
$b_2 > 0$



=

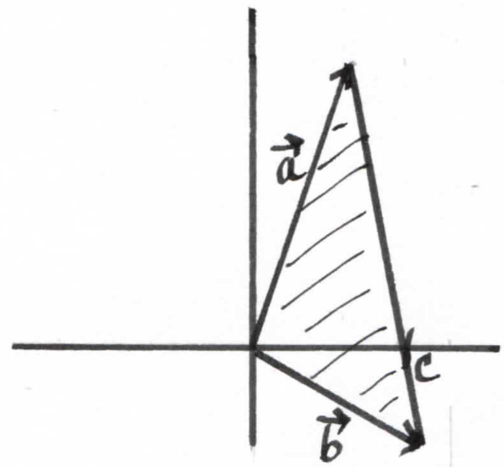


(minus)

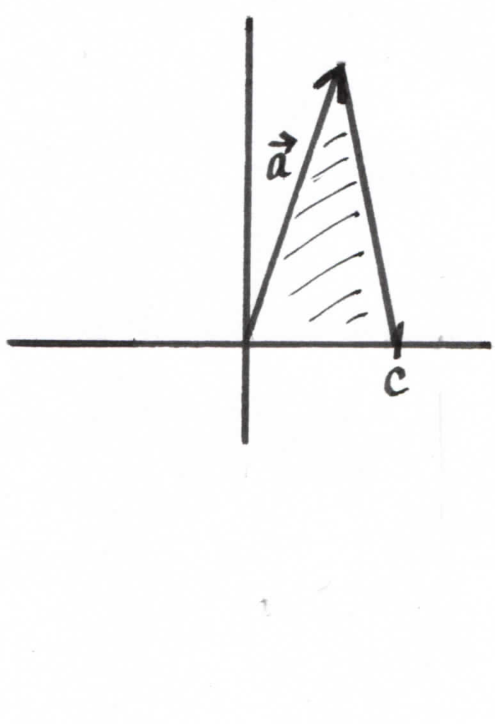


DRAWINGS 9.4 (continued)

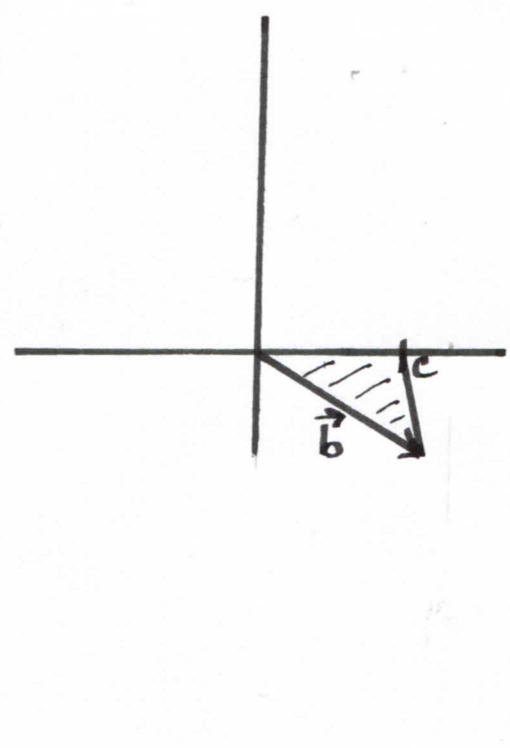
$b_2 < 0$



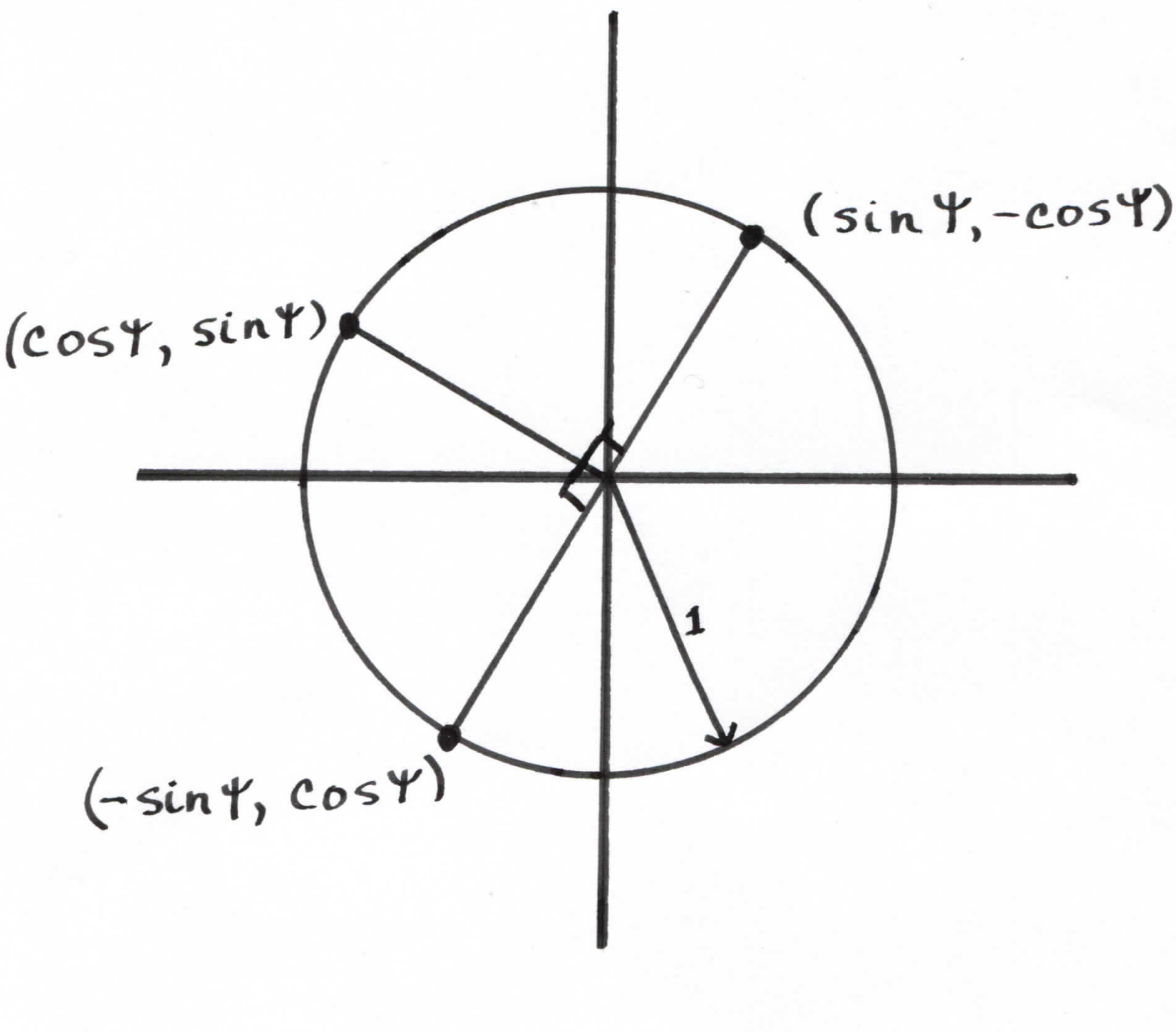
$=$



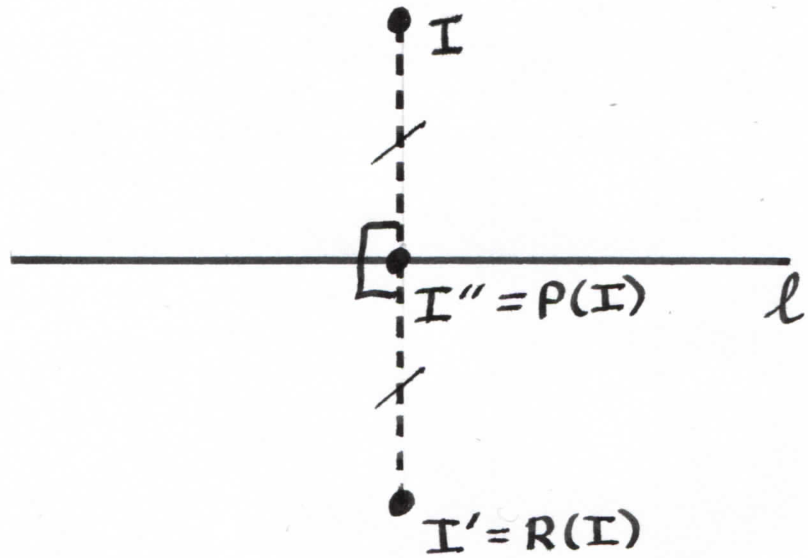
(plus)

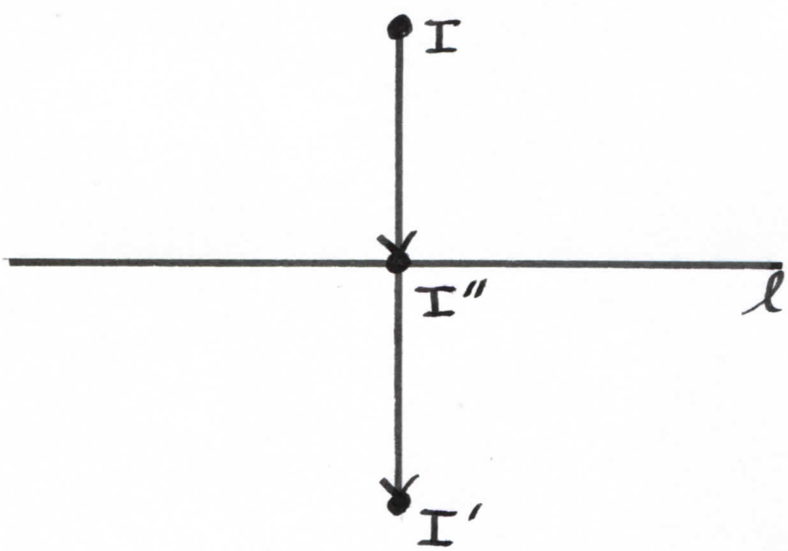


DRAWING 9.5



DRAWING 9.6





$$\overrightarrow{II''} = \overrightarrow{I''I'}$$

CHAPTER X: Congruence and Similarity of Triangles.

We would like simple (meaning requiring as little information as possible) ways of determining when two triangles are congruent or similar (Definitions 8.4). Theorem 10.1 is sufficient for our applications in Chapters XI, XII, and XIII. Theorems 10.4 and 10.6 address natural questions that arise from Theorem 10.1 (see 10.3 and 10.5), while Theorem 10.7 discusses information (two sides and an angle not between the two sides) with more uncertain conclusions; in general, there might be no such triangle, precisely one (up to congruence) or two (up to congruence) possible triangles satisfying the information given.

Here (Theorem 10.1) are the traditional sufficient conditions for congruence, along with their traditional acronyms (**SAS, SSS, and AAS**). Those acronyms imply the *order* of angles or sides: SAS means the angle with specified measure is between the two sides whose lengths are specified, while AAS means the side whose length is specified might not be between the angles whose lengths are specified.

Theorem 10.1. Suppose T_1 and T_2 are triangles, T_1 has sides S_1, S_2, and S_3, T_2 has sides S_4, S_5, and S_6, and, for $j = 1, 2, 3, 4, 5, 6$, θ_j is the measure of the angle opposite S_j. See DRAWING 10.1 at the end of this chapter.

(a) Any one of the following conditions (SAS, SSS, AAS) implies that T_1 and T_2 are congruent.

SAS. $\|S_1\| = \|S_4\|, \|S_2\| = \|S_5\|$ and $\theta_3 = \theta_6$. See DRAWING 10.2 at the end of this chapter.

Informally, agreement on two sides and the angle between the two sides implies congruence.

SSS. $\|S_1\| = \|S_4\|, \|S_2\| = \|S_5\|$, and $\|S_3\| = \|S_6\|$. See DRAWING 10.3 at the end of this chapter.

Informally, agreement on all sides implies congruence.

AAS. $\|S_1\| = \|S_4\|, \theta_1 = \theta_4$, and $\theta_2 = \theta_5$. See DRAWING 10.4 at the end of this chapter.

Informally, agreement on two angles and a side implies congruence; note that AAS follows from ASA, by Proposition 3.9.

(b) **AAA or AA.** $\theta_1 = \theta_4, \theta_2 = \theta_5$, and $\theta_3 = \theta_6$ (the last equality follows automatically from the first two, by Proposition 3.9) implies that T_1 and T_2 are similar. See DRAWING 10.5 at the end of this chapter.

Informally, agreement of all angles implies similarity.

Proof: SAS: Suppose two triangles T_1 and T_2 each have a side of length a and a side of length b, with the measure of the angle between them equal to ψ; that is, $\|S_1\| = a = \|S_4\|, \|S_2\| = b = \|S_5\|, \theta_3 = \theta_6 = \psi$ in DRAWING 10.1 at the end of this chapter.

We claim that both triangles are congruent to the triangle, call it T_3, with vertices $(0,0), (a,0)$, and $b(\cos\psi, \sin\psi)$. We will describe this with T_1; the same argument applies to T_2. See DRAWINGS 10.6 at the end of this chapter.

If ψ is the measure of the clockwise angle from S_2 to S_1, then a translation and rotation makes T_1 congruent to T_3. See DRAWINGS 10.7 at the end of this chapter.

If ψ is the measure of the counterclockwise angle from S_2 to S_1, then a reflection (e.g., through the side S_1) makes T_1 congruent to the triangle at the beginning of DRAWINGS 10.7, which we have already shown is congruent to T_3, thus T_1 is congruent to T_3. See DRAWINGS 10.8 at the end of this chapter.

Either way, this proves our claim. Since T_1 and T_2 are each congruent to T_3, it follows that T_1 is congruent to T_2, as desired. $\square$

SSS: Denoting $c \equiv \|S_3\|, a \equiv \|S_1\|, b \equiv \|S_2\|$, by the Law of Cosines,

$$c^2 = a^2 + b^2 - 2ab\cos\theta_3,$$

thus

$$\theta_3 = \cos^{-1}\left[\frac{a^2 + b^2 - c^2}{2ab}\right] = \theta_6,$$

and we may apply SAS. □

AAS: Since the sum of the measures of the angles in each triangle is π, we also have

$$\theta_3 = \theta_6,$$

thus, by the Law of Sines,

$$\frac{\|S_j\|}{\sin\theta_j}$$

is constant, for $j = 1, 2, 3, 4, 5, 6$, so that we may apply SSS.

AAA: Let $R \equiv \frac{\|S_1\|}{\|S_4\|}$, and let T_3 be T_2 magnified by R (see Definitions 8.1(d)) By AAS, T_3 is congruent to T_1, thus by Theorem 8.3(b) and Definitions 8.4, T_2 is similar to T_1. □

Example 10.2. Here is a different proof of Theorem 7.10(d) → (a), (b), and (c). See DRAWINGS 10.9 at the end of this chapter, where the hypothesis of (d) implies the agreement on a side and two angles of triangles PSR and SQR, so that AAS implies the congruence of triangles PSR and SQR, which implies all of (a)–(d) in Theorem 7.10. □

Discussion 10.3. Another way to view SAS, SSS, and AAS is in terms of what information is sufficient to uniquely determine a triangle, up to congruence. Most of the following is a restatement of Theorem 10.1. We will put off until later (Theorem 10.7) the most interesting case, not mentioned in Theorem 10.1, of specifying two sides and the measure of an angle *not* between the two sides.

Theorem 10.4. Each of the following sets of conditions (with no additional conditions) is sufficient to uniquely specify a triangle (if it exists; see Theorem 10.6); that is, specify all measures of angles and lengths of sides, up to congruence.

(a) SAS; that is, the lengths of two sides and the measure of the interior angle between them, is specified.

(b) SSS; that is, the length of each side is specified.

(c) AAS; that is, the length of a side and the measures of two angles are specified.

(d) Specify numbers a, b, c such that the triangle is formed by vectors $\vec{a}, \vec{b}$ such that $\|\vec{a}\| = a, \|\vec{b}\| = b$ and $\vec{a} \cdot \vec{b} = c$, as in DRAWING 2.8(a) at the end of Chapter II.

(e) Specify vectors $\vec{a}, \vec{b}$, such that the triangle is formed by $\vec{a}$ and $\vec{b}$, as in DRAWING 2.8(a) at the end of Chapter II.

Proof: The constructions for (a)–(c) are contained in the proof of Theorem 10.1(a).

(d) is equivalent to (a), by Theorem 6.9.

The specifications of (e) clearly imply those of (d), which we have already shown specifies a unique triangle. □

More Discussion 10.5. Another concern, that arguably should precede uniqueness of triangles as in Discussion 10.3 and Theorem 10.4, is the existence of a triangle having specified sides and angles.

At first glance, there appear to be at least six numbers involved in describing a triangle: three (measures of) angles and three (lengths of) sides. But Theorem 10.4 implies that these six numbers cannot be chosen independently; in Theorem 10.4(a)–(d), choosing only three of those six numbers uniquely determines the remaining numbers. In fact, even those three numbers in SAS, SSS, and AAS cannot be chosen completely independently.

Theorem 10.6. Each of (a)–(e) below contains necessary and sufficient restrictions on the corresponding specification in Theorem 10.4, so that the specified triangle exists.

(a) All specified lengths are positive and, denoting by θ the specified measure of the specified angle, $0 < \theta < \pi$.

(b) All specified lengths are positive and each side has length less than the sum of the lengths of the other sides.

(c) The specified length and both specified angle measures are positive and the sum of the two specified angle measures is less than π.

(d) The fraction $|\frac{c}{ab}|$ is less than 1 and both a and b are greater than zero.

(e) The vectors $\vec{a}$ and $\vec{b}$ are not parallel.

Proof: (b) is certainly geometrically believable. To prove it is necessary and sufficient algebraically, let s_1, s_2, s_3 be the lengths of the sides of the desired triangle, and for $j = 1, 2, 3$, let θ_j be the measure of the angle opposite the side of length s_j. Note that we get measures of angles from lengths via the Law of Cosines; e.g., to get θ_1, use

$$s_1^2 = s_2^2 + s_3^2 - 2s_2 s_3 \cos\theta_1,$$

so that, since $|\cos\theta_1| < 1$, θ_1 exists if and only if

$$-1 < \frac{s_2^2 + s_3^2 - s_1^2}{2s_2 s_3} < 1 \iff -2s_2s_3 < s_1^2 - (s_2^2 + s_3^2) < 2s_2s_3 \iff (s_2 - s_3)^2 < s_1^2 < (s_2 + s_3)^2 \iff$$

$$|s_2 - s_3| < s_1 < (s_2 + s_3) \iff -s_1 < s_2 - s_3 < s_1 \text{ and } s_1 < s_2 + s_3 \iff$$

$$s_2 < s_1 + s_3,\ s_3 < s_1 + s_2 \ \text{ and } s_1 < s_2 + s_3;$$

that is, θ_1 exists if and only if each side has length less than the sum of the lengths of the other sides. An identical argument shows that same result for θ_2 and θ_3.

For the sufficiency of (a), (c), (d), and (e), we will construct vectors $\vec{a}, \vec{b}$ as in DRAWING 2.8(a) at the end of Chapter II, with $\vec{a} \equiv < a, 0 >$ on the positive x axis, $\vec{b}$ in the upper halfplane $y > 0$, so that $\vec{a}, \vec{b}$, and $(\vec{a} - \vec{b})$ will form the sides of our desired triangles; see DRAWINGS 10.10 at the end of this chapter.

(e) The vectors $\vec{a}, \vec{b}$ and $(\vec{a} - \vec{b})$ will enclose an area if and only if $\vec{a}$ and $\vec{b}$ are not parallel (see DRAWING 2.8(a) at the end of Chapter II).

(d) and (a). Let

$$\vec{a} \equiv < a, 0 >,\ \ \vec{b} \equiv b < \cos\theta, \sin\theta >,$$

where, for (a), lengths a and b and angle measure θ are specified, and, for (d), $\theta \equiv \cos^{-1}(\frac{c}{ab})$ (see Theorem 6.9).

(c) Let s_3 be the specified length of a side. Since the sum of measures of the angles in the triangle is π, our condition guarantees that all measures of angles are specified; denote said measures by $\theta_j, j = 1, 2, 3$, where θ_3 is the measure of the angle opposite the side of length s_3. Worrying about the Law of Sines tells us to define

$$s_j \equiv (\sin\theta_j)\left(\frac{s_3}{\sin\theta_3}\right),$$

for $j = 1, 2$.

In DRAWINGS 10.10 at the end of this chapter, let $\vec{a} \equiv < s_3, 0 >, \vec{b} \equiv s_2 < \cos\theta_1, \sin\theta_1 >$, as in (d) and (a) with $a \equiv s_3, \theta \equiv \theta_1$. The extra twist needed here is to show that

$$(\vec{a} - \vec{b}) = s_1 < \cos(\theta_2), -\sin(\theta_2) > \quad (*),$$

so that we will have our specified angle measures; see DRAWINGS 10.11 at the end of this chapter, where we have drawn in both the known angle measure θ_1 and the as yet unknown angle measures ψ_3 and ψ_2.

The calculation for (*) follows:

$$\vec{b} + s_1 < \cos(\theta_2), -\sin(\theta_2) >= s_2 < \cos(\theta_1), \sin(\theta_1) > + \; s_1 < \cos(\theta_2), -\sin(\theta_2) >$$

$$= (\sin\theta_2)\left(\frac{s_3}{\sin\theta_3}\right) < \cos(\theta_1), \sin(\theta_1) > + \; (\sin\theta_1)\left(\frac{s_3}{\sin\theta_3}\right) < \cos(\theta_2), -\sin(\theta_2) >$$

$$= \frac{s_3}{\sin(\theta_3)}\left[< \sin(\theta_2)\cos(\theta_1) + \sin(\theta_1)\cos(\theta_2), 0 >\right] = \frac{s_3}{\sin(\theta_3)} < \sin(\theta_2 + \theta_1), 0 >,$$

by Proposition 6.4(ii); finally, since $\theta_1 + \theta_2 + \theta_3 = \pi$, we have

$$\vec{b} + s_1 < \cos(\theta_2), -\sin(\theta_2) >= \frac{s_3}{\sin(\theta_3)} < \sin(\pi - \theta_3), 0 >=< s_3, 0 >\equiv \vec{a},$$

by 6.3(ii) and (iv), and (*) follows, as in DRAWINGS 10.11 at the end of this chapter.

Now we will use the dot product to determine the measure ψ_3 of the angle opposite $\vec{a}$:

$$\cos(\psi_3) = \frac{-s_2 < \cos(\theta_1), \sin(\theta_1) > \cdot \; s_1 < \cos(\theta_2), -\sin(\theta_2) >}{\|s_2 < \cos(\theta_1), \sin(\theta_1) > \|\|s_1 < \cos(\theta_2), -\sin(\theta_2) > \|} = - < \cos(\theta_1), \sin(\theta_1) > \cdot < \cos(\theta_2), -\sin(\theta_2) >$$

$$= -\left[\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)\right] = -\cos(\theta_1 + \theta_2),$$

by Proposition 6.4(i), so that, since $\theta_1 + \theta_2 + \theta_3 = \pi$,

$$\cos(\psi_3) = -\cos(\pi - \theta_3) = \cos(\theta_3),$$

by 6.3(i) and (iii).

This implies that $\psi_3 = \theta_3$, so that $\psi_2 = \pi - (\theta_3 + \theta_1) = \theta_2$, and we have the desired triangle. See DRAWINGS 10.11 at the end of this chapter.

The necessity of (a), (c), (d), and (e) is quickly addressed. The necessity of (e) was shown simultaneously with the sufficiency. The necessity of (d) is the Cauchy inequality (HWVI.1). The necessity of both (a) and (c) follow from the fact that the sum of measures of angles in a triangle is π. □

Specifying two sides and an angle is much different when the angle is not between the two sides.

Theorem 10.7. (SSA) Suppose, for a potential triangle, the lengths of two sides and the measure of an angle not between the two sides are specified. Denote by θ_1 the specified angle measure, with $0 < \theta_1 < \pi$, by s_1 the length of the side opposite θ_1, and by s_2 the other specified length. See DRAWING 10.12 at the end of this chapter.

Then the following table gives the number of triangles $(0, 1,$ or $2)$, up to congruence, that have the specified lengths and angle measures.

	$\theta_1 < \frac{\pi}{2}$	$\theta_1 \geq \frac{\pi}{2}$
$s_1 < s_2 \sin\theta_1$	0	0
$s_1 = s_2 \sin\theta_1$	1	0
$s_2 > s_1 > s_2 \sin\theta_1$	2	0
$s_1 = s_2$	1	0
$s_1 > s_2$	1	1

Proof: If the desired triangle exists, then s_3, from DRAWING 10.12 at the end of this chapter, must be positive and satisfy the Law of Cosines

$$s_1^2 = s_2^2 + s_3^2 - 2s_2s_3\cos\theta_1;$$

this may be written as a quadratic equation in the variable s_3:

$$s_3^2 - (2s_2\cos\theta_1)s_3 + (s_2^2 - s_1^2) = 0 \qquad (*).$$

Conversely, if there exists a positive real solution of (*), call it s_3, then Theorem 10.6(a) implies the existence of a triangle with sides of lengths s_2 and s_3 and angle of measure θ_1 between the sides of lengths s_2 and s_3; the remaining side would then have length s_1, as in DRAWING 10.12 at the end of this chapter, because the Law of Cosines above, that (*) is equivalent to, specifies s_1 as its unique positive solution

$$s_1 = \sqrt{s_2^2 + s_3^2 - 2s_2s_3\cos\theta_1}.$$

Thus a triangle with the specified lengths and angle measures corresponds to a solution, s_3, of (*) that is real and positive. Throughout the proof, we will refer to such solutions.

By the quadratic formula (Definitions 0.6), solutions of (*) have the form

$$s_3 = \frac{1}{2}\left[2s_2\cos\theta_1 \pm \sqrt{(2s_2\cos\theta_1)^2 - 4(s_2^2 - s_1^2)}\right] = \left[s_2\cos\theta_1 \pm \sqrt{(s_2\cos\theta_1)^2 - (s_2^2 - s_1^2)}\right]$$
$$= \left[s_2\cos\theta_1 \pm \sqrt{s_1^2 - (s_2\sin\theta_1)^2}\right].$$

Denote by s_3^+ and s_3^- the two solutions

$$s_3^+ = \left[s_2\cos\theta_1 + \sqrt{s_1^2 - (s_2\sin\theta_1)^2}\right], \quad s_3^- = \left[s_2\cos\theta_1 - \sqrt{s_1^2 - (s_2\sin\theta_1)^2}\right].$$

We will find it convenient to begin with some preliminary factoids:

(1) s_3^+ and s_3^- are both not real if and only if at least one of s_3^+ and s_3^- is not real if and only if $s_1 < s_2\sin\theta_1$.

(2) $s_3^+ = s_3^-$ if and only if $s_1 = s_2\sin\theta_1$.

(3) $|s_2\cos\theta_1| < \sqrt{s_1^2 - (s_2\sin\theta_1)^2}$ if and only if $s_1 > s_2$.

Proof of (1): (1) is equivalent to $\left(s_1^2 - (s_2\sin\theta_1)^2\right)$ being negative, which is equivalent to

$$s_1^2 < (s_2\sin\theta_1)^2;$$

since s_1, s_2, and $\sin\theta_1$ are positive, this is equivalent to $s_1 < s_2\sin\theta_1$.

Proof of (2): (2) is equivalent to $\left(s_1^2 - (s_2\sin\theta_1)^2\right)$ equalling zero; as in (1), this is equivalent to $s_1 = s_2\sin\theta_1$.

Proof of (3): $|s_2\cos\theta_1| < \sqrt{s_1^2 - (s_2\sin\theta_1)^2}$ if and only if

$$s_2^2\cos^2\theta_1 < s_1^2 - s_2^2\sin^2\theta_1 \iff s_1^2 > s_2^2(\cos^2\theta_1 + \sin^2\theta_1);$$

by 6.3(v) and the positivity of s_1 and s_2, this is equivalent to $s_1 > s_2$.

Now let's investigate the entries (0, 1, or 2) in the table stated in the theorem via solutions of (*).

First, note that Factoid (1) gives us the first row of the desired table: $s_1 < s_2\sin\theta_1$ implies no real solutions of (*).

For the remainder of the proof, assume $s_1 \geq s_2\sin\theta_1$. Factoid (1) implies that we have a real, positive solution of (*) if and only if $s_3^+ > 0$, and that solution is unique if and only if, in addition, either $s_3^- \leq 0$ or $s_3^- = s_3^+$.

First let's address the remaining rows of the right column of the table we wish to prove: assume $\theta_1 \geq \frac{\pi}{2}$. This is equivalent to $\cos\theta_1 \leq 0$, so that we may now rewrite Factoid (3) as follows. Since $s_2\cos\theta_1 \leq 0, |s_2\cos\theta_1| = -s_2\cos\theta_1$, thus

$$|s_2\cos\theta_1| < \sqrt{s_1^2 - (s_2\sin\theta_1)^2} \iff s_2\cos\theta_1 > -\sqrt{s_1^2 - (s_2\sin\theta_1)^2},$$

so that Factoid (3) is now

$$s_1 > s_2 \iff s_2\cos\theta_1 > -\sqrt{s_1^2 - (s_2\sin\theta_1)^2},$$

which is equivalent to $s_3^+ > 0$. This explains all the zeroes in the right column of the table, when $s_1 \leq s_2$, and, since $\cos\theta_1 \leq 0$ clearly implies $s_3^- \leq 0$, explains the "1" in the bottom of the right column.

For the remainder of the proof, assume $\theta_1 < \frac{\pi}{2}$. This implies that $s_2\cos\theta_1 > 0$, thus $s_3^+ > 0$; we are guaranteed at least one real, positive solution of (*). Factoid (2) implies that the solution is unique when $s_1 = s_2\sin\theta_1$, hence we get a "1" in that row of the table.

Again because $s_2\cos\theta_1 > 0$, Factoid (3) becomes

$$s_1 > s_2 \iff s_2\cos\theta_1 < \sqrt{s_1^2 - (s_2\sin\theta_1)^2} \iff s_3^- < 0.$$

Also

$$s_1 = s_2 \iff \sqrt{s_1^2 - (s_2\sin\theta_1)^2} = \sqrt{s_1^2 - (s_1\sin\theta_1)^2} \iff \sqrt{s_1^2 - (s_2\sin\theta_1)^2} = \sqrt{s_1^2(\cos\theta_1)^2}$$
$$\iff s_3^- = 0,$$

thus

$$s_1 \geq s_2 \iff s_3^- \leq 0.$$

Combining this with Factoid (2), we get two real, positive solutions of (*) when $s_2 > s_1 > s_2\sin\theta_1$, and a unique solution when $s_1 \geq s_2$. □

Remark 10.8. The proof of Theorem 10.7 is purely algebraic. Thus it carries authority, but might not provide intuition. See DRAWINGS 10.13 at the end of this chapter for some geometric intuitive rationalization of the table in Theorem 10.7. Our perspective in DRAWINGS 10.13 is to begin with s_2 and θ_1, then try to visualize what values of s_1 will allow the side of length s_3 to appear.

In DRAWINGS 10.13 at the end of this chapter, we denoted by $\epsilon \equiv \sqrt{s_1^2 - (s_2\sin\theta_1)^2}$, so that

$$s_3 = s_2\cos\theta_1 \pm \epsilon;$$

that is, in the terminology from the proof of Theorem 10.7,

$$s_3^+ = s_2\cos\theta_1 + \epsilon \quad \text{and} \quad s_3^- = s_2\cos\theta_1 - \epsilon.$$

Also note that $s_2\sin\theta_1$ is the shortest distance from where the sides of length s_2 and s_1 meet to the desired missing side of length s_3 (see the first picture in DRAWINGS 10.13 at the end of this chapter).

Examples 10.9. Which of the following drawn triangles in DRAWINGS 10.14 at the end of this chapter are possible? Which of them has more than one possibility (up to congruence)?

Solutions. Use Theorem 10.7, with $s_2 = 10$. For (a) thru (e), use the column in Theorem 10.7 labelled "$\theta_1 < \frac{\pi}{2}$", so that $\sin\theta_1 = \frac{1}{2}$, giving us a possible unique triangle for (b), (d), and (e), two possible triangles for (c), and no possible triangle for (a). For (f) and (g), use the column labelled "$\theta_1 \geq \frac{\pi}{2}$, to conclude there is no triangle as in (f), a unique triangle in (g).

Examples 10.10. (a) In DRAWING 10.15 at the end of this chapter, find x, the length of the lower horizontal line. Assume the two horizontal lines are parallel.

(b) Suppose that, when an eight foot tall Frankenstein monster is five feet from a lamppost, the distance from the foot of the monster to the end of his shadow is four feet. How tall is the lamppost? Assume that both the monster and the lamppost are perpendicular to the ground.

Solutions. (a) See DRAWING 10.16 at the end of this chapter. By Proposition 3.6, angle measures θ_1 and θ_2 are equal, thus, by Theorem 10.1(b) (AAA), triangles ACD and ABE are similar. This implies, by Proposition 8.5, that

$$\frac{x}{10} = \frac{5}{5+7} \to x = \frac{50}{12} = \frac{25}{6}.$$

(b) See DRAWING 10.17 at the end of this chapter, where x is the height of the lamppost. By Theorem 10.1(b), triangles ACD and ABE are similar, thus, by Proposition 8.5,

$$\frac{x}{4+5} = \frac{8}{4} \to x = 18.$$

HOMEWORK

HWX.1. In each part of DRAWINGS 10.18 at the end of this chapter, determine how many (up to congruence) triangles have the given lengths of sides and measures of angles.

HWX.2. Which of the pairs of triangles in DRAWINGS 10.19 at the end of this chapter are guaranteed by Theorem 10.1, from the information given, to be congruent?

HWX.3. Use congruence (Theorems 10.1 and 10.7) to prove the other parts (see Example 10.2) of Theorem 7.10.

HWX.4. Some children are equipped with graph paper, a protractor (for measuring angles on a piece of paper), and an astrolabe (for measuring angles of elevation). The goal is to get the height of a tree from measurements on the ground, without trigonometry.

(a) Fourteen paces from the tree, we measure the angle of elevation, call it θ, to the top of the tree. On a piece of graph paper we construct that angle of elevation and the paces, with each pace represented by a side of a square; see DRAWING 10.20 at the end of this chapter. Use similar triangles to estimate the height of the tree.

(b) Same as (a), except there is poison ivy that you must avoid in a disc of radius seven paces with the tree at the center, and we measure the angle of elevation both at the edge of the poison ivy and nine paces further away from the tree than the poison ivy; see DRAWING 10.21 at the end of this chapter.

HOMEWORK ANSWERS

HWX.1. (a) one (Theorem 10.7) (b) none (Theorem 7.10) (c) none (Theorem 10.7) (d) one (Theorem 10.7) (e) two (Theorem 10.7) (f)–(j) none (Theorem 10.7) (k) one (Theorem 10.7) (ℓ) one (Theorems 10.4(a) and 10.6(a)) (m) one (Theorem 10.7 or Pythagorean theorem) (n) none (Theorem 10.7 or Pythagorean theorem) (o) infinitely many (other side forming vertex of angle measure sixty degrees could be anything) (p) none (Theorem 10.6(b)) (q) infinitely many (could magnify triangle by any positive factor) (r) one (Theorems 10.4(b) and 10.6(b)).

HWX.2. (a), (d), (g).

HWX.3. See Examples 10.2 and DRAWINGS 10.9 at the end of this chapter, for the meaning of the points P, Q, R, and S. The congruence of the triangles PSR and SQR imply all the assertions (a)–(d) in Theorem 7.10.

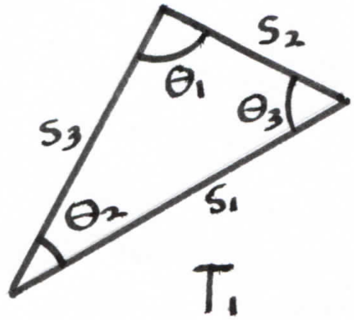
See DRAWINGS 10.22 at the end of this chapter for (a) implying the desired congruence, hence (b), (c), and (d); (b) implying the desired congruence, hence (a), (c), and (d); and (c) implying the desired congruence, hence (a), (b), and (d).

HWX.4. Count squares; for (a), look at DRAWING 10.20 at the end of this chapter, for (b), look at DRAWING 10.21 at the end of this chapter.

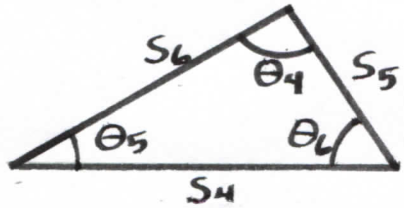
(a) 17 paces high.

(b) 20 paces high.

DRAWING 10.1

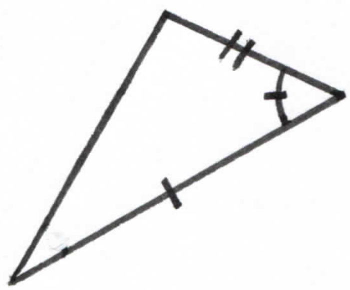


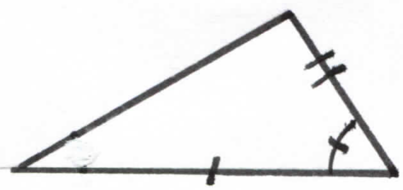
T_1



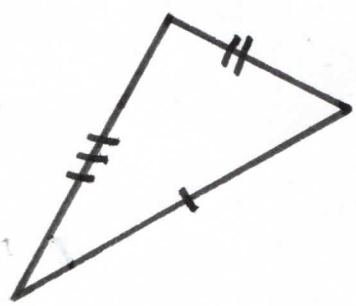
T_2

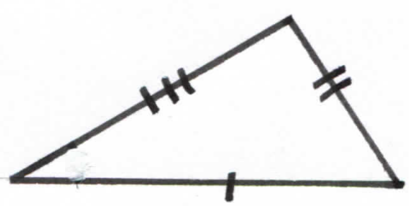
DRAWING 10.2



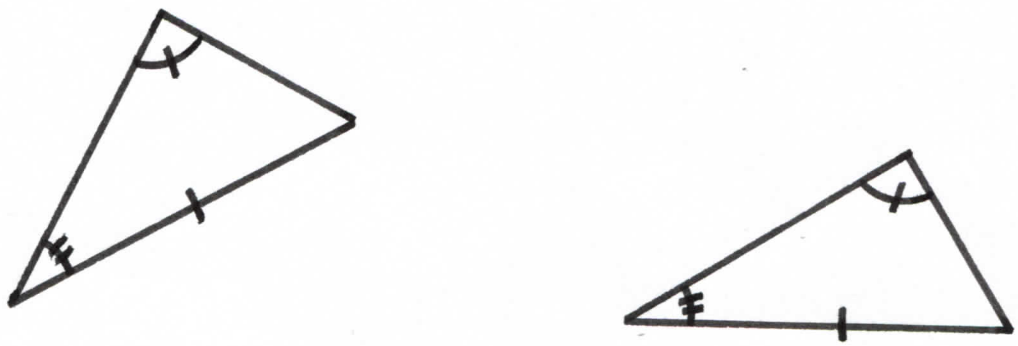


DRAWING 10.3

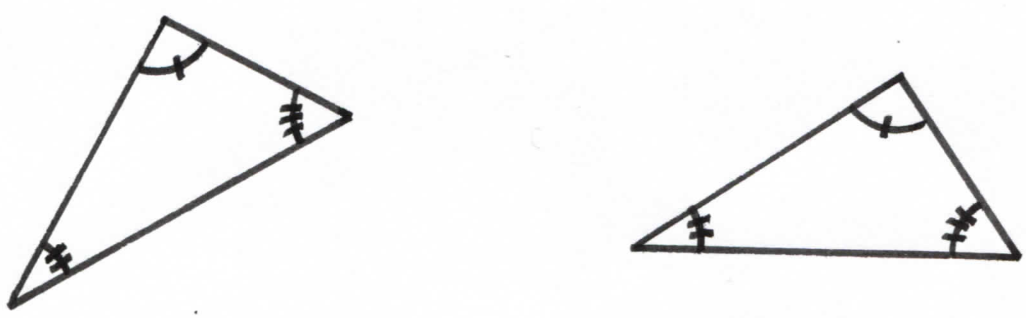




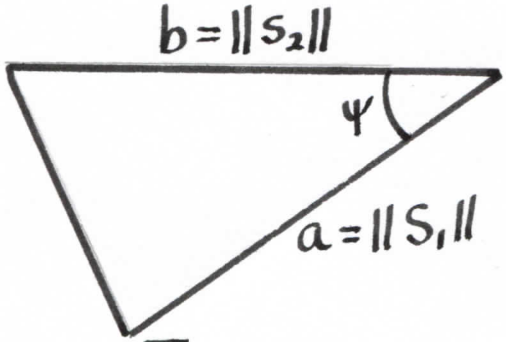
DRAWING 10.4



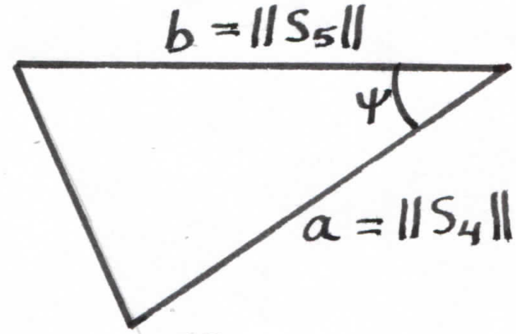
DRAWING 10.5



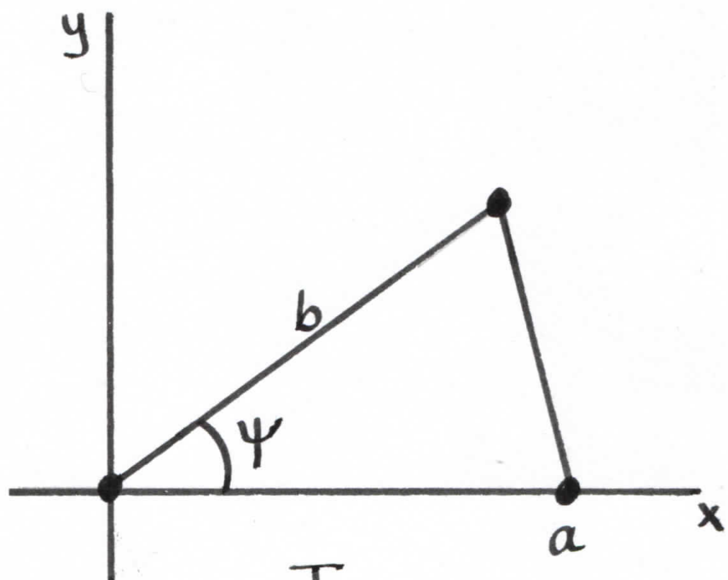
DRAWINGS 10.6



T_1

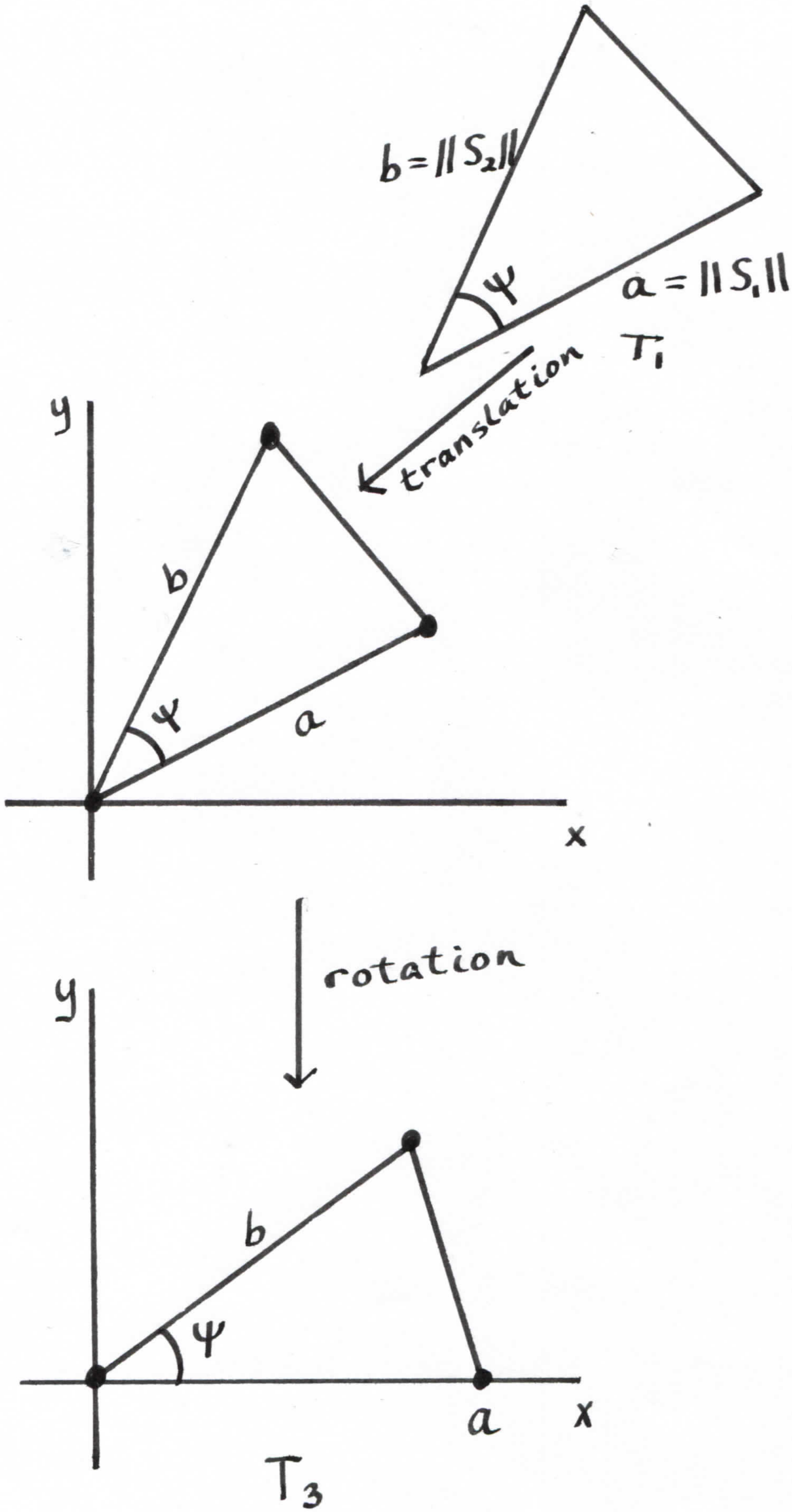


T_2

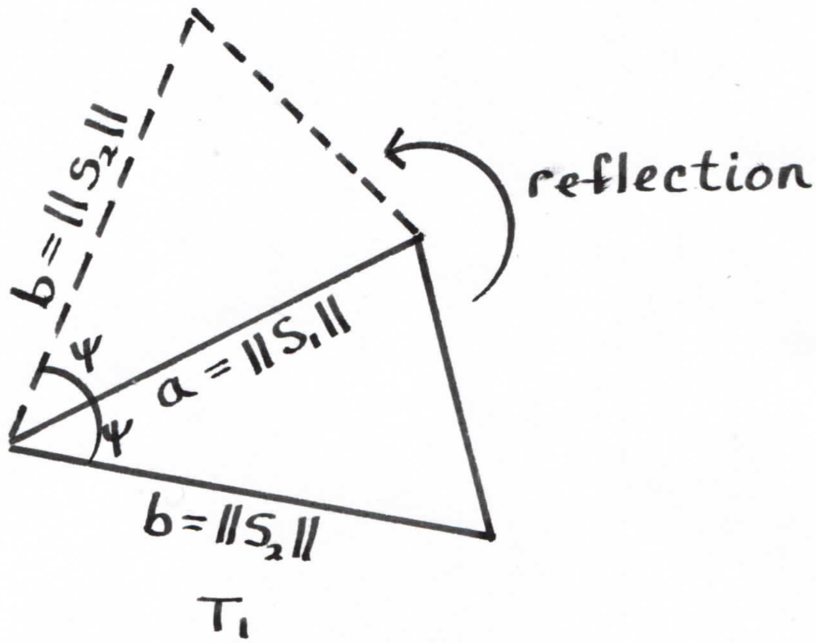


T_3

DRAWINGS 10.7



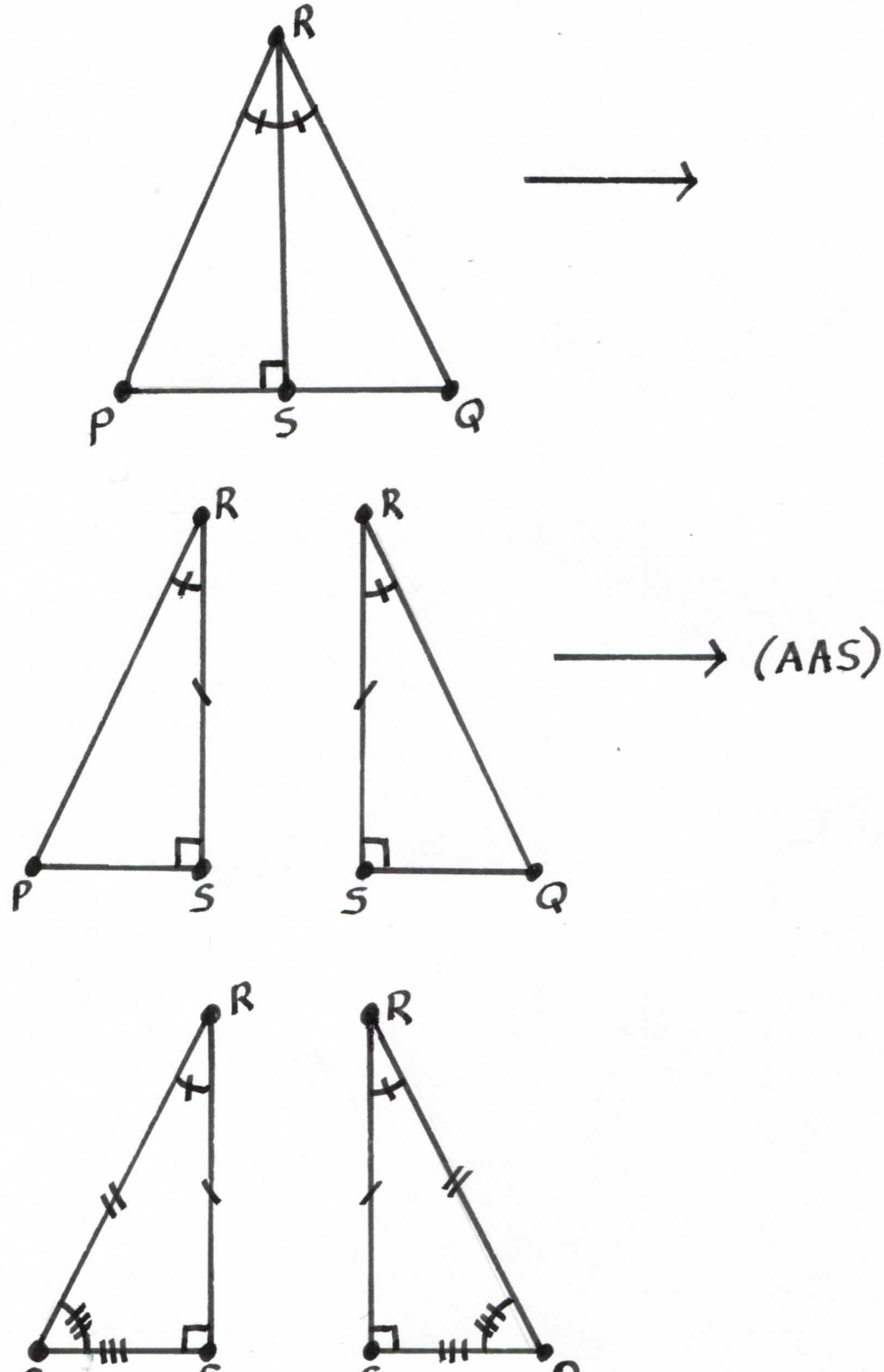
DRAWINGS 10.8



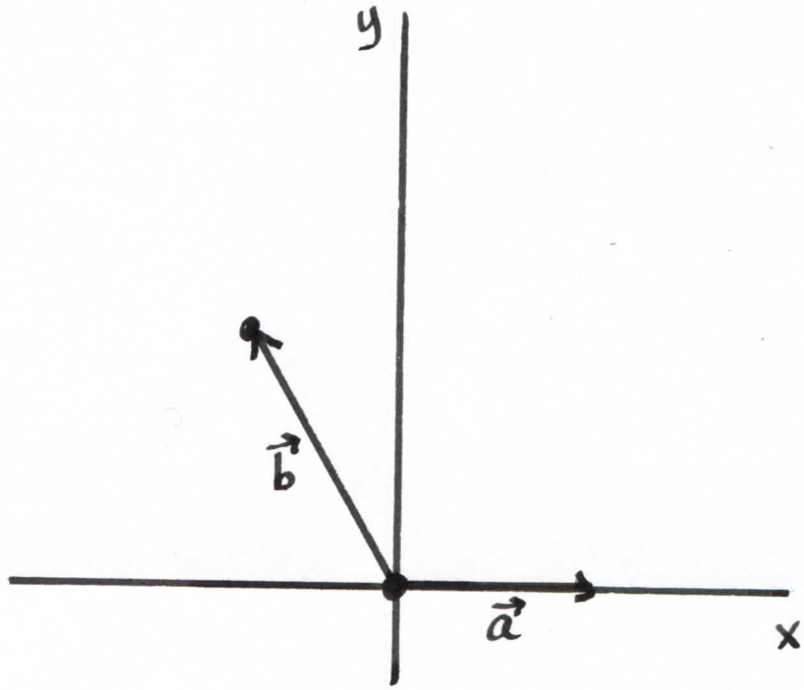
Now (after reflection) the triangle is congruent to the triangle at the beginning of DRAWINGS 10.7.

DRAWINGS 10.9

(d)



DRAWINGS 10.10



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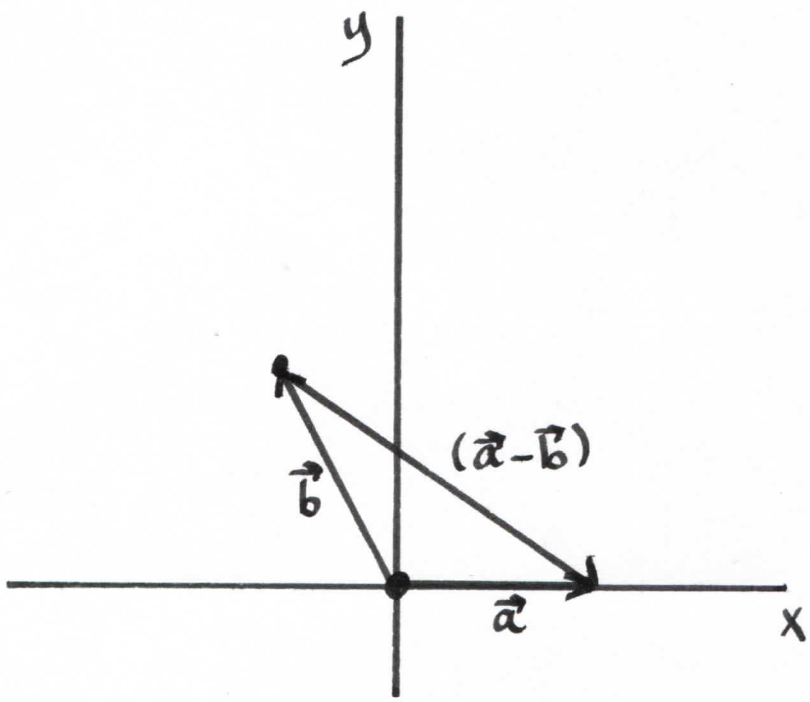
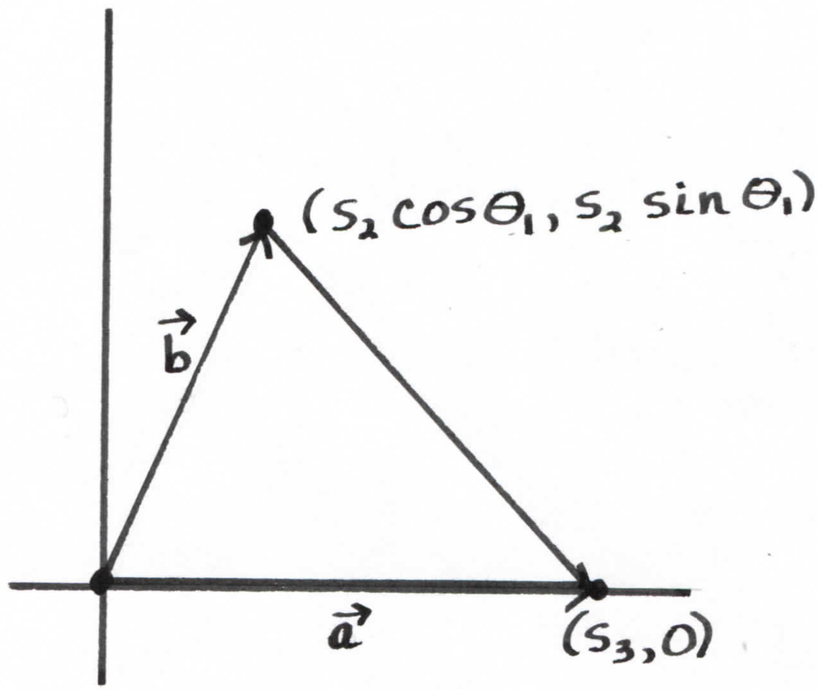
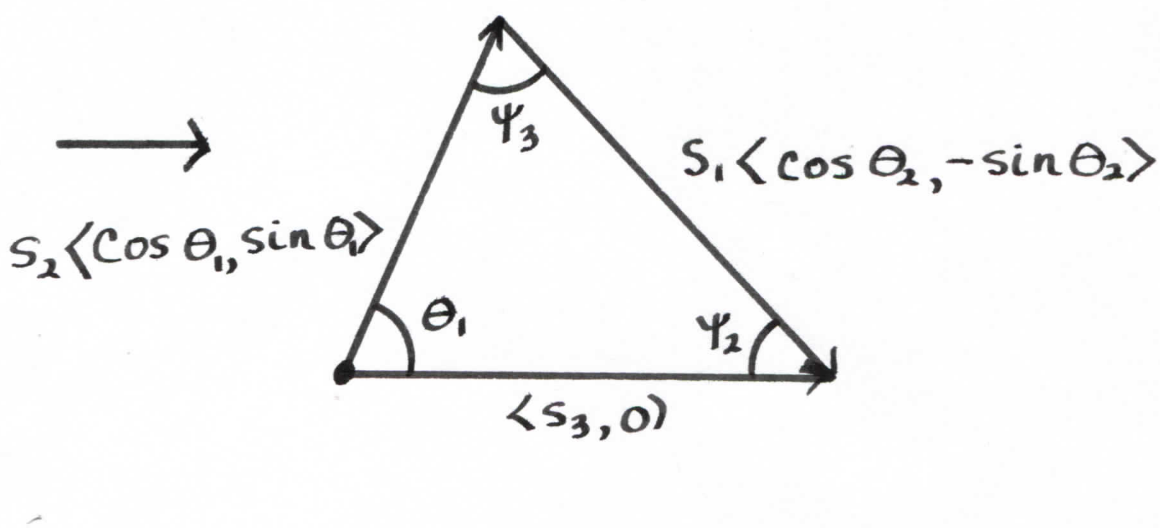
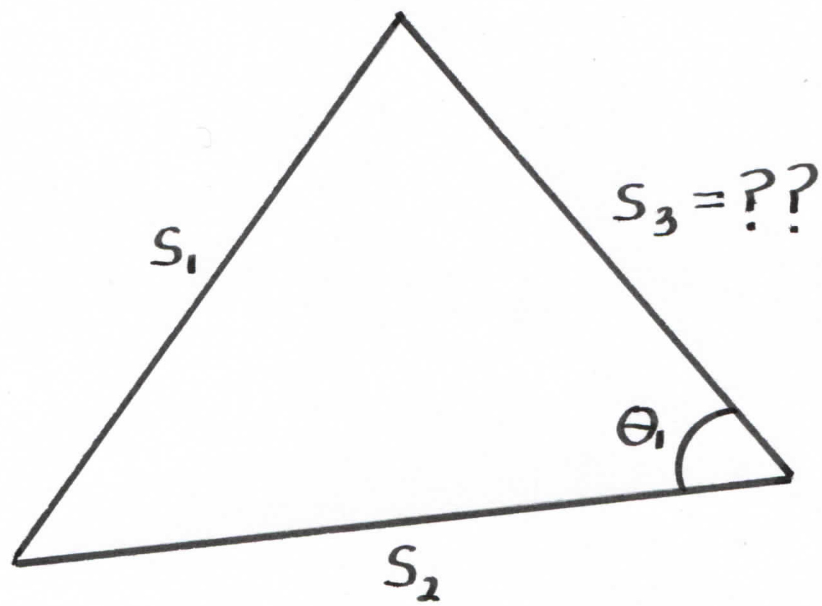


DRAWINGS 10.11

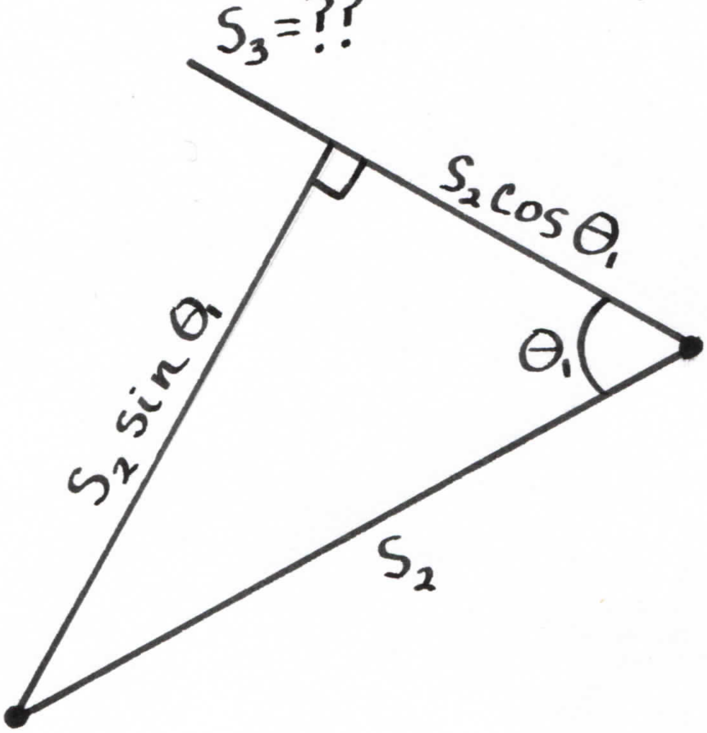




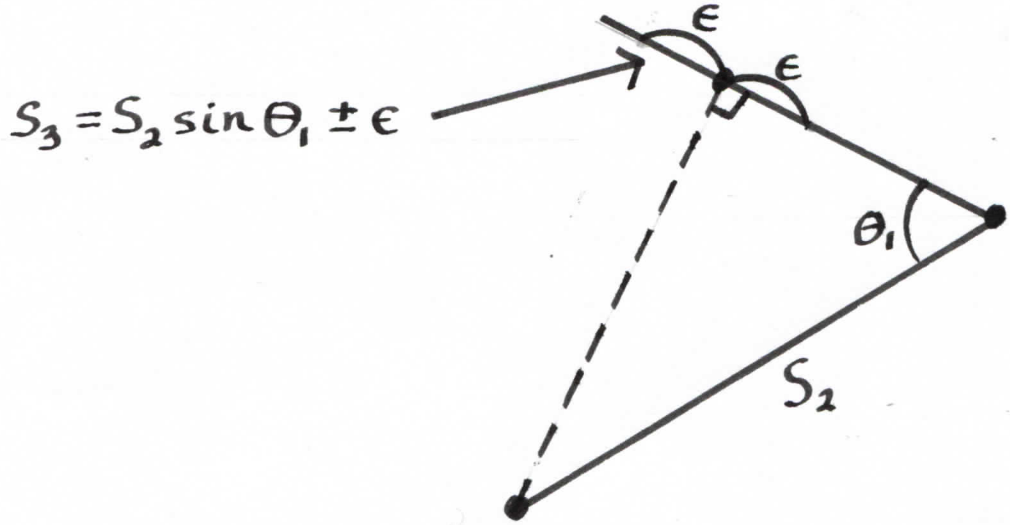
DRAWING 10.12

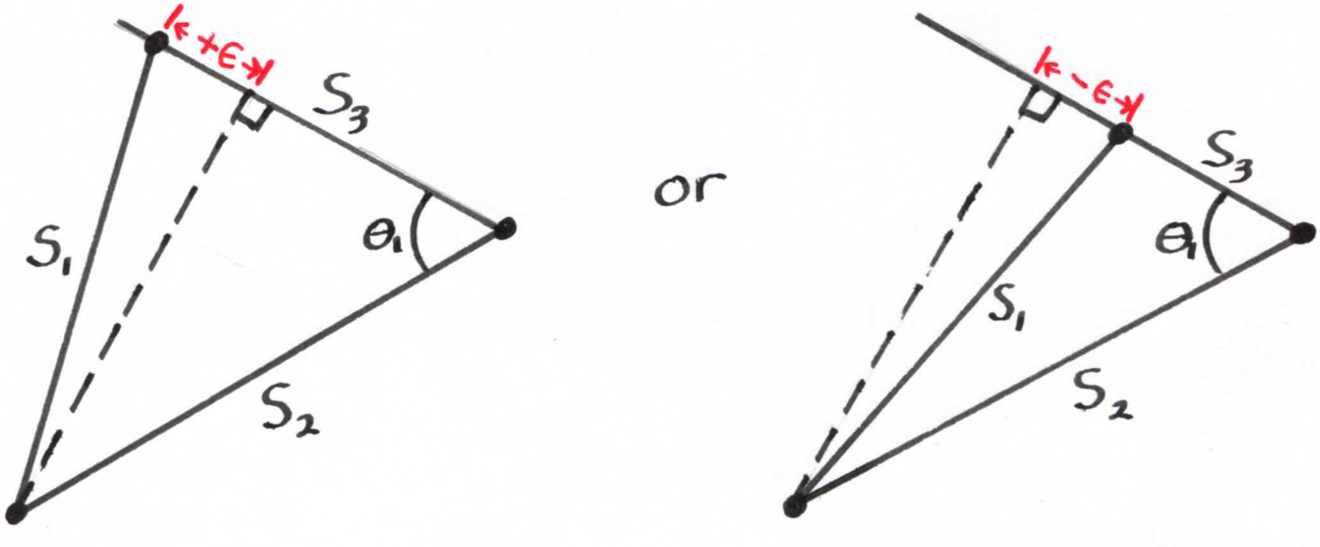


DRAWINGS 10.13(1) ($\theta_1 < \frac{\pi}{2}$)



DRAWINGS 10.13(2) $(\theta_1 < \frac{\pi}{2})$

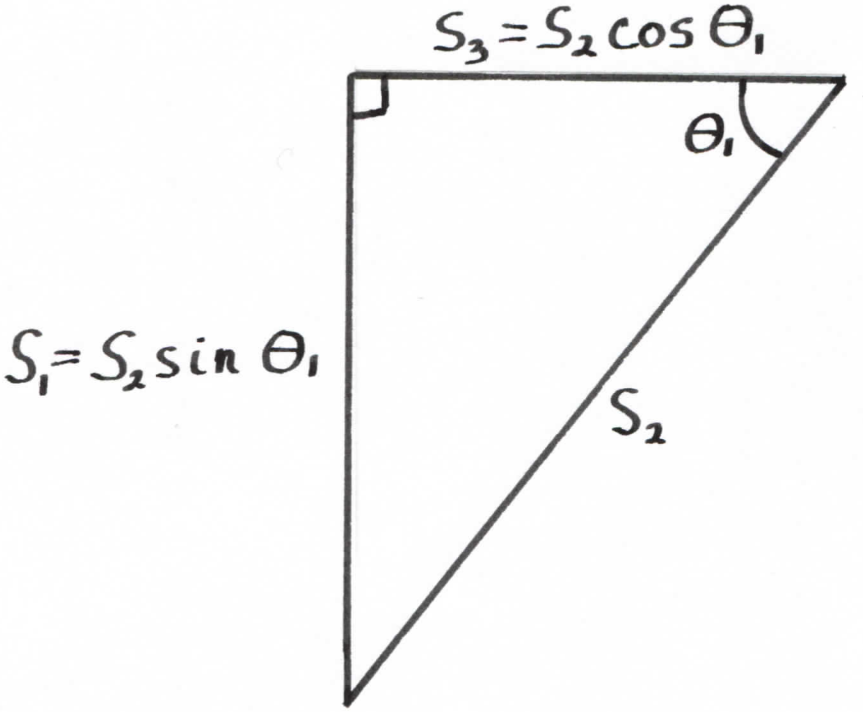




$\longrightarrow S_1 \geq S_2 \sin\theta_1$

DRAWINGS 10.13(3) $(\theta_1 < \frac{\pi}{2})$

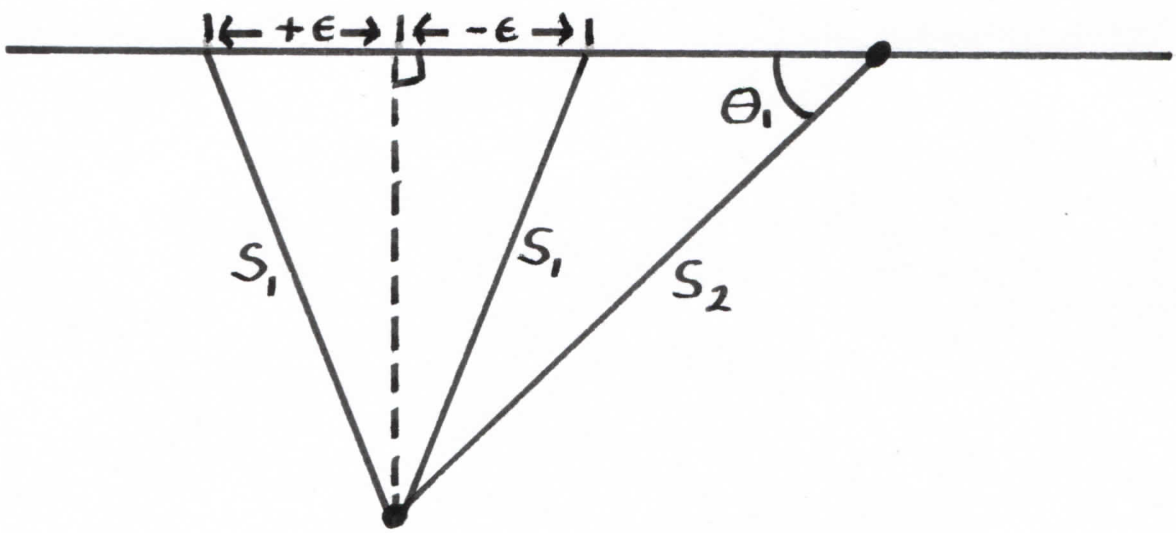
$$S_1 = S_2 \sin\theta_1$$



<u>Unique</u> triangle

DRAWINGS 10.13(4) $\left(\theta_1 < \frac{\pi}{2}\right)$

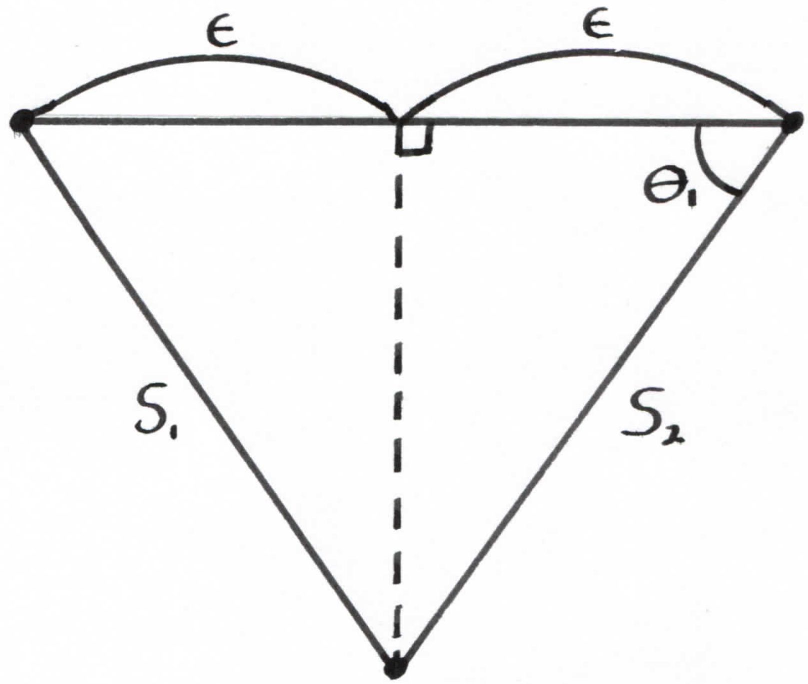
$$S_2 > S_1 > S_2 \sin\theta_1$$



$$S_3 = S_2 \cos\theta_1 \pm \epsilon \quad (2 \text{ solutions})$$

DRAWINGS 10.13(5) ($\theta_1 < \frac{\pi}{2}$)

$$S_1 = S_2$$

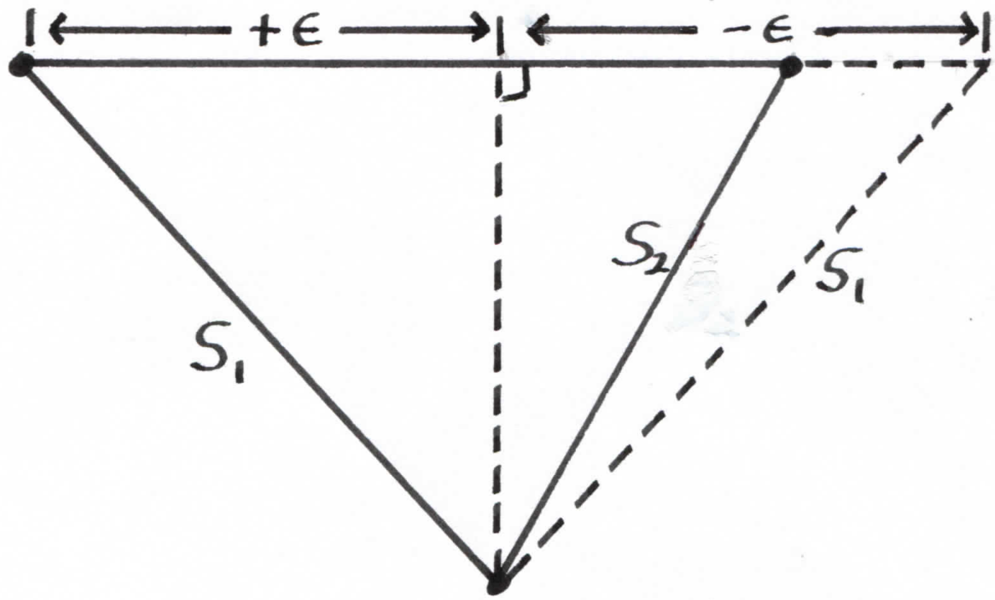


$\longrightarrow$ $S_2 \cos \theta_1 = \epsilon$ $\longrightarrow$ $S_2 \cos \theta_1 - \epsilon = 0$

$\longrightarrow$ $S_3 = S_2 \cos \theta_1 + \epsilon$ is the only solution (see DRAWINGS 10.13(4))

DRAWINGS 10.13(6) $(\theta_1 < \frac{\pi}{2})$

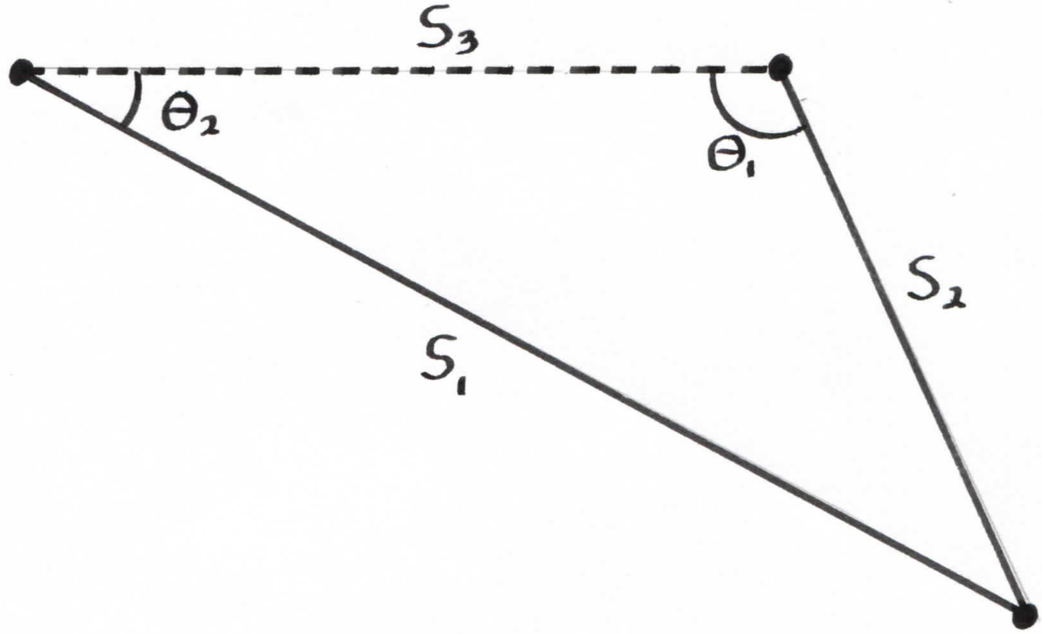
$S_1 > S_2$



$S_3 = S_2 \cos\theta_1 + \epsilon$ (1 solution)

(see DRAWINGS 10.13(4))

DRAWINGS 10.13(7) $(\theta_1 \geq \frac{\pi}{2})$



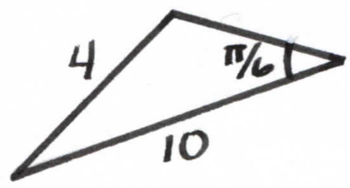
MUST HAVE $S_1 > S_2$, since $\theta_1 > \theta_2$

$\longrightarrow S_3 = S_2 \cos\theta_1 + \epsilon$ (1 solution),

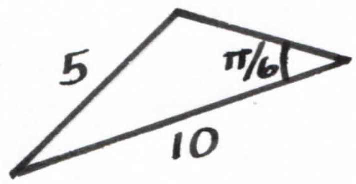
as in DRAWINGS 10.13(6)

DRAWINGS 10.14

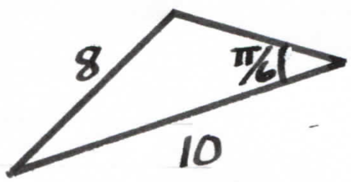
(a)



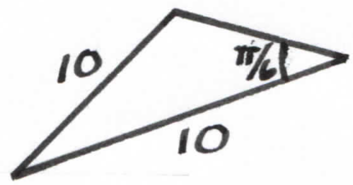
(b)



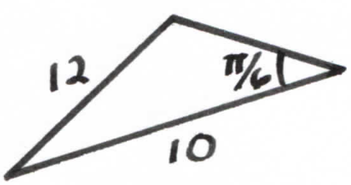
(c)



(d)

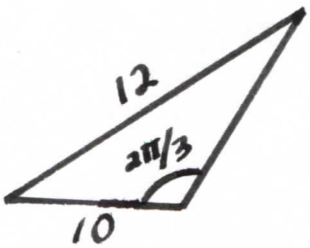


(e)

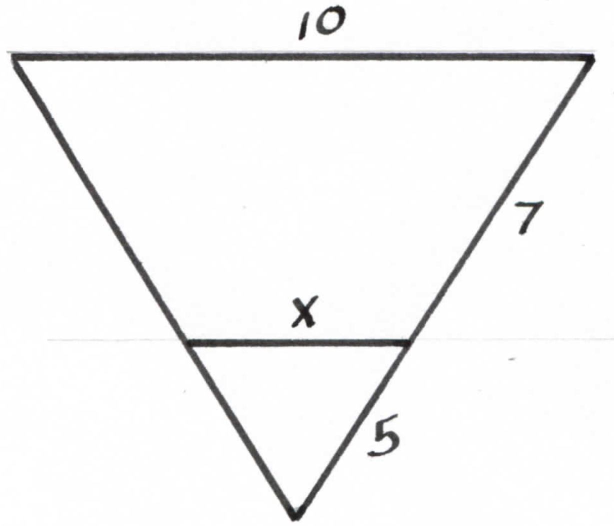


(f)

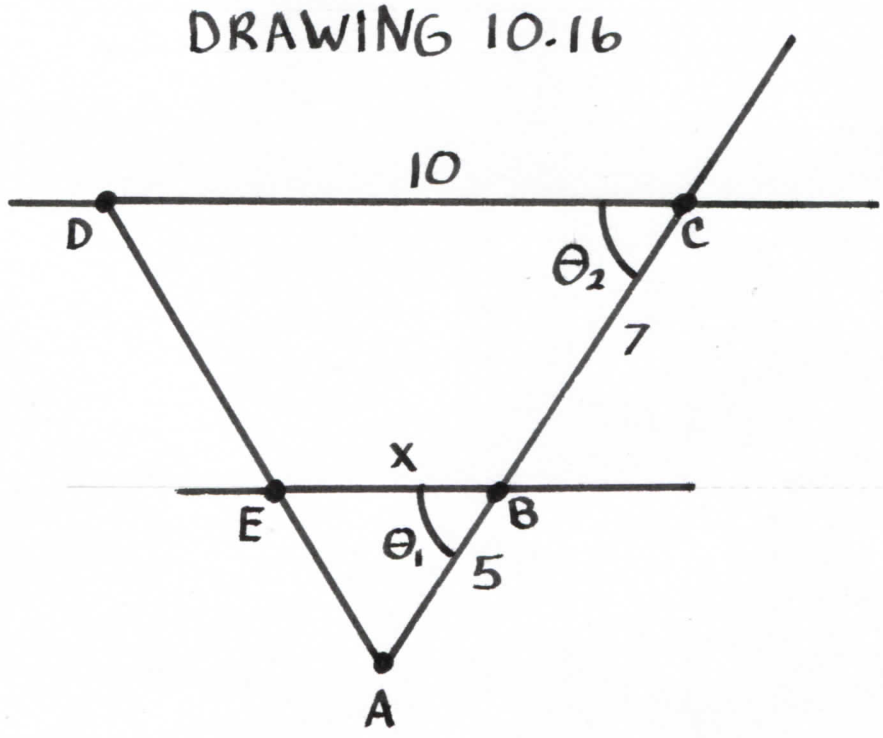
(g)



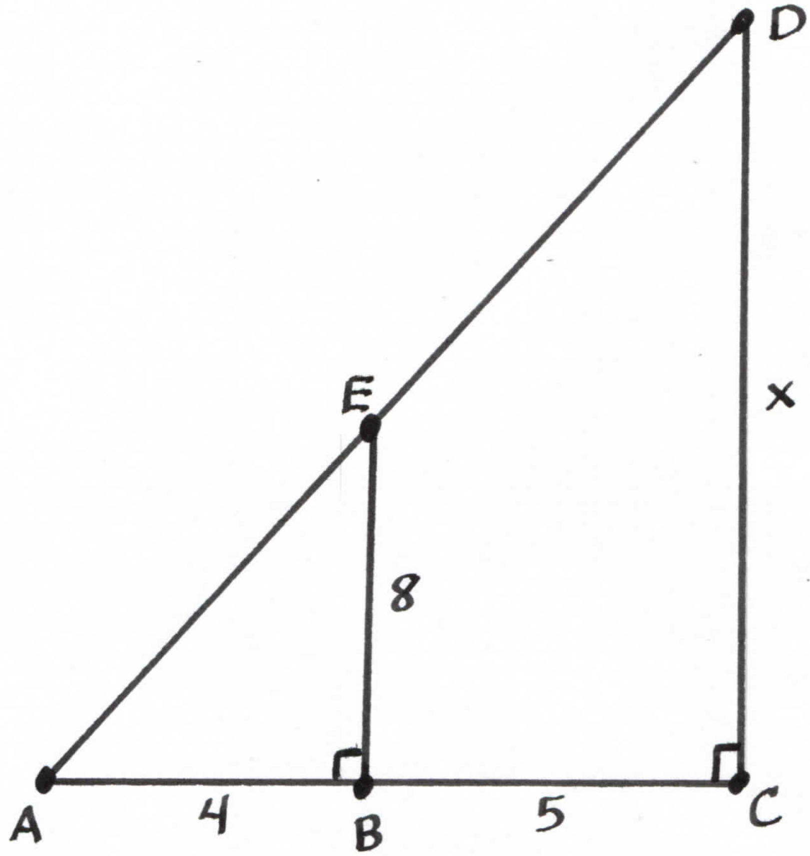
DRAWING 10.15



DRAWING 10.16

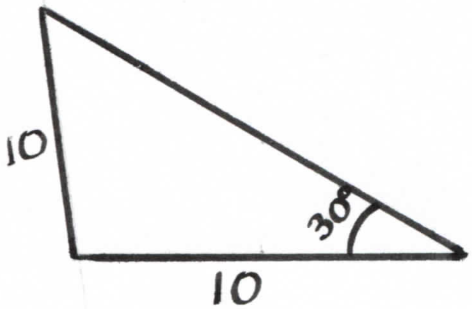


DRAWING 10.17

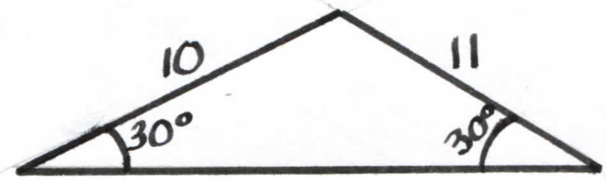


DRAWINGS 10.18

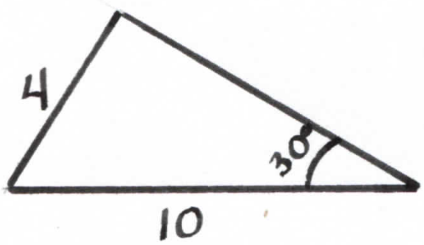
(a)



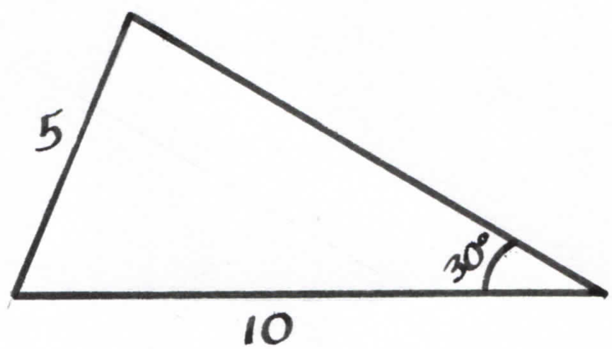
(b)



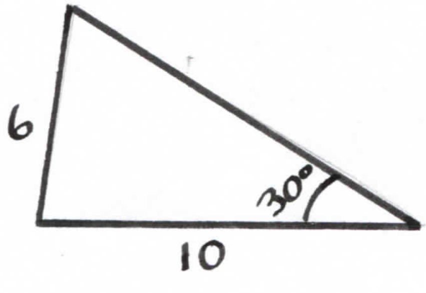
(c)



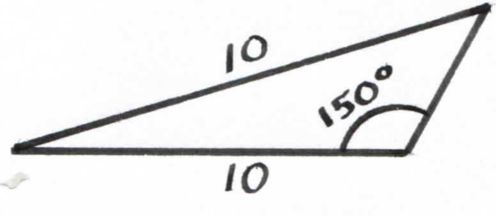
(d)



(e)

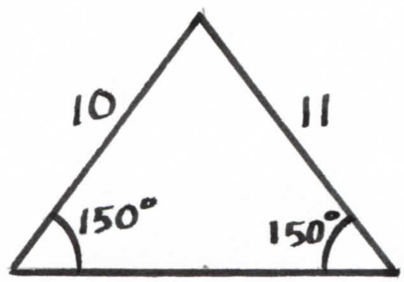


(f)

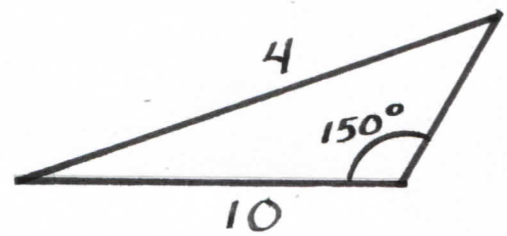


DRAWINGS 10.18 (continued)

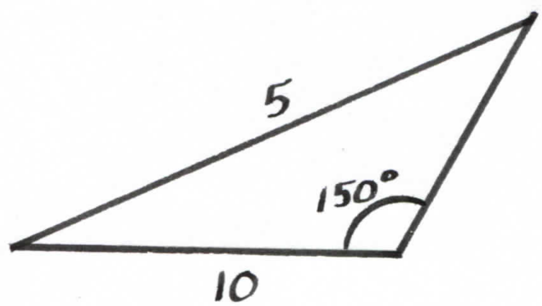
(g)



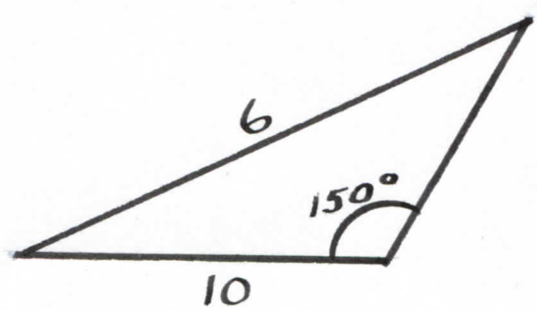
(h)



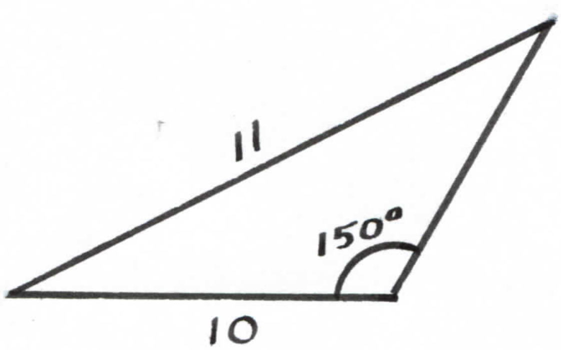
(i)



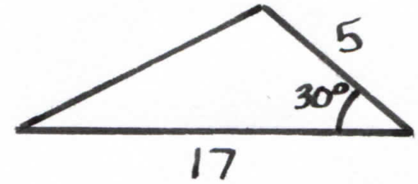
(j)



(k)

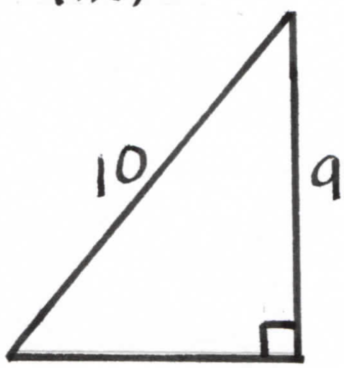


(l)

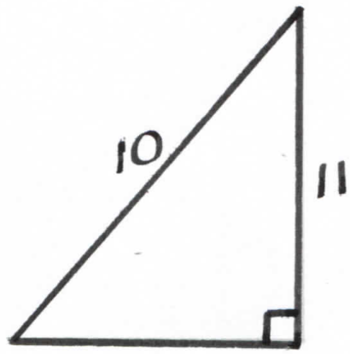


DRAWINGS 10.18 (continued)

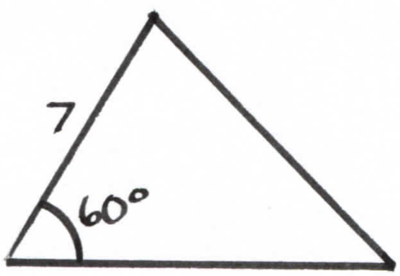
(m)



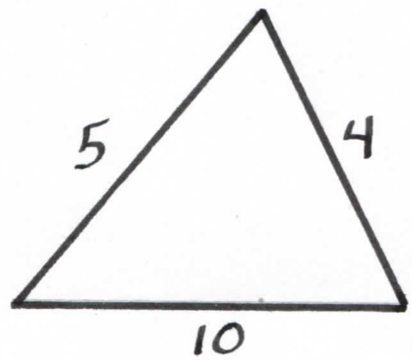
(n)



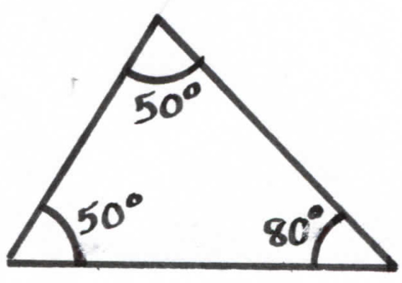
(o)



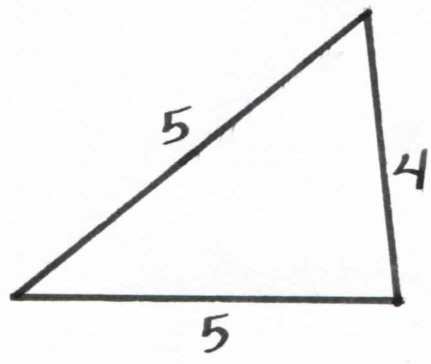
(p)



(q)



(r)



DRAWINGS 10.19

(a)

(b)

(c)

(d)

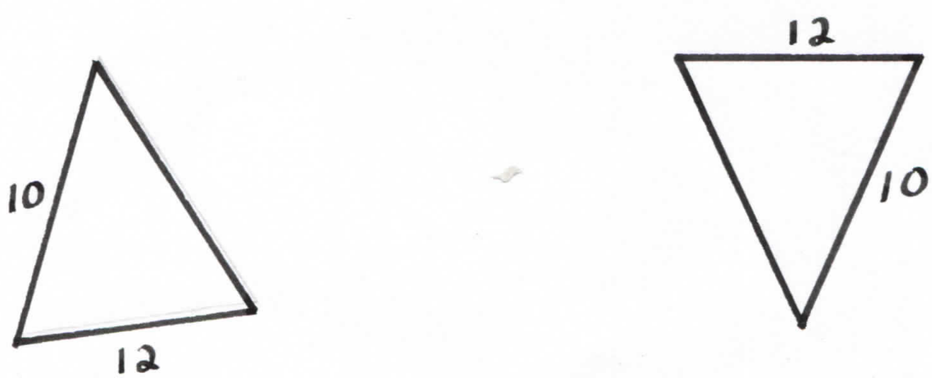
DRAWINGS 10.19 (continued)

(e)

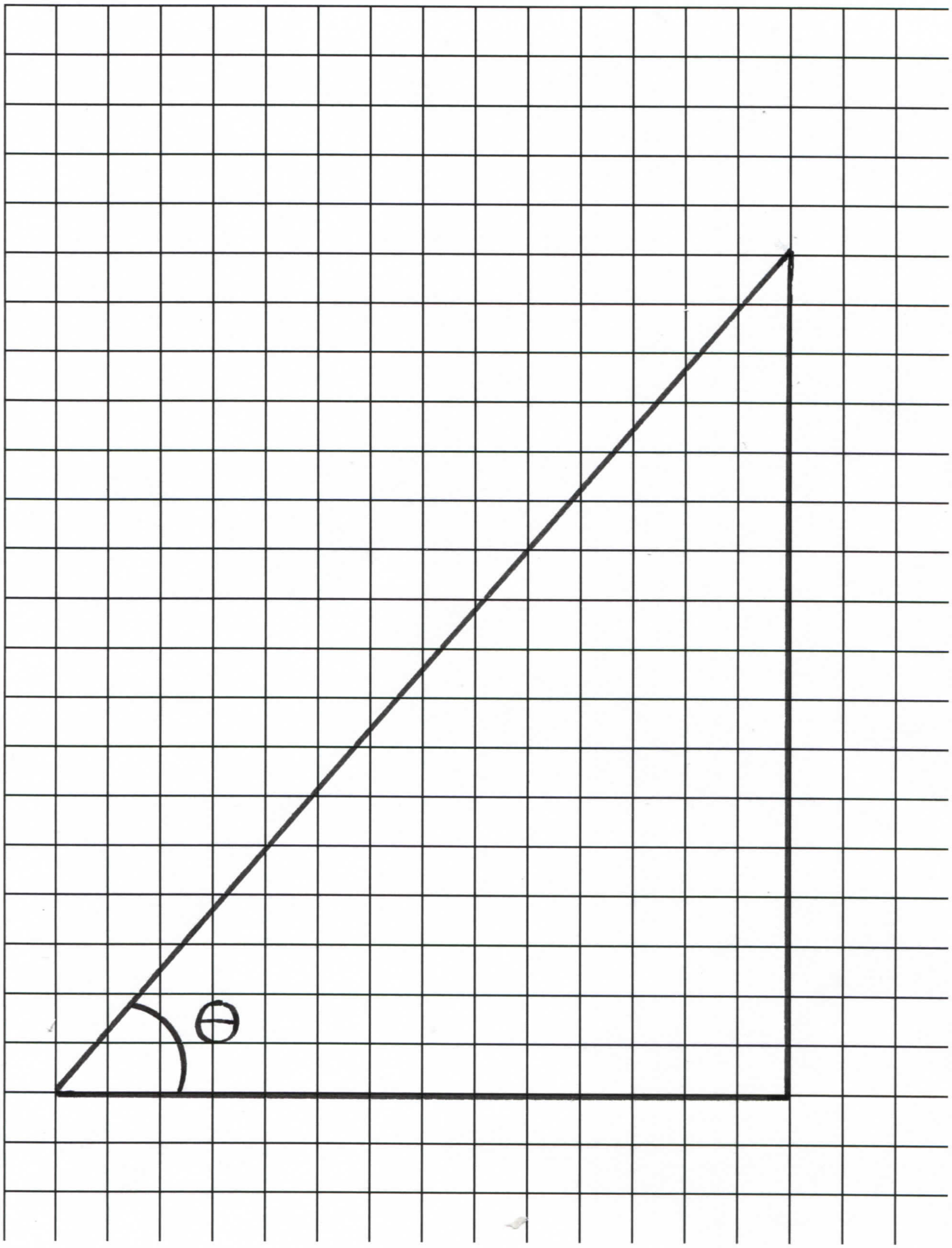
(f)

(g)

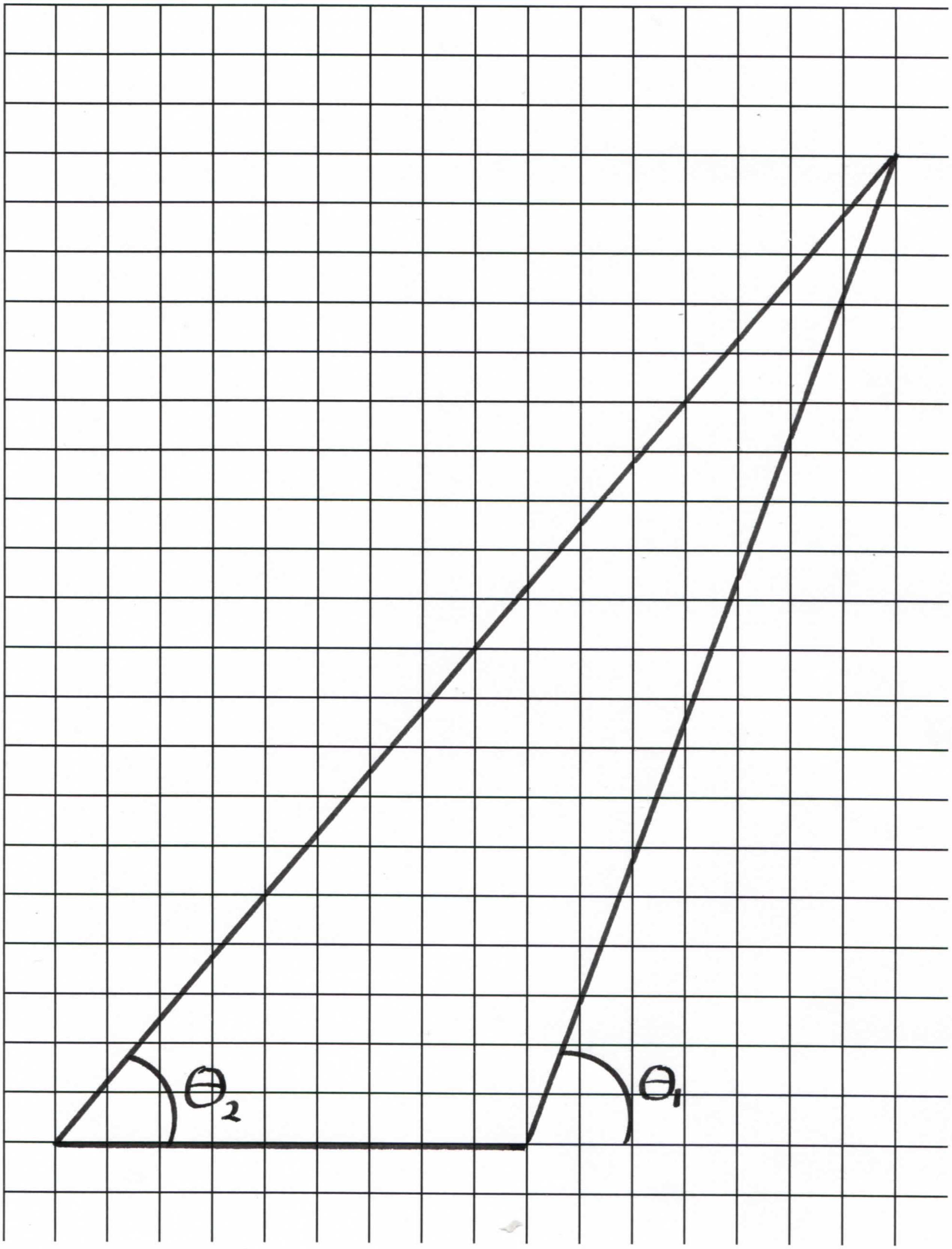
(h)



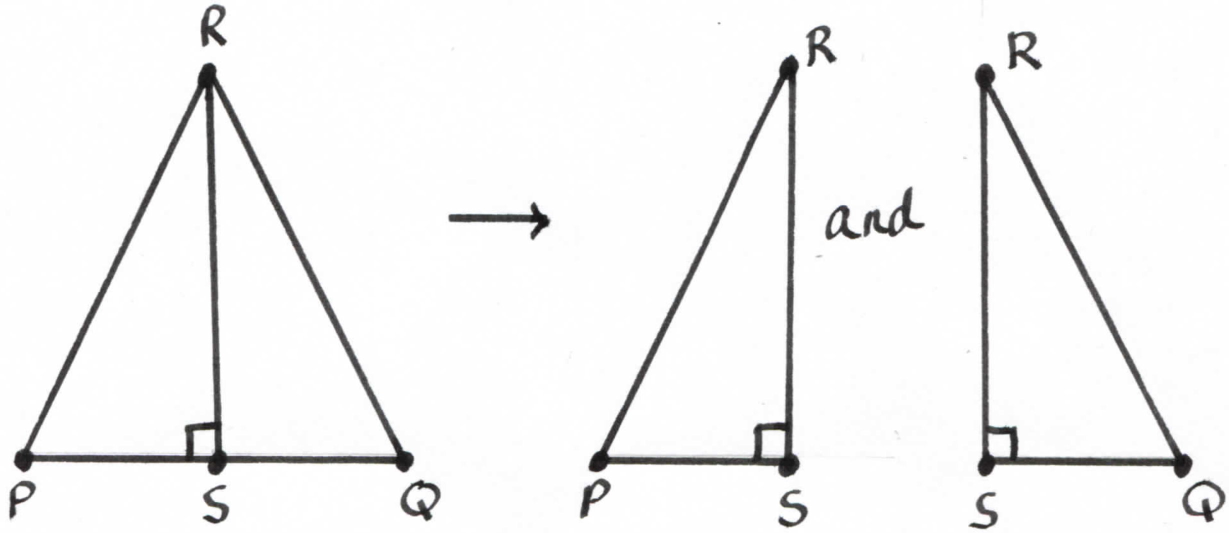
DRAWING 10.20



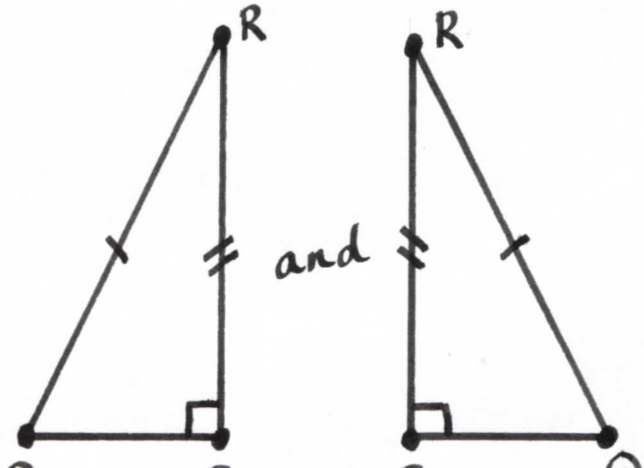
DRAWING 10.21



DRAWINGS 10.22



(a)



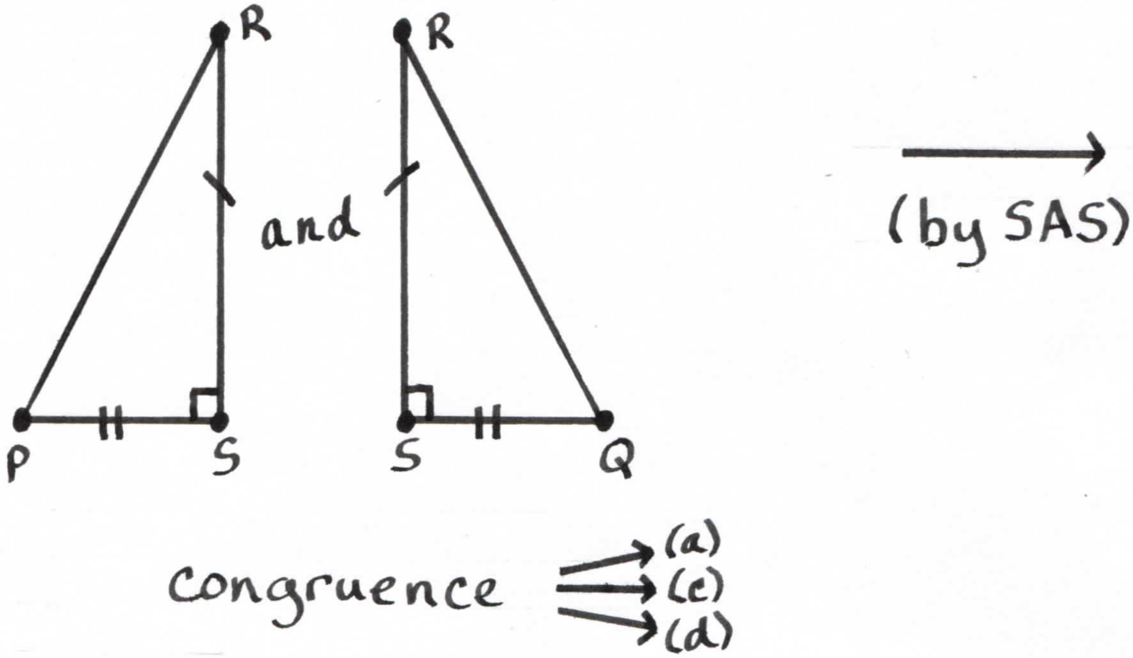
$\longrightarrow$

(by Theorem 10.7, $S_1 > S_2$, $\theta_1 \geq \frac{\pi}{2}$)

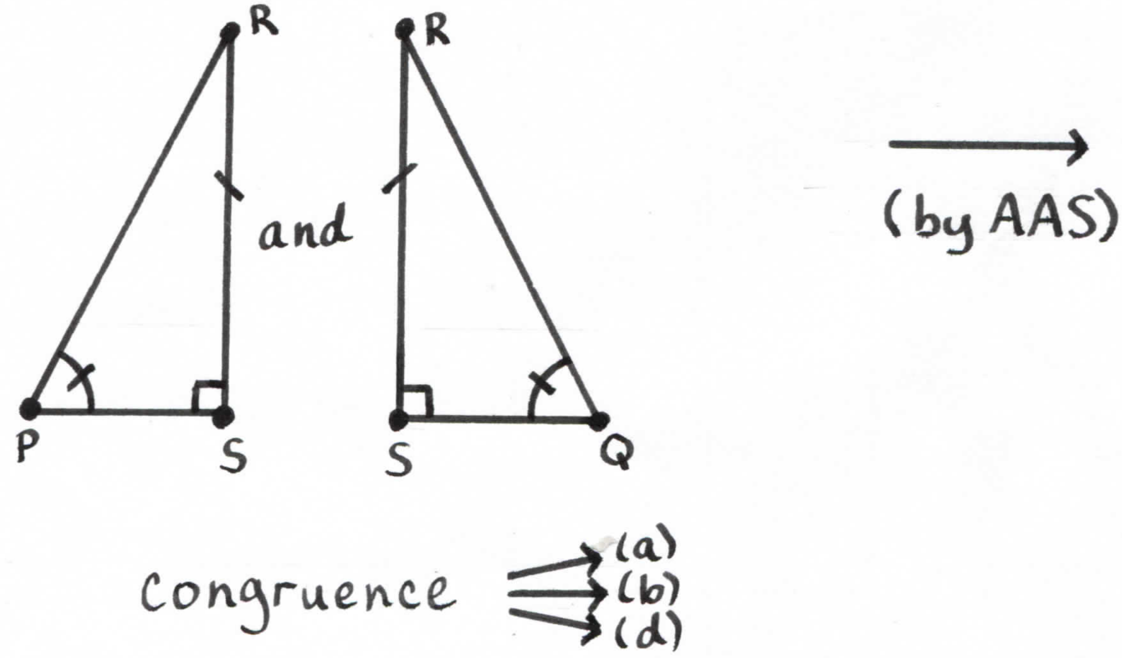
congruence $\longrightarrow$ (b)
$\longrightarrow$ (c)
$\longrightarrow$ (d)

DRAWINGS 10.22 (continued)

(b)



(c)



CHAPTER XI: More Quadrilateral Results.

First we would like to address converses to some parallelogram results in Chapter III. Proposition 11.1 is a converse of Proposition 3.6, while Proposition 11.3 is a converse of Proposition 3.8, and Proposition 11.3, which should be compared to Proposition 3.5, is a converse of Proposition 3.3.

Proposition 11.4 will characterize parallelograms whose diagonals bisect their interior angles. Proposition 11.5 shows a surprising amount of information following from small conditions on a quadrilateral, that will provide the key for most of our constructions in the next chapter.

Proposition 11.1. Suppose ℓ_1, ℓ_2, ℓ_3 are lines, with ℓ_3 intersecting both ℓ_1 and ℓ_2. If, in DRAWING 11.1 at the end of this chapter, $\theta_4 = \theta_5$, then ℓ_1 and ℓ_2 are parallel.

Proof: For $j = 1, 2, 3$, let $e^{i\theta_j}$ be a unit direction vector, in complex form, for ℓ_j, oriented as in DRAWING 11.1 at the end of this chapter. By Proposition 6.6 and Theorem 6.9, $\theta_4 = (\theta_3 - \theta_1)$ and $\theta_5 = (\theta_3 - \theta_2)$. If $\theta_4 = \theta_5$, then $\theta_1 = \theta_2$, so that ℓ_1 and ℓ_2 have the same direction vector. This is saying (Definitions 2.1) that ℓ_1 and ℓ_2 are parallel. See DRAWING 11.2 at the end of this chapter. □

Proposition 11.2. If a quadrilateral has nonadjacent interior angles of equal measure, as in DRAWING 11.3 at the end of this chapter, then it is a parallelogram.

Proof: In DRAWING 11.3 at the end of this chapter, extend the left side of the quadrilateral (to a line ℓ_3) and the top and bottom of the quadrilateral (to lines ℓ_1 and ℓ_2), as in DRAWING 11.4 at the end of this chapter.

By Corollary 3.10, $2\pi = 2\theta_1 + 2\theta_2$ (see DRAWING 11.3 at the end of this chapter), thus $(\theta_1 + \theta_2) = \pi = (\theta_2 + \theta_3)$ (see DRAWING 11.4 at the end of this chapter). Thus $\theta_1 = \theta_3$ in DRAWING 11.4 at the end of this chapter; Proposition 11.1 now implies that ℓ_1 and ℓ_2, hence the top and bottom of the quadrilateral, are parallel. An identical argument shows that the right and left sides of the quadrilateral are parallel; this means we have a parallelogram. □

Proposition 11.3. If a quadrilateral has nonconsecutive sides of equal length, then it is a parallelogram.

Proof: Add a diagonal to the quadrilateral, as in DRAWING 11.5 at the end of this chapter.

The two triangles formed are congruent, by SSS (Theorem 10.1(a)). Since sides must be mapped, by the rigid motions defining congruence, to sides of equal length, the congruent angles are as drawn in DRAWING 11.6 at the end of this chapter.

By Proposition 11.2, the result follows. □

Proposition 11.4. A diagonal in a parallelogram bisects opposite interior angles if and only if the parallelogram is a rhombus (Definitions 5.4).

Proof: See DRAWING 11.7 at the end of this chapter, where we have drawn the picture of a diagonal in a parallelogram guaranteed by HWIII.2, along with lengths of sides. Focusing on the left triangle, Theorem 7.10(a) $\iff$ (c) tells us that $s_1 = s_2 \iff \theta_1 = \theta_2$, as desired. □

Recall that we had another characterization of a rhombus, in Proposition 5.5.

Proposition 11.5. Form a quadrilateral from the isosceles sides of two isosceles triangles, with vertices P, Q, R, S, as in DRAWING 11.8 at the end of this chapter.

Add on line segments $\overrightarrow{QS}$ and $\overrightarrow{PR}$ between opposite vertices, as in DRAWING 11.9 at the end of this chapter.

Then $\overrightarrow{QS}$ is the perpendicular bisector of $\overrightarrow{PR}$ and bisects the angle at the vertex Q, as in DRAWING 11.10 at the end of this chapter.

Proof: See DRAWINGS 11.11 at the end of this chapter for the following chain of reasoning.

By SSS (Theorem 10.1(a)), triangle SPQ is congruent to triangle SRQ. This implies that $\overrightarrow{QS}$ bisects the angle at Q. The remaining results follow from Theorem 7.10, focusing on the isosceles triangle RPQ, or SAS (Theorem 10.1(a)) applied to the pair of triangles in the second-to-last drawing in DRAWINGS 11.11 at the end of this chapter. $\square$

Examples 11.6. For each of the parallelograms in DRAWINGS 11.12 at the end of this chapter, fill in side lengths and angle measurements, where possible.

Solutions. See DRAWINGS 11.13 at the end of this chapter.

(a) Propositions 3.3 and 3.8.

(b) Propositions 3.3, 3.8, and 11.4.

(c) Proposition 3.3 and HWIII.2 or Proposition 3.6.

(d) Propositions 3.8, 3.9, and 11.4.

(e) Propositions 3.8, 5.5, and 11.4.

(f) Propositions 3.8, 3.13, and 5.6.

(g) Propositions 3.13, 5.6, and 11.4, and the Pythagorean theorem.

Examples 11.7. Find x in each of the drawings of parallelograms in DRAWINGS 11.14 at the end of this chapter.

Solutions. (a) By HWIII.2 or Proposition 3.6, $x = (2x - \frac{\pi}{3})$, so $x = \frac{\pi}{3}$.

(b) By Proposition 11.4, $(3x - \frac{\pi}{2}) = (x + \pi)$, so $x = \frac{3\pi}{4}$.

(c) By Proposition 3.8, $(x - \pi) + (2x - \pi) = \pi$, so $x = \pi$.

(d) By Proposition 3.8, $(3x + \pi) = (2x + \frac{4\pi}{3})$, so $x = \frac{\pi}{3}$.

(e) By Propositions 5.6 and 3.13, $2(x + 5) = 2(2x)$, so $x = 5$.

(f) By Proposition 3.13, $(3x - 14) = x$, so $x = 7$.

(g) By Proposition 5.5, $(2x + 9) = (x + 20)$, so $x = 11$.

HOMEWORK

HWXI.1. For each of the parallelograms in DRAWINGS 11.15 at the end of this chapter, fill in side lengths and angle measurements, where possible.

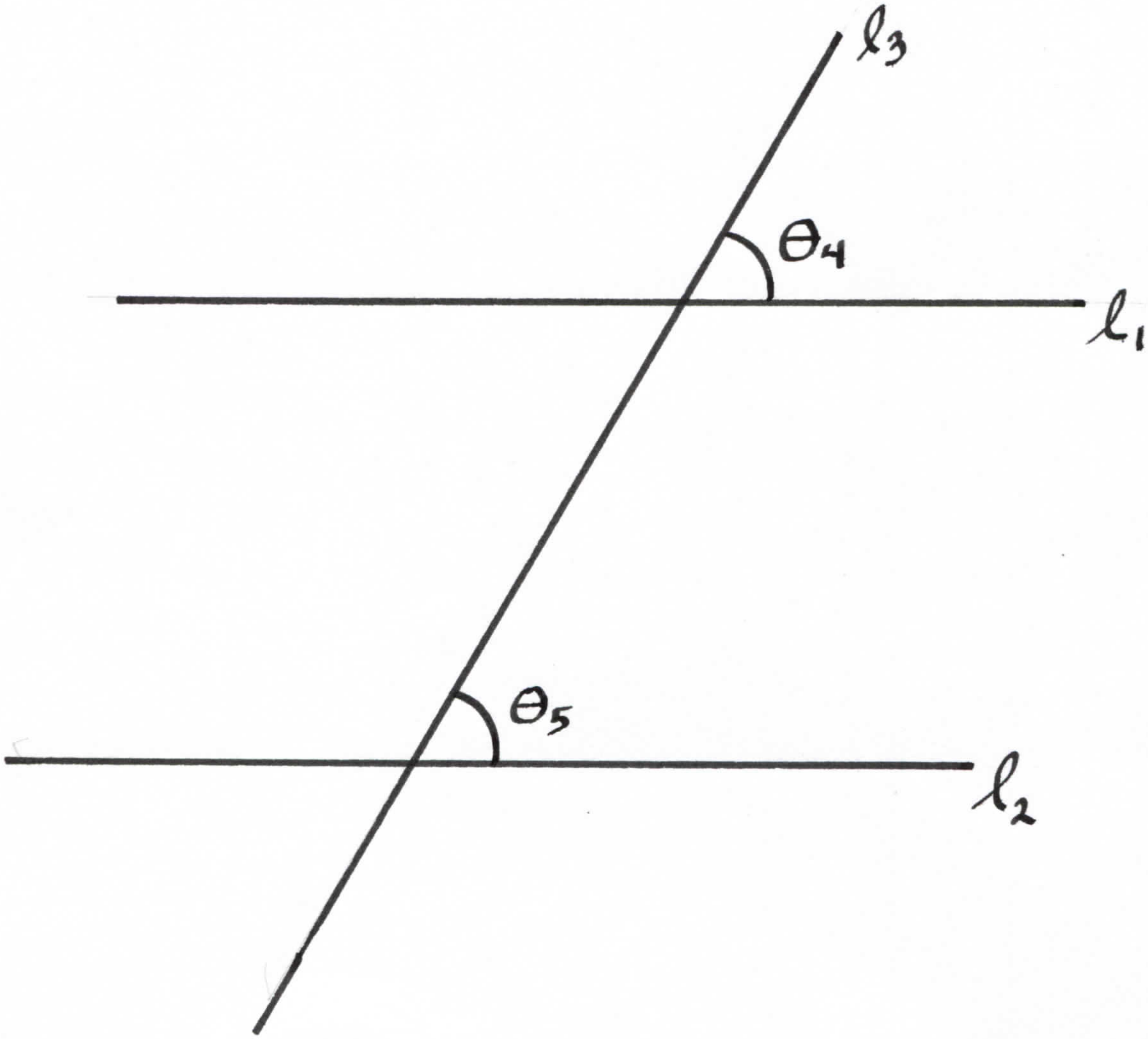
HWXI.2. Find x in each of the drawings of parallelograms in DRAWINGS 11.16 at the end of this chapter.

HOMEWORK ANSWERS

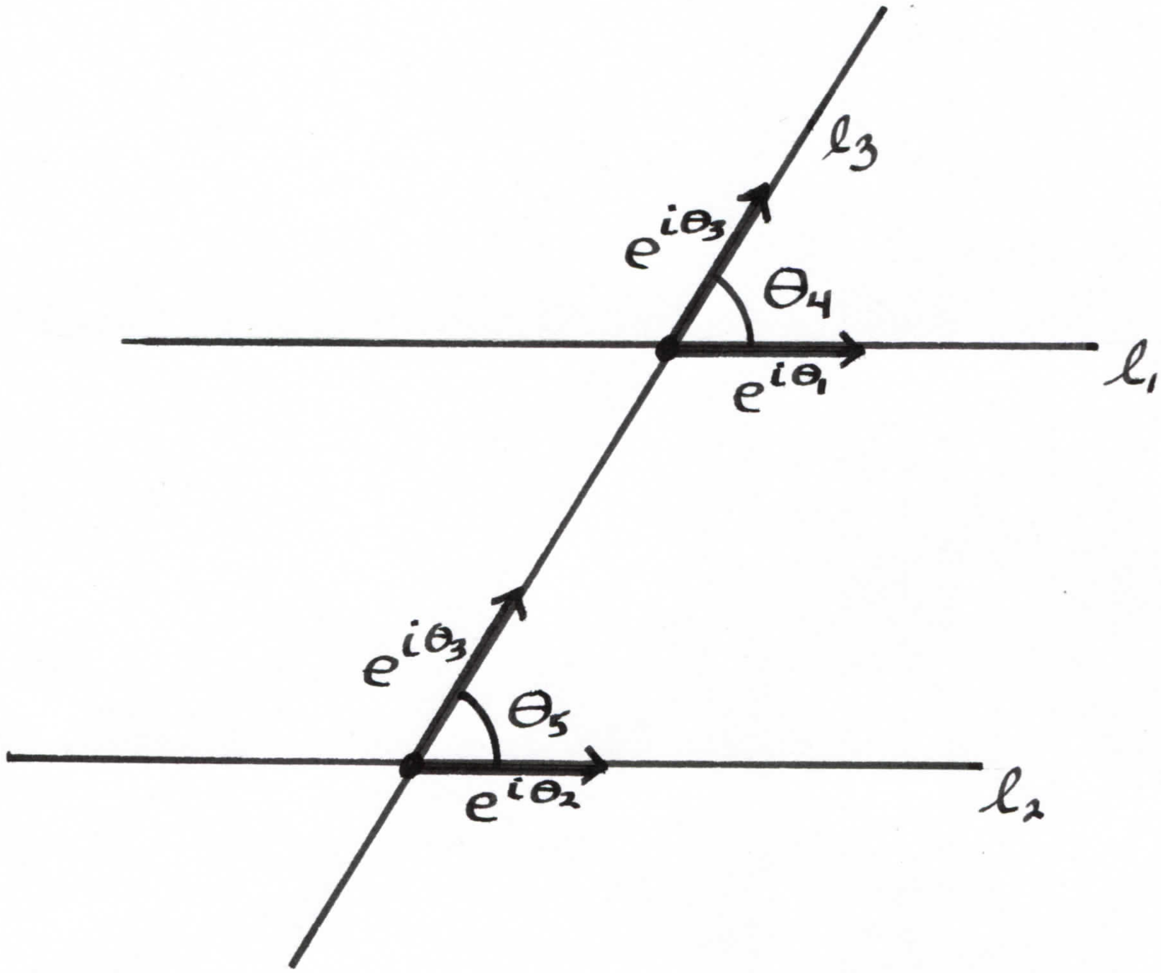
HWXI.1. See DRAWINGS 11.17 at the end of this chapter.

HWXI.2. (a) $\frac{\pi}{8}$ (b) $\frac{2\pi}{3}$ (c) 3 (d) 70 degrees (e) $\frac{\pi}{4}$ (f) $\frac{\pi}{6}$.

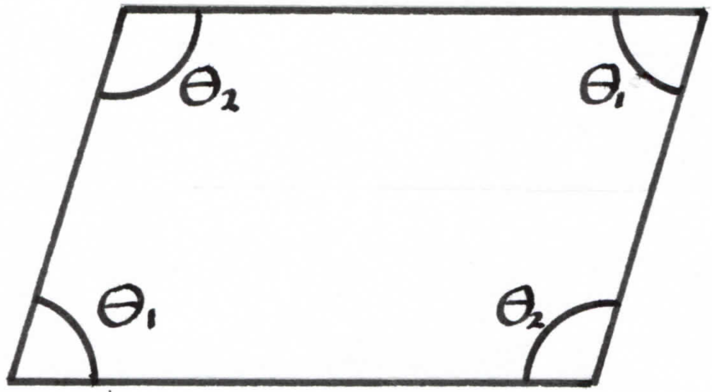
DRAWING 11.1



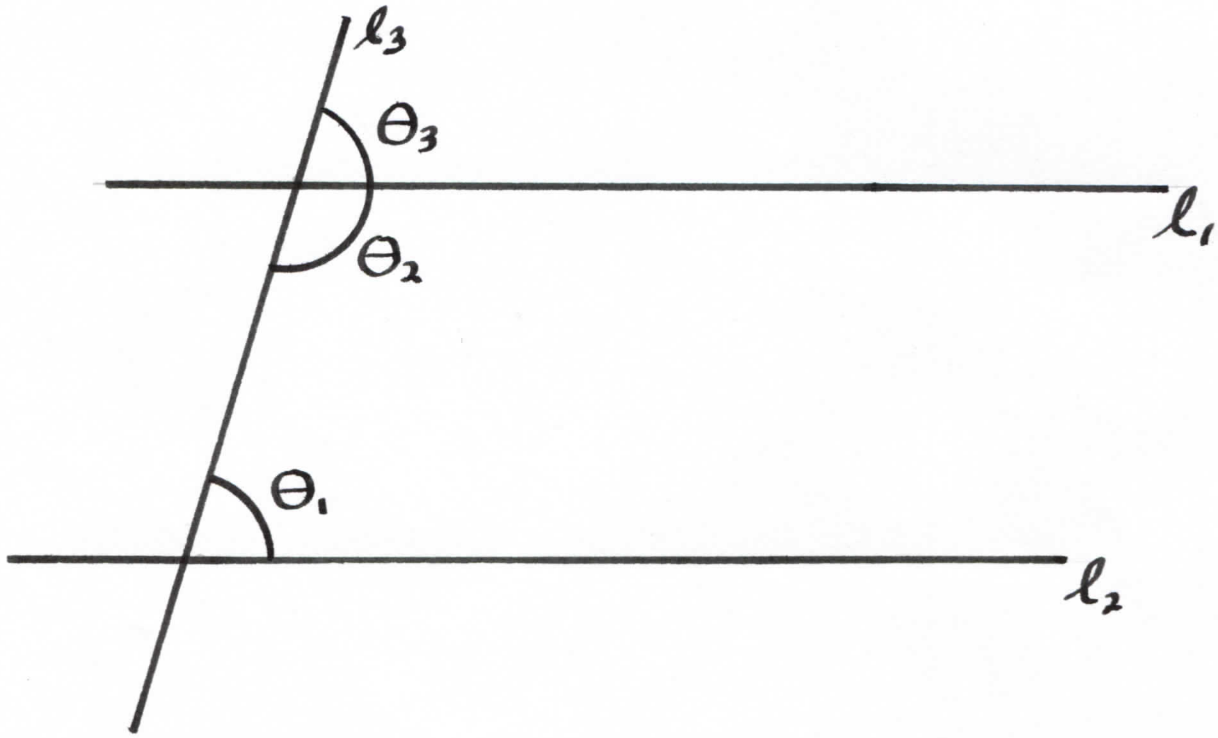
DRAWING 11.2


ℓ_3
$e^{i\theta_3}$
θ_4
ℓ_1
$e^{i\theta_1}$
$e^{i\theta_3}$
θ_5
ℓ_2
$e^{i\theta_2}$

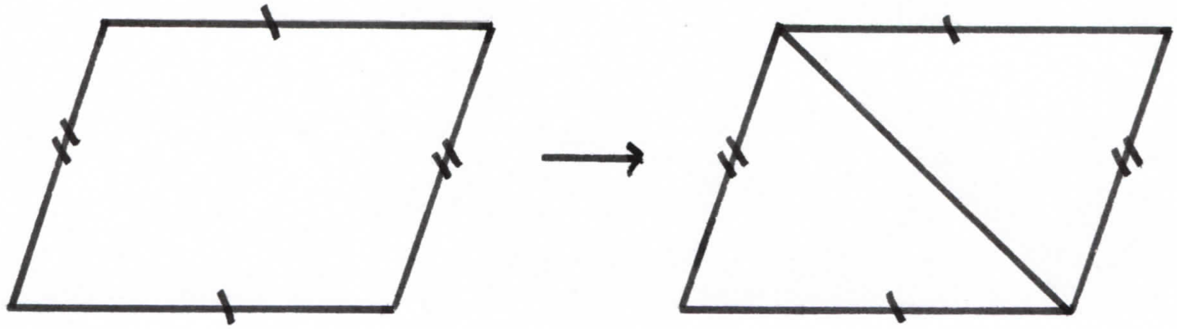
DRAWING 11.3



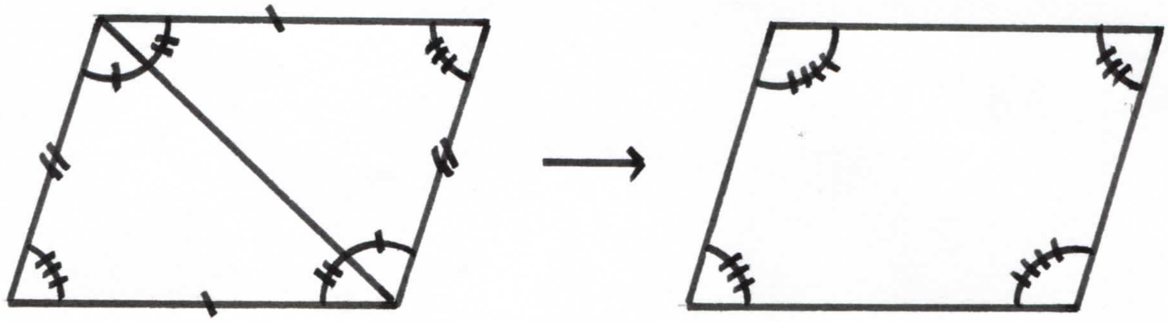
DRAWING 11.4



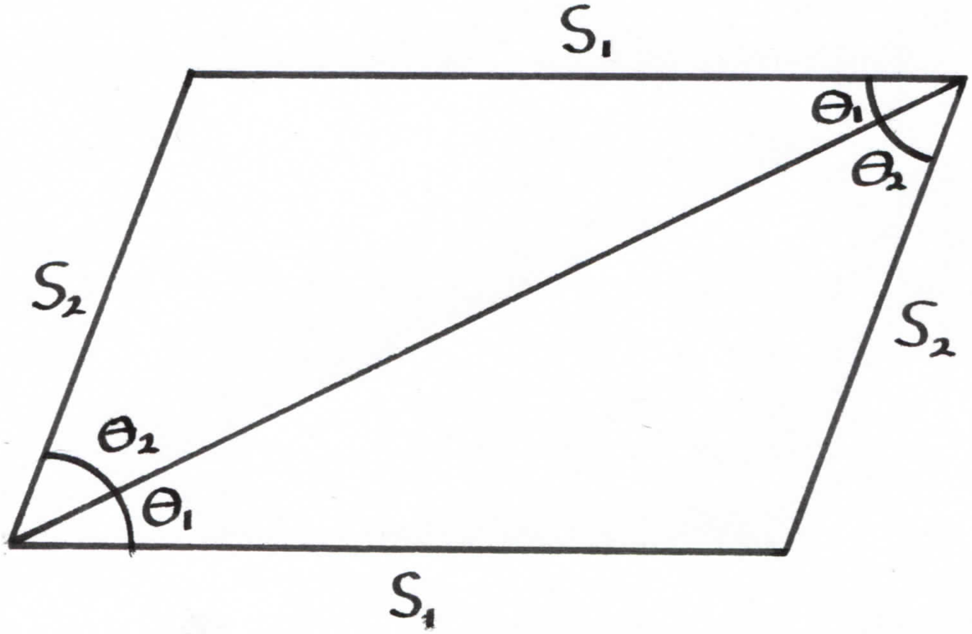
DRAWING 11.5



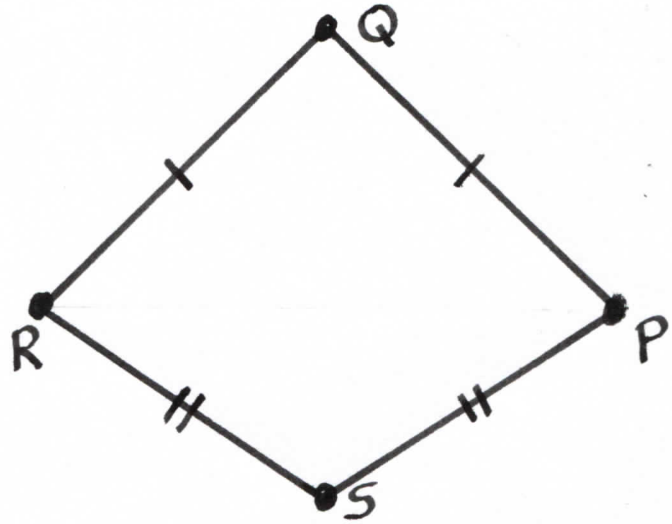
DRAWING 11.6



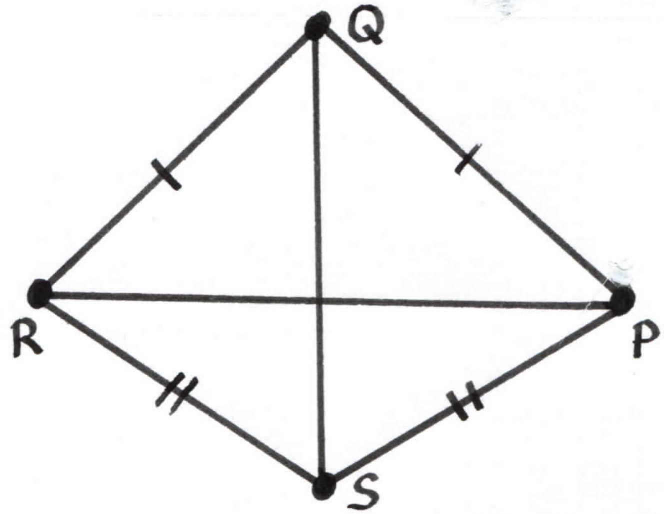
DRAWING 11.7



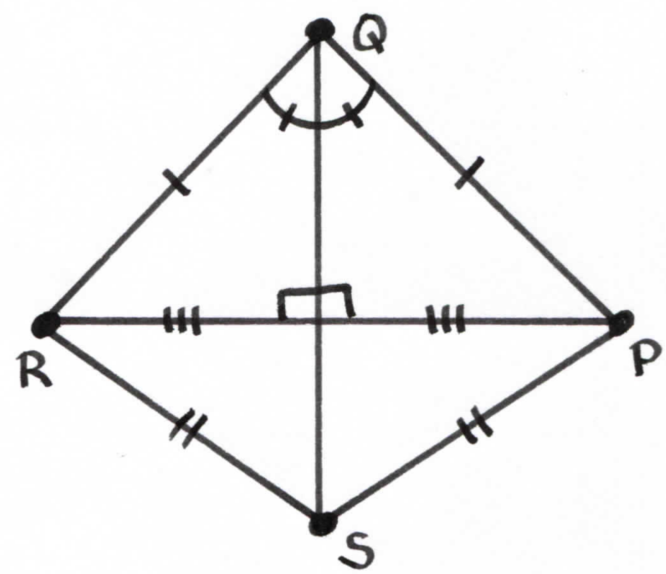
DRAWING 11.8



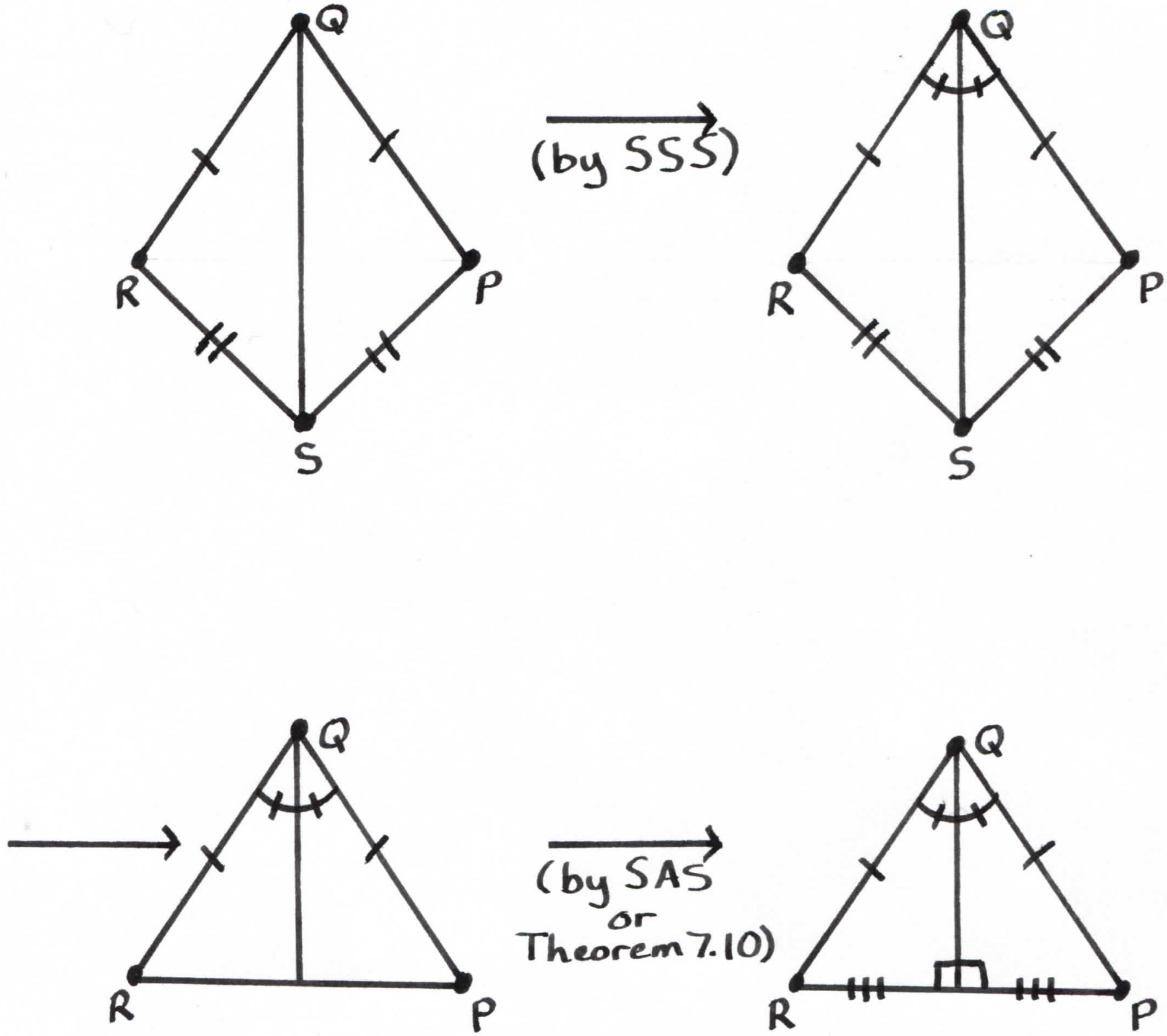
DRAWING 11.9



DRAWING 11.10

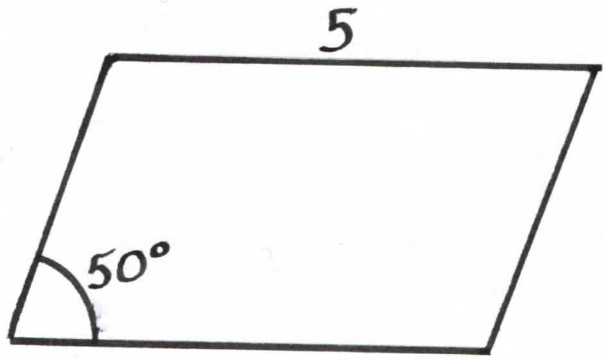


DRAWINGS 11.11

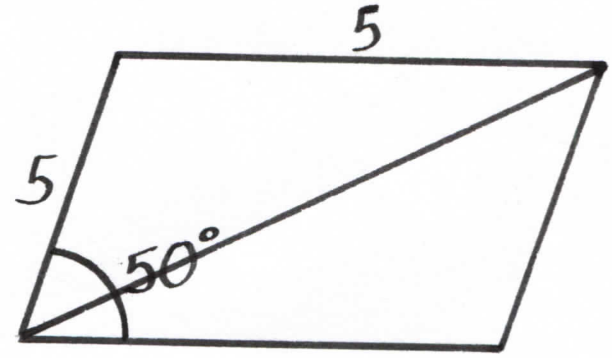


DRAWINGS 11.12

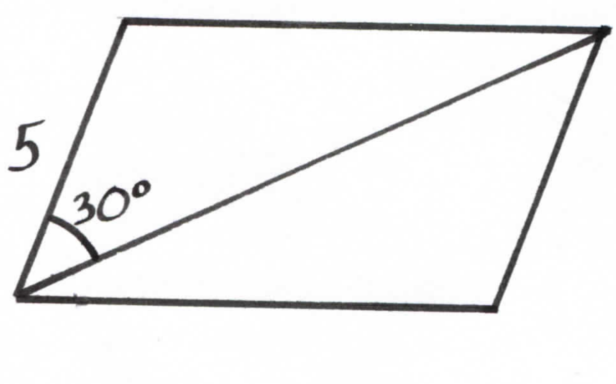
(a)



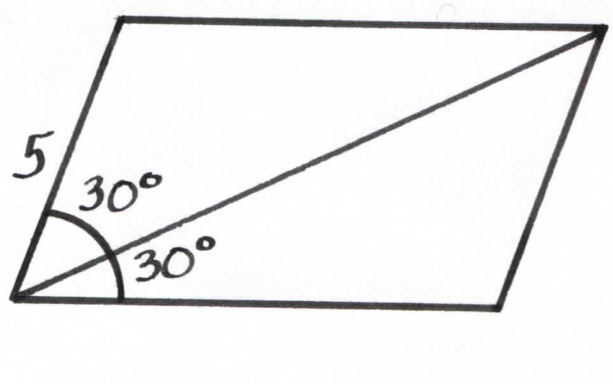
(b)



(c)

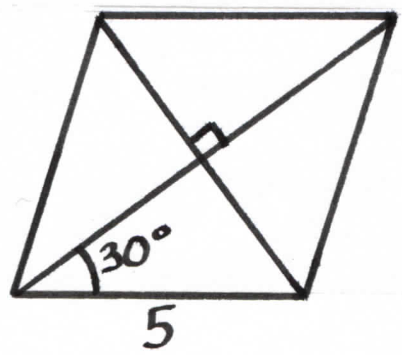


(d)

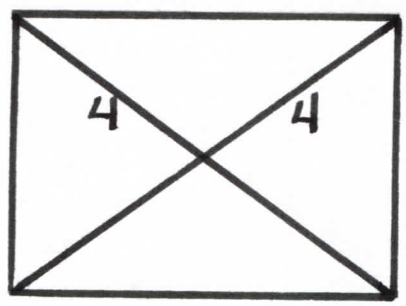


DRAWINGS 11.12 (continued)

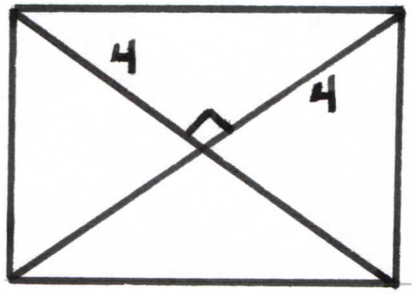
(e)



(f)

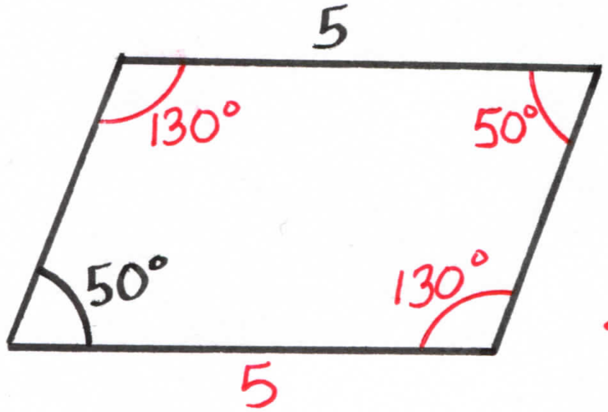


(g)

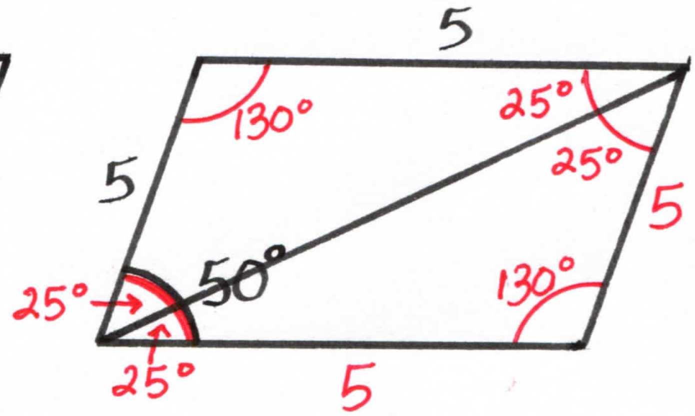


DRAWINGS 11.13

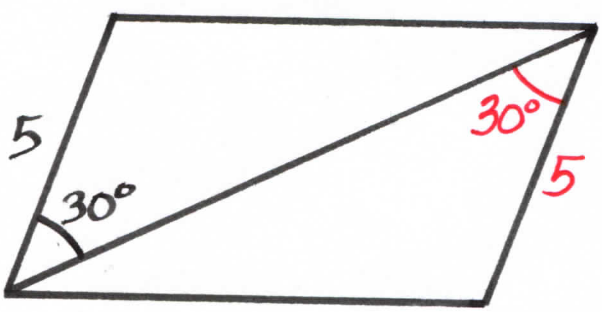
(a)



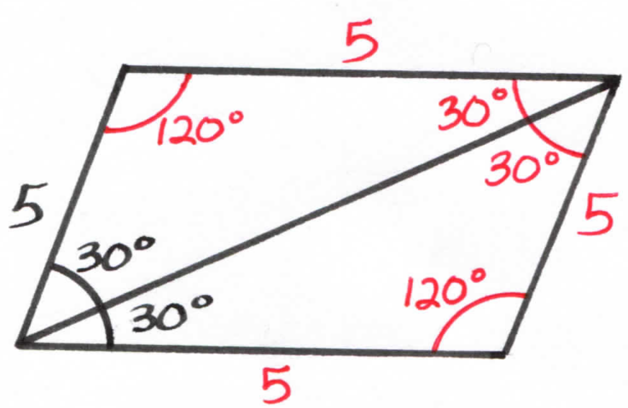
(b)



(c)

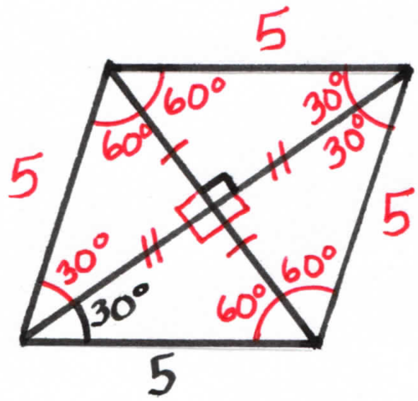


(d)



DRAWINGS 11.13 (continued)

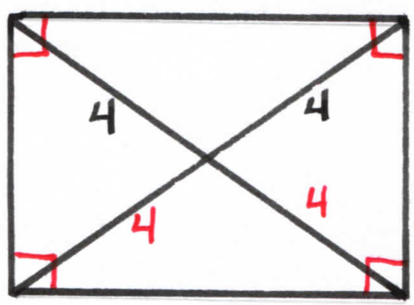
(e)



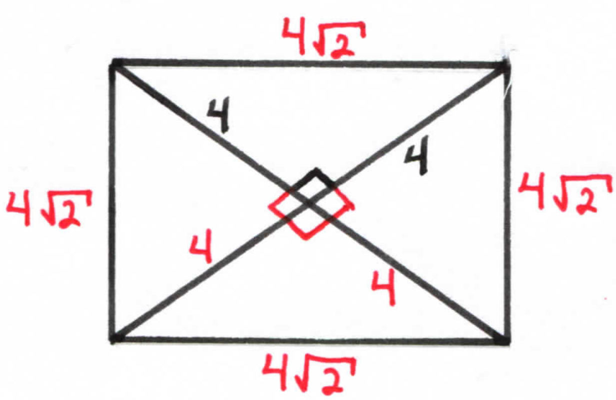
$/ = \frac{5}{2}$

$// = 5\left(\frac{\sqrt{3}}{2}\right)$

(f)



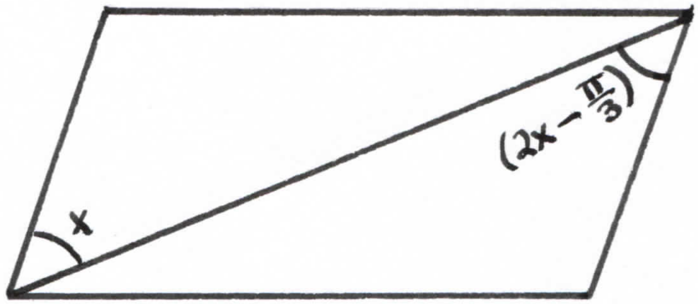
(g)



all angles at vertices of square are 45°

DRAWINGS 11.14

(a)



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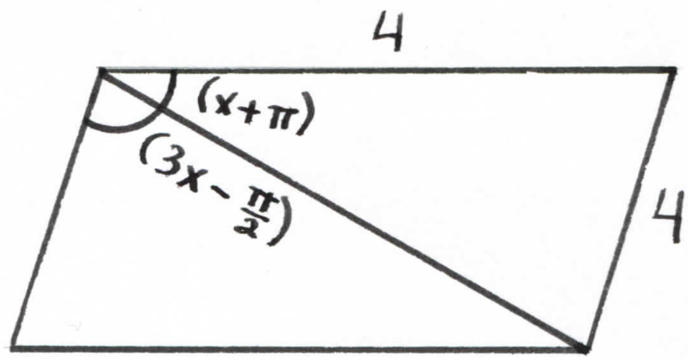

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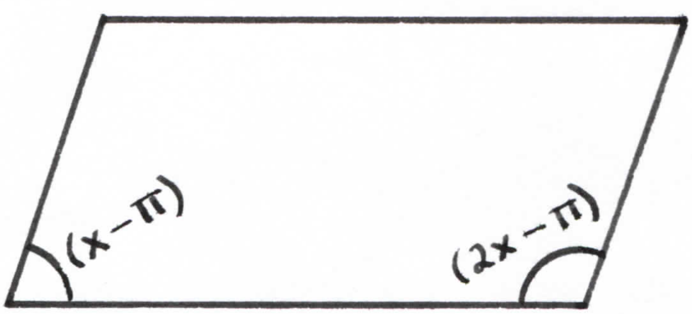
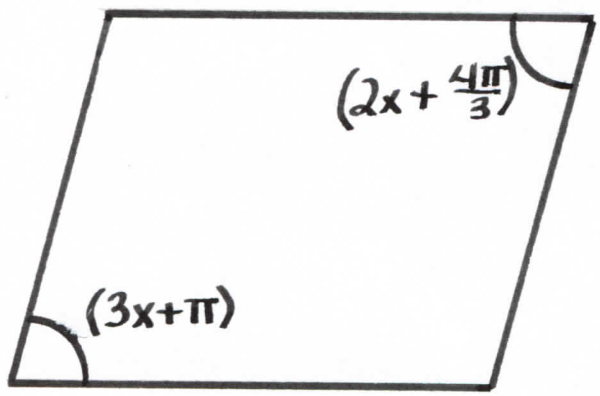
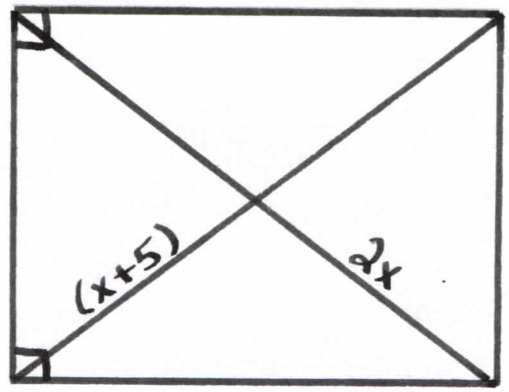


DRAWINGS 11.14 (continued)

(d)



(e)



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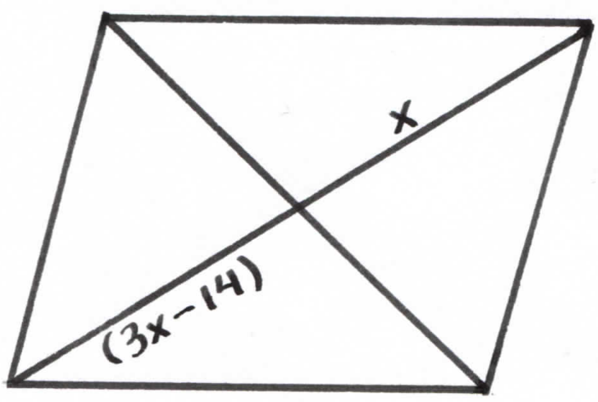
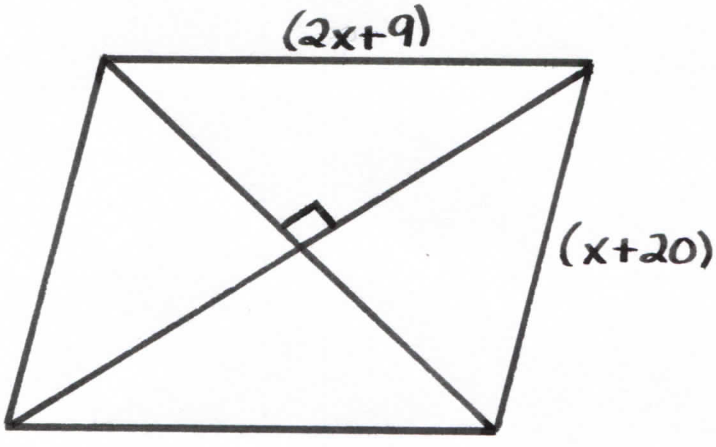
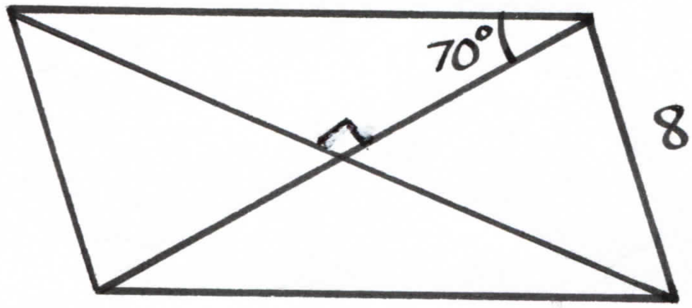


(g)



DRAWINGS 11.15

(a)



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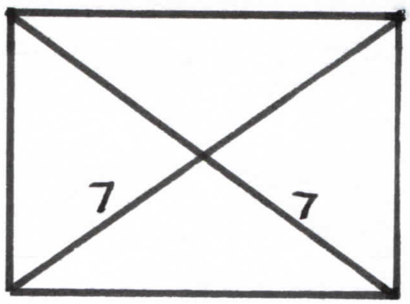

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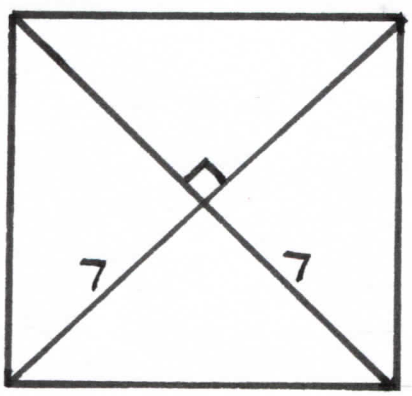
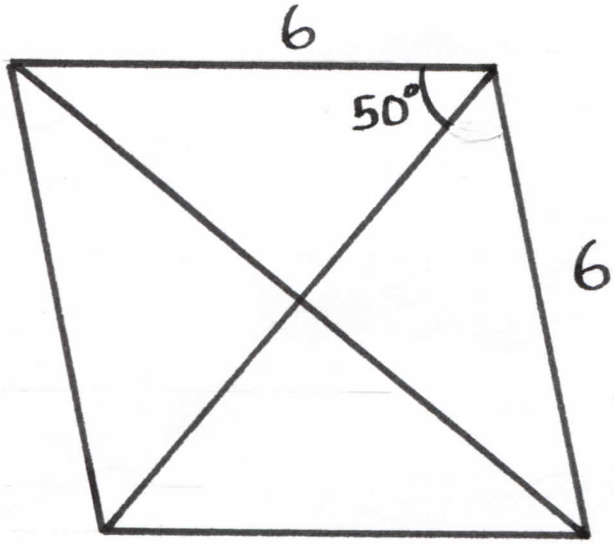
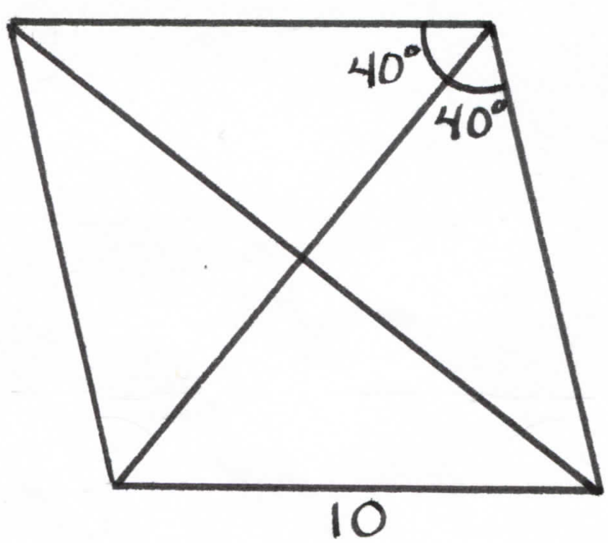


DRAWINGS 11.15 (continued)

(d)

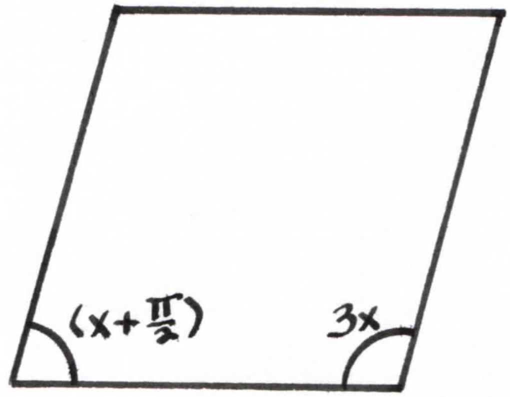


(e)

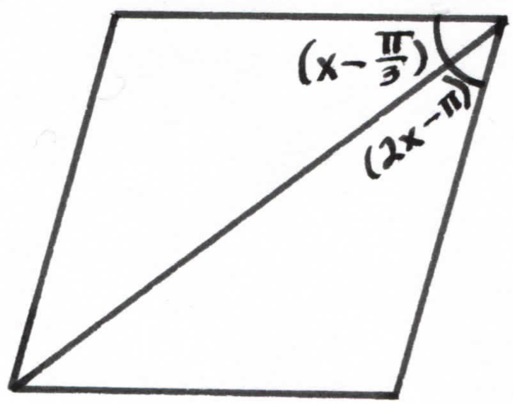


DRAWINGS 11.16

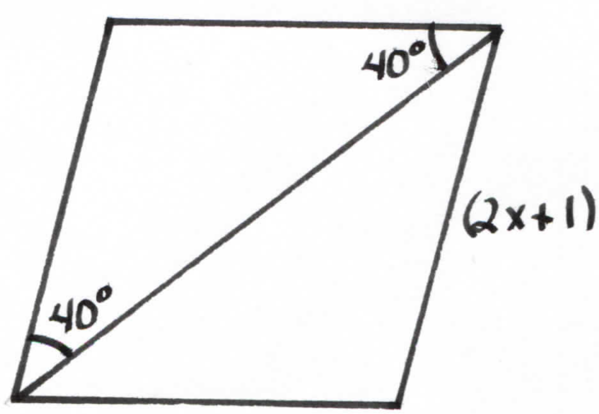
(a)



(b)

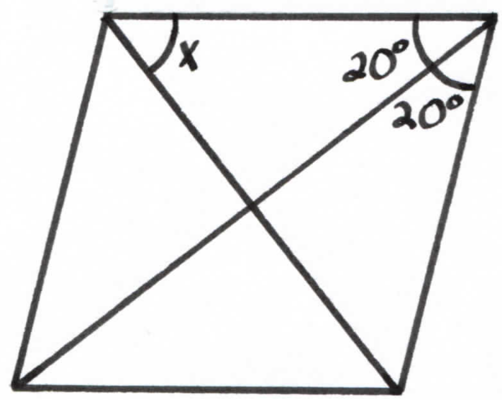


(c)

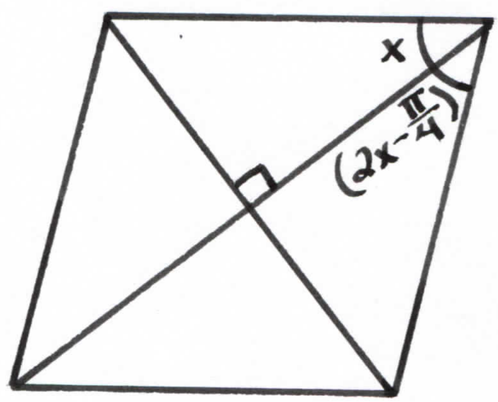


DRAWINGS 11.16 (continued)

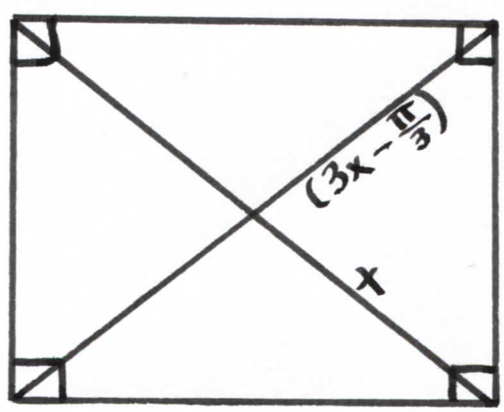
(d)



(e)

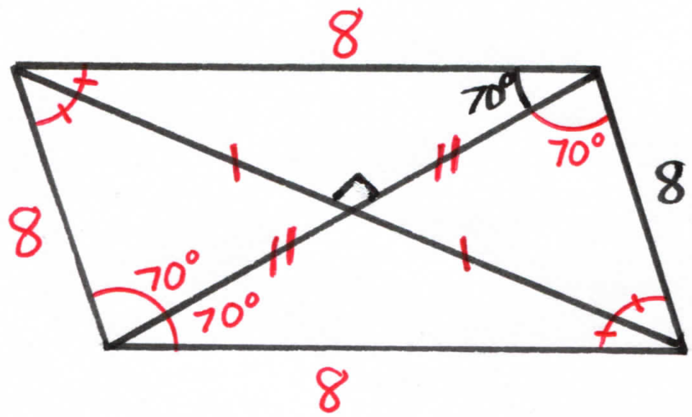


(f)



DRAWINGS 11.17

(a)

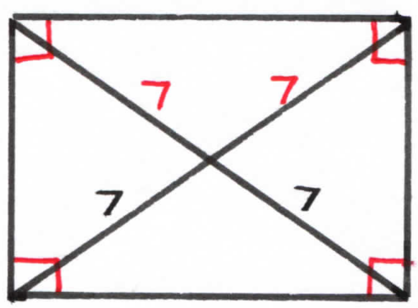


$/ = 8\sin(70°)$

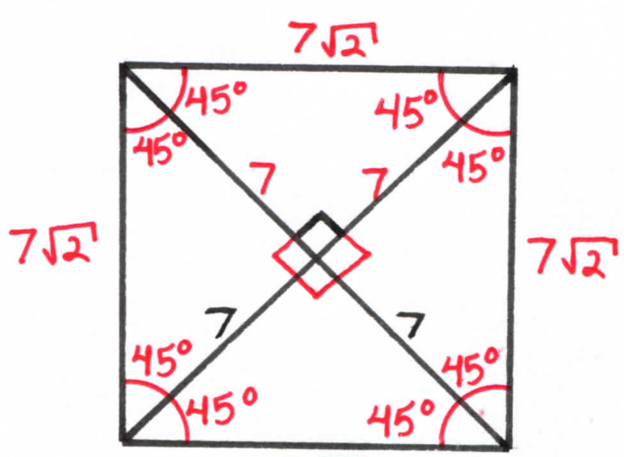
$// = 8\cos(70°)$

$\nmid = 20°$ (Proposition 3.9)

(b)

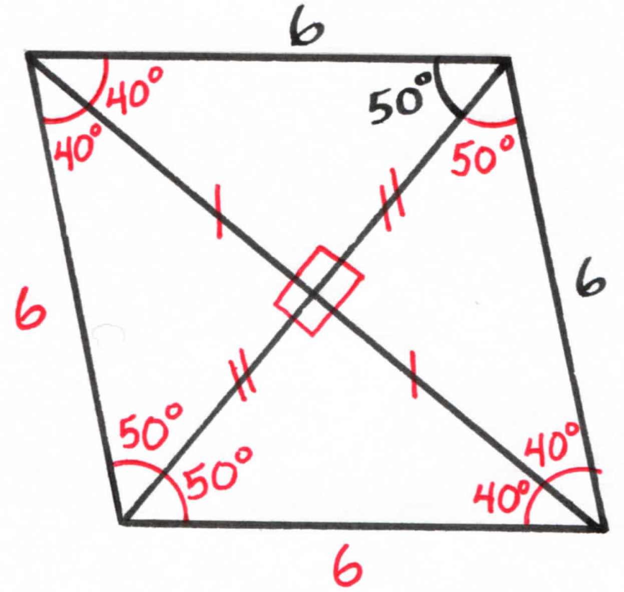


(c)



DRAWINGS 11.17 (continued)

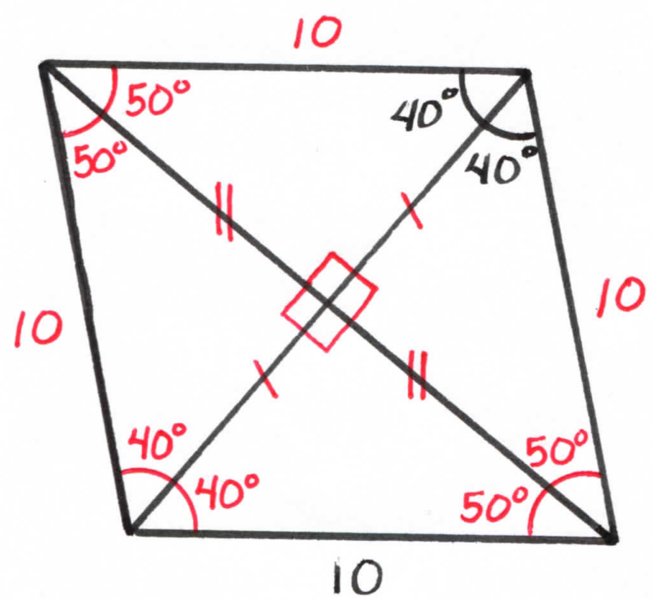
(d)



$/ = 6\sin(50°)$

$// = 6\cos(50°)$

(e)



$/ = 10\cos(40°)$

$// = 10\sin(40°)$

CHAPTER XII: Area and Volume.

In this chapter, we will first derive comfortable formulas for areas of triangles, parallelograms, trapezoids (Definitions 12.2) and closed sectors of discs (Definitions 2.18). We will assume nothing from the Appendices nor the results of Chapter IX for our area formulas. Theorem 8.3 will be assumed.

We will also define *volume* and give formulas for volumes of spheres, (generalized) cylinders, and (generalized) cones. Our presentation of volume will be less rigorous than we have done for length, area, and angle measure, and our formulas will require calculus, as in Appendix One, for their derivations.

Proposition 12.1. The area of either triangle formed by drawing a diagonal in a parallelogram, as in DRAWING 12.1 at the end of this chapter, is half the area of the parallelogram.

Proof: Label the vertices of the parallelogram P, Q, R, S, as in DRAWING 12.1 at the end of this chapter.

By SSS, triangle PQR is congruent to triangle PRS, thus they have equal area (Theorem 8.3). Since the area of the parallelogram is the sum of the areas of the two triangles, the result follows. □

Definitions 12.2. A **trapezoid** is a quadrilateral with (at least) one pair of opposite sides parallel. The **height** of a trapezoid is the distance between two parallel opposite sides (visualize the trapezoid with parallel sides horizontal), that is, the distance between any point on one of the parallel opposite sides to the line containing the opposite side; see Definition 5.1 and Theorem 5.2.

Theorem 12.3. The area of a trapezoid equals the height times the average of the lengths of the two parallel sides; that is,

$$\frac{1}{2}h(b_1 + b_2),$$

where h is the distance between the parallel sides of lengths b_1 and b_2, as in DRAWING 12.2 at the end of this chapter.

Proof: Label the vertices of the trapezoid P, Q, R, S, as in DRAWING 12.3 at the end of this chapter.

By translating, reflecting, rotating and relabeling the parallel sides if necessary (by Theorem 8.3 this will not change area), assume the parallel side of length b_2 is on the positive x axis, with the vertex R at the origin and the x coordinate of Q nonnegative.

Still in DRAWING 12.3, let

$$T \equiv \mathrm{proj}_{\overrightarrow{RS}}(Q), \quad U \equiv \mathrm{proj}_{\overrightarrow{RS}}(P),$$

and label lengths

$$h \equiv \|\overrightarrow{QT}\| = (\text{by Corollary 5.3})\|\overrightarrow{PU}\|, \quad c_1 \equiv \|\overrightarrow{RT}\|, \quad c_3 \equiv \|\overrightarrow{SU}\|,$$

$c_2 \equiv$ the remaining horizontal distance on the base of the trapezoid in DRAWING 12.3 at the end of this chapter.

There are three relevant areas to manipulate:

$$A_1 \equiv \text{area of the triangle with vertices } Q, R, T,$$

$$A_2 \equiv \text{ area of the quadrilateral with vertices} P, Q, T, U,$$

and

$$A_3 \equiv \text{area of the triangle with vertices } P, S, U.$$

Since $\overrightarrow{QT}$ and $\overrightarrow{TS}$ are orthogonal, while Theorem 5.2 implies that $\overrightarrow{QT} = \overrightarrow{PU}$, the quadrilateral with vertices P, Q, T, U is a rectangle, thus we may use Definitions 2.4(ii) and Proposition 12.1 for all three areas, to get

$$A_1 = \frac{1}{2}c_1 h, \quad A_2 = hb_1 \quad \text{and} \quad A_3 = \frac{1}{2}c_3 h.$$

We now need three cases, the case depending on the relationship, in DRAWING 12.3 at the end of this chapter, between S and U.

CASE 1: S to right of U (see DRAWING 12.4 at the end of this chapter).

$$\text{area of trapezoid} = A_1 + A_2 + A_3$$
$$= \frac{1}{2}c_1 h + hb_1 + \frac{1}{2}c_3 h = \frac{1}{2}h(c_1 + 2b_1 + c_3) = \frac{1}{2}h(b_1 + (c_1 + b_1 + c_3)) = \frac{1}{2}h(b_1 + b_2),$$

as desired.

CASE 2: S to left of U (see DRAWING 12.5 at the end of this chapter).

$$\text{area of trapezoid} = A_1 + A_2 - A_3$$
$$= \frac{1}{2}c_1 h + hb_1 - \frac{1}{2}c_3 h = \frac{1}{2}h(c_1 + 2b_1 - c_3) = \frac{1}{2}h(b_1 + (c_1 + b_1 - c_3)) = \frac{1}{2}h(b_1 + b_2),$$

as desired.

CASE 3: S equals U (see DRAWING 12.6 at the end of this chapter).

$$\text{area of trapezoid} = A_1 + A_2$$
$$= \frac{1}{2}c_1 h + hb_1 = \frac{1}{2}h(c_1 + 2b_1) = \frac{1}{2}h(b_1 + (c_1 + b_1)) = \frac{1}{2}h(b_1 + b_2),$$

as desired. $\square$

Notice, in our formula for trapezoid in Theorem 12.3, we get a formula for area of a triangle by setting $b_1 = 0$, which in turn, by Proposition 12.1, gives us a formula for area of a parallelogram (see DRAWINGS 12.7 at the end of this chapter and HW XII.1).

See Definitions 2.18 for the definitions of arc of a circle and sector of a disc.

Theorem 12.4. If $0 < \psi \leq 2\pi$ and a closed sector of a disc of radius R is determined by an arc of length $R\psi$, then the sector has area $\frac{1}{2}R^2\psi$. See DRAWING 12.8 at the end of this chapter.

Proof: By Theorem 8.3, we may assume that $R = 1$ and the disc is centered at the origin, so that our closed disc is

$$\{(x, y) \,|\, x^2 + y^2 \leq 1\} = \{re^{i\phi} \,|\, 0 \leq \phi \leq 2\pi,\, 0 \leq r \leq 1\}.$$

Again by Theorem 8.3 we may assume that the arc is counterclockwise from $(1, 0)$; that is, our arc is

$$\{e^{i\phi} \,|\, 0 \leq \phi \leq \psi\};$$

denote by S_ψ the closed sector determined by this arc.

Let $\theta \equiv \frac{\psi}{4}$. By Theorem 8.3 yet again, since our closed sector is the union of four closed sectors of equal area overlapping only on curves (for each sector, rotate by θ the closed sector determined by an arc of length θ), the area of our closed sector is four times the area of the closed sector determined by the arc

$$\{e^{i\phi} \,|\, 0 \leq \phi \leq \theta\}.$$

Let S_θ be the closed sector determined by the arc of length θ just described. See DRAWING 12.9 at the end of this chapter.

By Proposition APP1.9, the area of S_θ is $\frac{\theta}{2}$, thus the area of S_ψ is $4\left(\frac{\theta}{2}\right) = \frac{1}{2}\psi$, as desired. $\square$

Examples 12.5. In each of the figures in DRAWINGS 12.10 at the end of this chapter, get the area. All quadrilaterals are trapezoids; all curves that are not line segments are arcs of a circle.

Solutions. (a) By Proposition 12.1 the area of the unshaded triangle is 11 and the area of the parallelogram is 22.

(b) By Theorem 12.3, the area of the trapezoid is $\frac{1}{2} \times 10 \times (12 + 15) = 135$.

(c) By Theorem 12.4, the area is $\frac{1}{2}5^2\pi = \frac{25\pi}{2}$.

(d) By Theorem 12.4, the area is $\frac{1}{2}(10 \text{ feet})^2 \frac{\pi}{3} = \frac{50\pi}{3}$ feet squared.

(e) Here $R\frac{4\pi}{3} = 40\pi$, so the radius R is 30, thus by Theorem 12.4 the area is $\frac{1}{2}(30)^2 \frac{4\pi}{3} = \frac{1800\pi}{3} = 600\pi$.

Definitions 12.6. To discuss volume, we must consider, very analogously to Definition 0.1, $\mathbf{R}^3$ (reads "**R** three"), the set of all ordered triples of real numbers

$$\{(a, b, c) \,|\, a, b, c \text{ are real numbers}\}.$$

As with $\mathbf{R}^2$, the number a is the **x coordinate** of (a, b, c), b is the **y coordinate**; now, in addition, c is the **z coordinate**.

The **x-axis** is now the line $\{(x, 0, 0) \,|\, x \text{ is real}\}$, the **y-axis** is now the line $\{(0, y, 0) \,|\, y \text{ is real}\}$, and we additionally get the **z-axis** $\{(0, 0, z) \,|\, z \text{ is real}\}$.

The **xy plane** $\equiv \{(a, b, 0) \,|\, a, b \text{ are real numbers}\}$. Compare this to the definition of $\mathbf{R}^2$ (Definitions 0.1).

When we think of $\mathbf{R}^2$ or the xy plane as the surface of a flat, infinite earth, $\mathbf{R}^3$ becomes space, with the z coordinate measuring how far above the earth we are. Negative values of z place you below the surface of the earth. See DRAWING 12.11 at the end of this chapter.

Very analogously to area (Definitions 2.4(ii)), the **volume** of the set

$$[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \equiv \{(x, y, z) \,|\, a_1 \le x \le b_1, a_2 \le y \le b_2, a_3 \le z \le b_3\} \quad (b_k \ge a_k, k = 1, 2, 3)$$

is defined to be $(b_1 - a_1)(b_2 - a_2)(b_3 - a_3)$.

When $(b_1 - a_1) = (b_2 - a_2) = (b_3 - a_3)$, then $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ is a **cube**. In general, $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ is a three-dimensional analogue of a rectangle, sometimes called a **box**. See DRAWING 12.12 at the end of this chapter.

See Definitions APP1.10 for calculating more general volume.

Definitions 12.7. Here are the subsets of $\mathbf{R}^3$ of interest. Throughout, H and r are positive real numbers.

(a) A **cylinder**, of height H and radius r, is

$$\{(x, y, z) \,|\, 0 \le z \le H,\, x^2 + y^2 \le r^2\}.$$

See DRAWING 12.13 at the end of this chapter.

This is like the soup inside a metal can of soup that you might buy at a grocery store. The geometric property of interest is that, if you freeze the soup, then cut the can parallel to the base, you will always get the same cross section of frozen soup staring at you; in this case, the disc $x^2+y^2 \leq r$.

The cylinder is formed by putting many identical discs of infinitesimal thickness on top of each other.

We might want a more exotic cross section repeating itself.

(b) Suppose Ω is a subset of $\mathbf{R}^2$ and $H > 0$. Then

$$\{(x, y, z) \,|\, (x, y) \text{ is in } \Omega,\ 0 \leq z \leq H\}$$

is a **generalized cylinder** of height H with base Ω.

We visualize here copies of Ω stacked up to a height of H. See DRAWING 12.14 at the end of this chapter.

(c) A **cone**, of height H and radius r, is

$$\{(x, y, z) \,|\, 0 \leq z \leq H,\ x^2 + y^2 \leq \left(\frac{H-z}{H}\right)^2 r^2\}.$$

See DRAWING 12.15 at the end of this chapter.

As with (a), every cross section parallel to the xy plane is a disc, but the radii shrink linearly as z increases: the radius of the cross section at z is $\left(\frac{H-z}{H}\right) r$.

The top of this cone is the point $(0, 0, H)$, the base is the disc $x^2 + y^2 \leq r^2$ in the xy plane, as with a cylinder, and the rest of the cone is formed by throwing in all line segments from the top to the base.

Note that we could equivalently define a cone of height H and radius r as

$$\{(\left(\frac{H-z}{H}\right) x, \left(\frac{H-z}{H}\right) y, z) \,|\, 0 \leq z \leq H,\ x^2 + y^2 \leq r^2\}.$$

The base of this cone is the disc $\{(x, y) \,|\, x^2 + y^2 \leq r^2\}$.

As with a cylinder, we might want to vary the base of our cone. Note the similarity to the generalized cylinder in the following definition.

(d) Suppose Ω is a subset of $\mathbf{R}^2$ and $H > 0$. Then

$$\{(\left(\frac{H-z}{H}\right) x, \left(\frac{H-z}{H}\right) y, z) \,|\, (x, y) \text{ is in } \Omega,\ 0 \leq z \leq H\}$$

is a **generalized cone** of height H with base Ω.

Here the base is Ω in the xy plane, and, as z increases from 0 to H, the cross sections are crushed by the factor $\left(\frac{H-z}{H}\right)$ (what we called a *magnification* in Definitions 8.1(d)), up to the top $(0, 0, H)$.

As with (c), we can form our generalized cone by including all line segments from the top to the base.

When Ω is a polygon (Definitions 2.3), the generalized cone is called a **pyramid**. When Ω is a triangle, the generalized cone is called a **tetrahedron**.

(e) A **ball** of radius r, centered at a point (a, b, c), is

$$\{(x, y, z) \,|\, (x-a)^2 + (y-b)^2 + (z-c)^2 \leq r^2\}.$$

The following theorem refers to Definitions 12.7. Notice that passing from (generalized) cylinder to (generalized) cone always multiplies the volume by $\frac{1}{3}$.

Theorem 12.8. As in Definitions 12.7, $H > 0, r > 0$, and Ω is a subset of $\mathbf{R}^2$.

(a) The volume of the cylinder of height H and radius r is $\pi r^2 H$.

(b) The volume of the generalized cylinder of height H and base Ω is (area of Ω)H.

(c) The volume of the cone of height H and radius r is $\frac{\pi}{3} r^2 H$.

(d) The volume of the generalized cone of height H and base Ω is $\frac{1}{3}$(area of Ω)H.

(e) The volume of a ball of radius r is $\frac{4}{3}\pi r^3$.

Proof: APP1.11. $\square$

Examples 12.9. Get the volumes of each of the following.

(a) A ball of radius 6.

(b) A cylinder of height 10 and radius 5.

(c) A generalized cylinder with height 8 and base the triangle with vertices $(-1, 0, 0), (1, 3, 0)$, and $(4, 0, 0)$.

(d) A tetrahedron with vertices $(-1, 0, 0), (1, 3, 0), (4, 0, 0)$, and $(0, 0, 8)$.

Solutions. (a) $\frac{4}{3}\pi 6^3 = 288\pi$.

(b) $\pi(5^2)(10) = 250\pi$.

(c) Since the triangle has height 3 and base 5 (see DRAWING 12.16 at the end of this chapter), its area is $\frac{1}{2}(3)(5) = \frac{15}{2}$, thus the desired volume is $(\frac{15}{2})(8) = 60$.

(d) This is the generalized cone for the same height and base as the generalized cylinder of (c), thus the volume of the generalized cone is $(\frac{1}{3})(60) = 20$.

HOMEWORK

HWXII.1. Use Theorem 12.3 to prove the following, as drawn in DRAWINGS 12.7 at the end of this chapter.

(a) The area of a triangle is one-half the base times the height.

(b) The area of a parallelogram is the base times the height.

HWXII.2. Get the shaded areas in DRAWING 2.21 and DRAWING 2.25, both at the end of Chapter II.

HWXII.3. Get the area of the trapezoid (a), the area of the parallelogram (b), and the area of the shaded triangle in (b), all in DRAWINGS 12.17 at the end of this chapter.

HWXII.4. Get the volumes of each of the following.

(a) A cylinder of height 12 meters and radius 3 meters.

(b) A generalized cylinder with height 10 and base the parallelogram formed by $< 1, 2 >$ and $< 3, 4 >$. Use Proposition 9.7.

(c) A tetrahedron with height 10 and base the parallelogram formed by $< 1, 2 >$ and $< 3, 4 >$.

(d) A ball of radius 10 feet.

(e) A ball such that the area of the disc, whose boundary is the equator, is 100 meters squared. (By *equator* we mean that if, after translation, the ball is $\{(x, y, z) \,|\, x^2 + y^2 + z^2 \leq R^2\}$, then the equator is $\{(x, y, 0) \,|\, x^2 + y^2 = R^2\}$, the boundary of the disc $\{(x, y, 0) \,|\, x^2 + y^2 \leq R^2\}$.)

HOMEWORK ANSWERS

HWXII.1. (b) follows from Theorem 12.3, with $b_1 = b_2 = b$. (a) now follows from Corollary APP1.6 or Proposition 12.1.

HWXII.2. The shaded area in DRAWING 2.21 is $\frac{1}{2} \times 10^2 \times 240(\frac{\pi}{180}) = \frac{200\pi}{3}$. The shaded area in DRAWING 2.25 is $\frac{1}{2} \times (8 \text{ ft })^2 \times \frac{11\pi}{12} = \frac{88\pi}{3}$ ft squared .

HWXII.3. (a) $\frac{1}{2} \times 6 \times (16 + 11) = 81$.

(b) $6 \times 13 = 78$.

(c) $\frac{1}{2} \times 78 = 39$.

HWXII.4. (a) $\pi \times (12 \text{ meters}) \times (3 \text{ meters})^2 = 108\pi$ meters cubed.

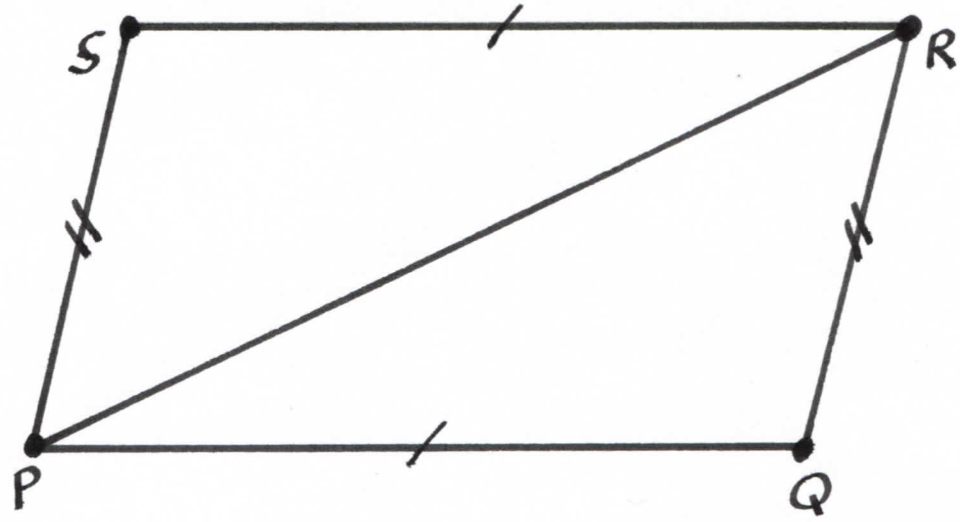
(b) The area of the base is (see Proposition 9.7) $|\det \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}| = 2$, so our volume is 20.

(c) $\frac{20}{3}$.

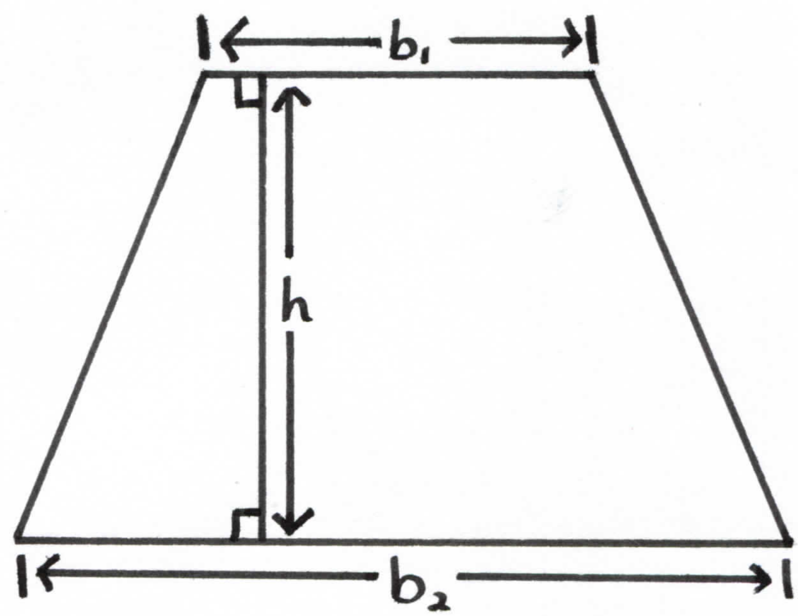
(d) $\frac{4\pi}{3}(10 \text{ feet})^3 = \frac{4{,}000\pi}{3}$ feet cubed.

(e) Let R be the radius of the ball. We have $100 = \pi R^2$, so $R = \frac{10}{\sqrt{\pi}}$ meters, thus our volume is $\frac{4}{3}\pi \left(\frac{10}{\sqrt{\pi}}\right)^3$ meters cubed $= \frac{4{,}000}{3\sqrt{\pi}}$ meters cubed.

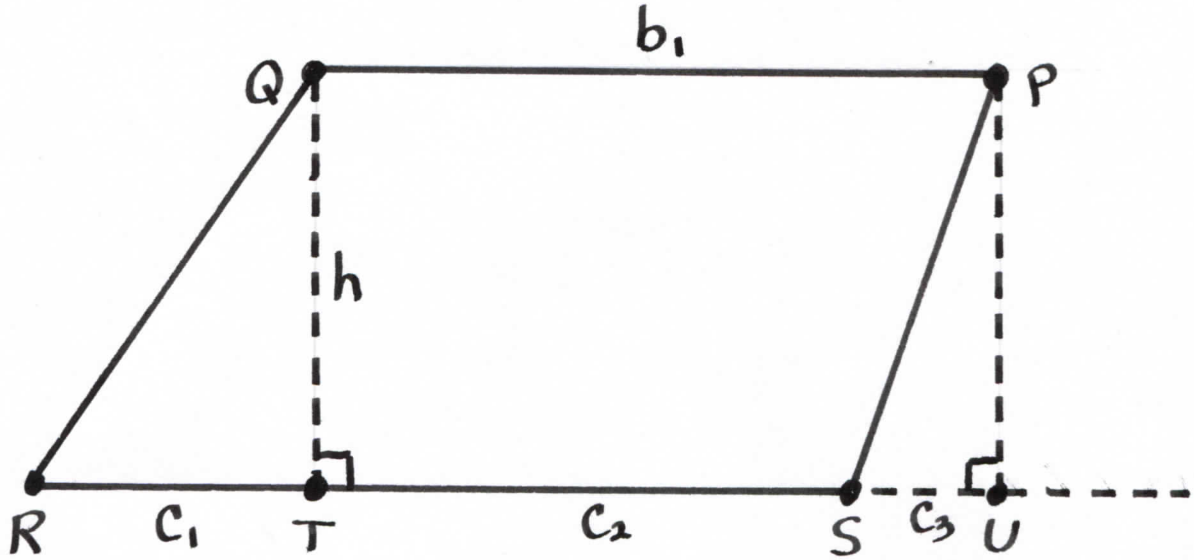
DRAWING 12.1



DRAWING 12.2



DRAWING 12.3



$T \equiv \text{proj}_{\overrightarrow{RS}}(Q)$; $U \equiv \text{proj}_{\overrightarrow{RS}}(P)$

$A_1 \equiv$ area of $QRT = \frac{1}{2} c_1 h$

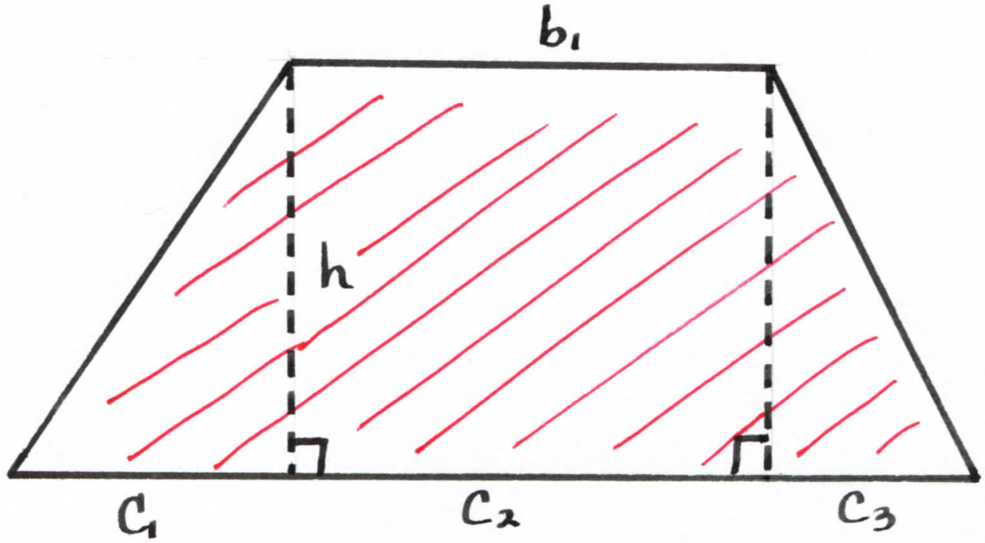
$A_2 \equiv$ area of $PQTU = h b_1$

$A_3 \equiv$ area of $PSU = \frac{1}{2} c_3 h$

b_2 = base of trapezoid

DRAWING 12.4

CASE 1: S to right of U



$$\longrightarrow A = A_1 + A_2 + A_3$$

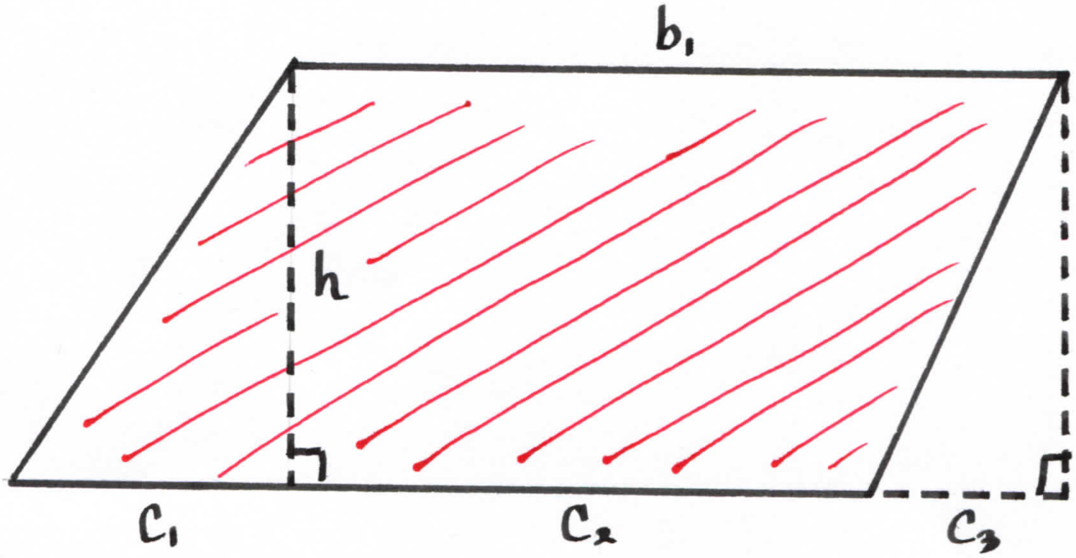
$$= \tfrac{1}{2}h(c_1 + 2b_1 + c_3)$$

$$= \tfrac{1}{2}h(b_1 + (c_1 + b_1 + c_3))$$

$$= \tfrac{1}{2}h(b_1 + b_2)$$

DRAWING 12.5

CASE 2: S to left of U



$$\longrightarrow \quad A = A_1 + A_2 - A_3$$

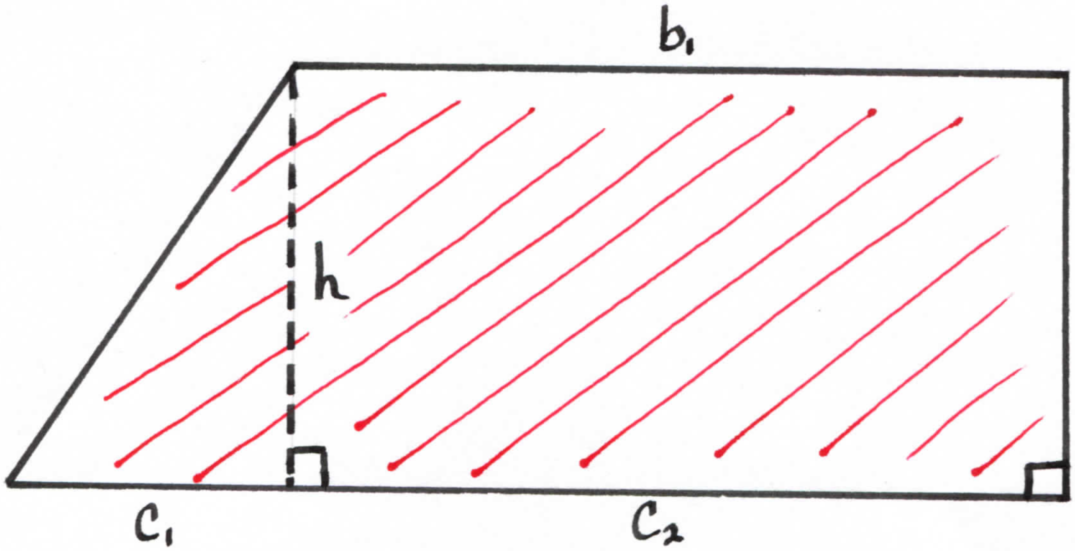
$$= \tfrac{1}{2}h\,(c_1 + 2b_1 - c_3)$$

$$= \tfrac{1}{2}h(b_1 + (c_1 + b_1 - c_3))$$

$$= \tfrac{1}{2}h\,(b_1 + b_2)$$

DRAWING 12.6

CASE 3: $S = U$



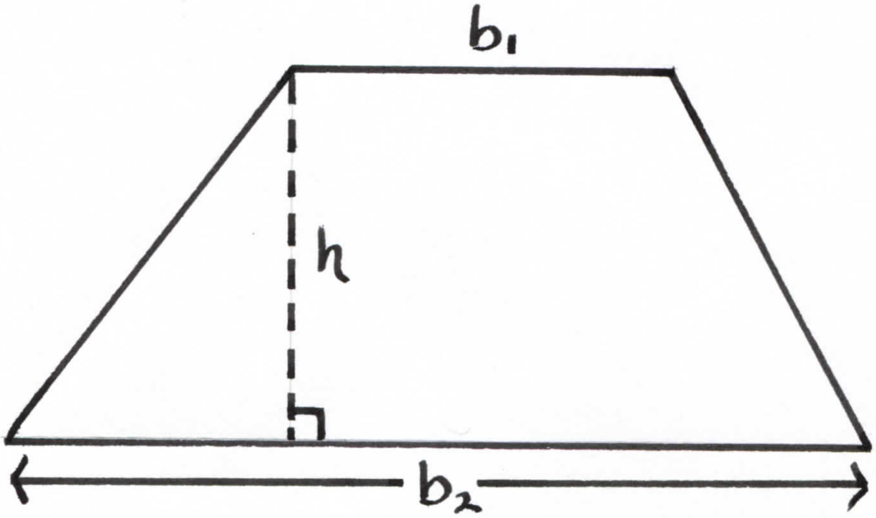
$$\longrightarrow \quad A = A_1 + A_2$$

$$= \tfrac{1}{2} h \,(c_1 + 2 b_1)$$

$$= \tfrac{1}{2} h \,(b_1 + (c_1 + b_1))$$

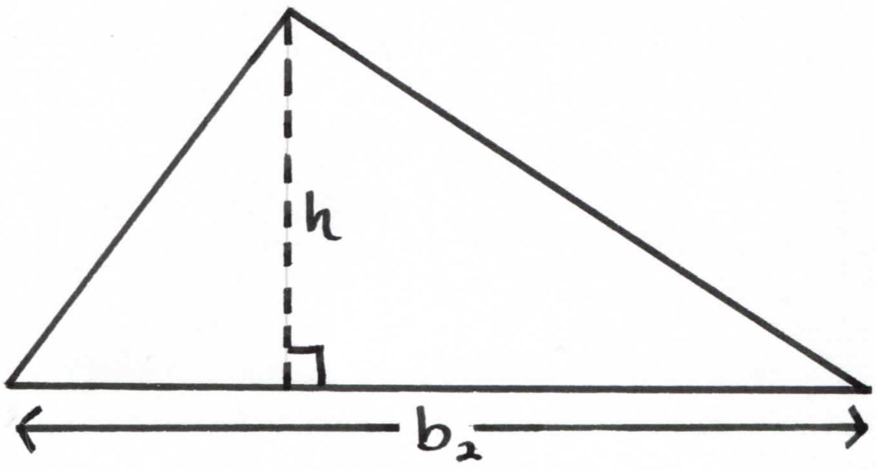
$$= \tfrac{1}{2} h \,(b_1 + b_2)$$

DRAWINGS 12.7



$\longrightarrow$

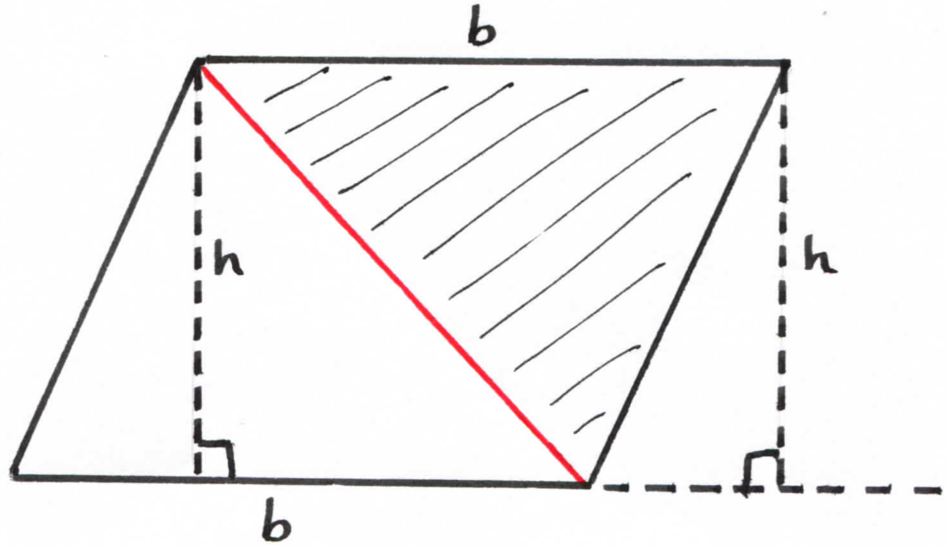
$(b_1 = 0)$



$\text{area} = \frac{1}{2} h\, b_2$

($\frac{1}{2}$ height times base)

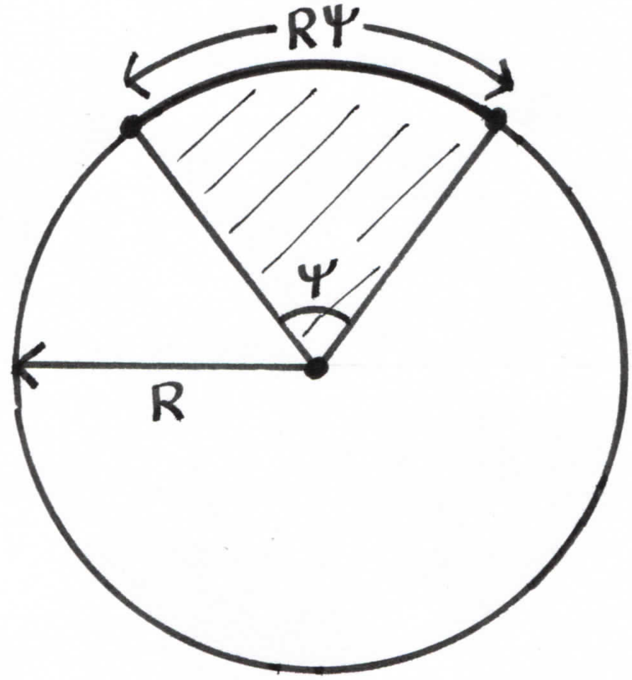
DRAWINGS 12.7 (continued)



Area of parallelogram equals base times height $= bh$.

Area of shaded triangle is one half base times height $= \frac{1}{2} bh$.

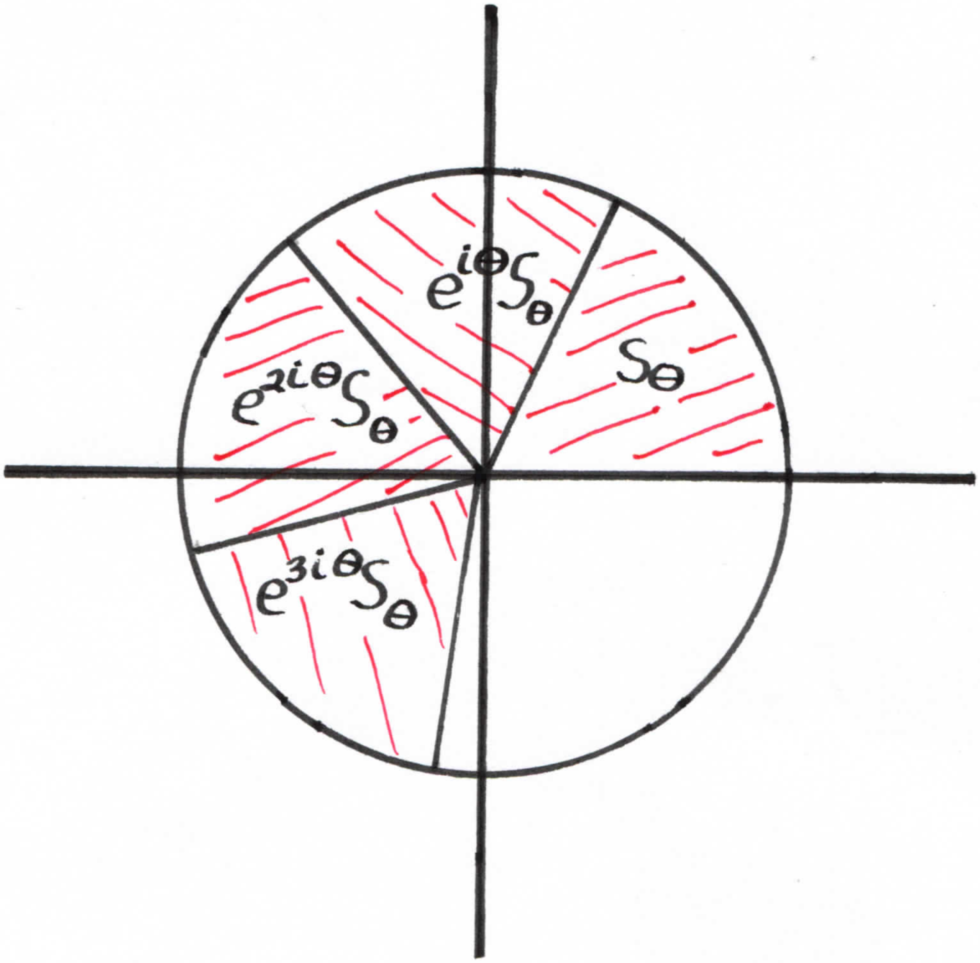
DRAWING 12.8



Shaded area is $\frac{1}{2}R^2\Psi$.

(Ψ is in radians.)

DRAWING 12.9

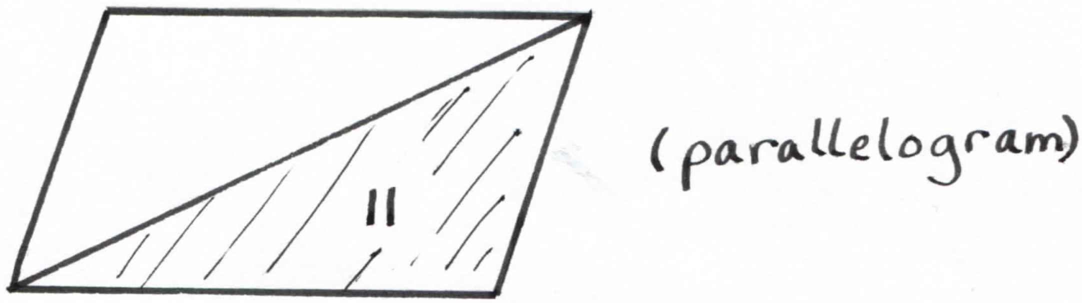


$$\theta = \frac{\psi}{4}$$

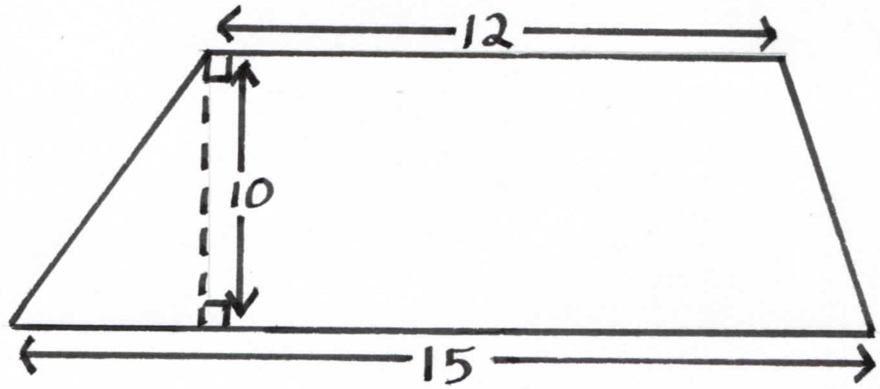
Total shaded area is S_ψ.

DRAWINGS 12.10

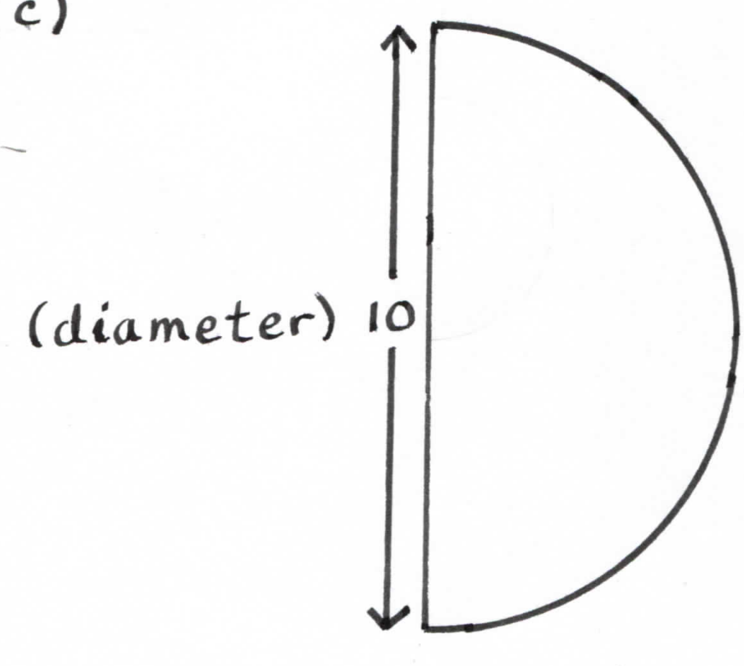
(a)



(b)

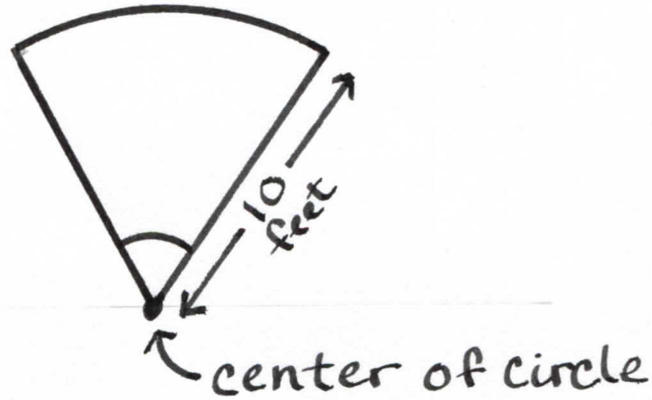


(c)

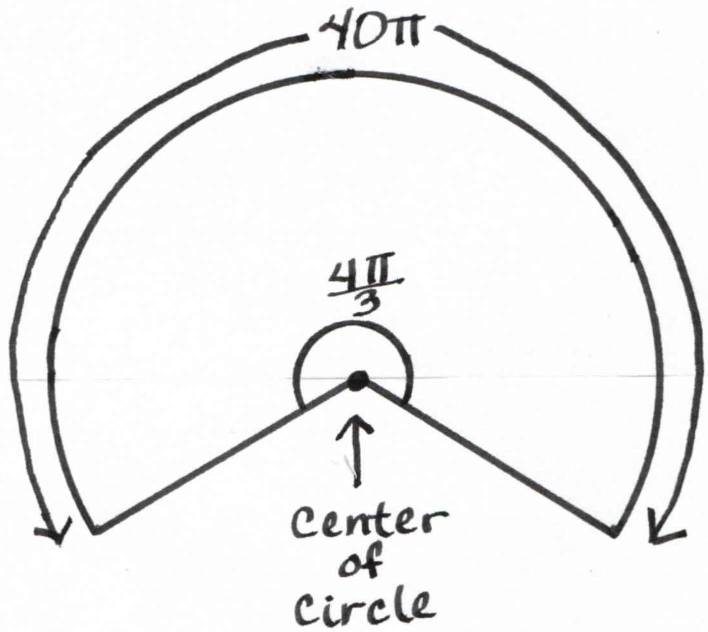


DRAWINGS 12.10 (continued)

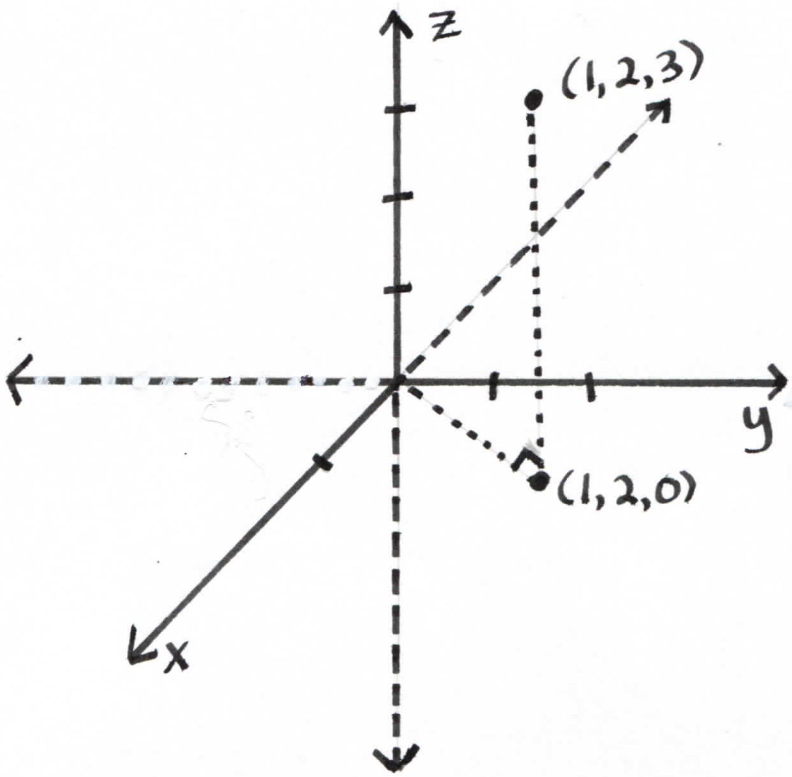
(d)



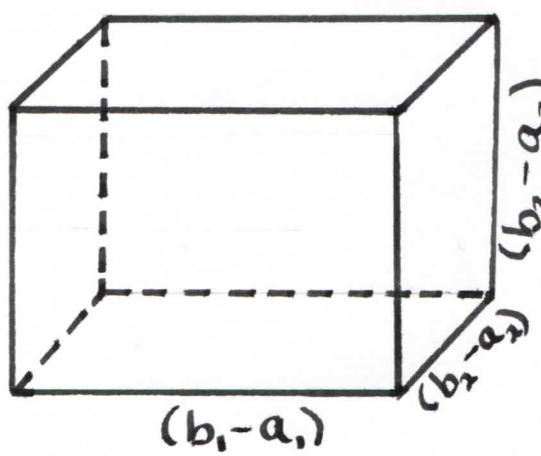
(e)



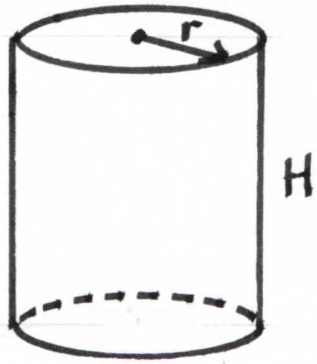
DRAWING 12.11



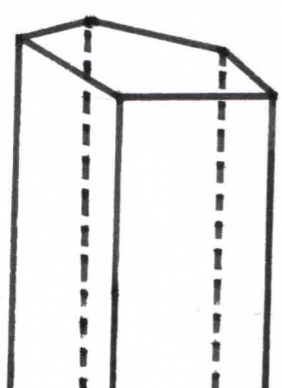
DRAWING 12.12



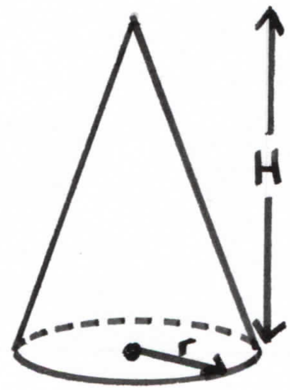
DRAWING 12.13



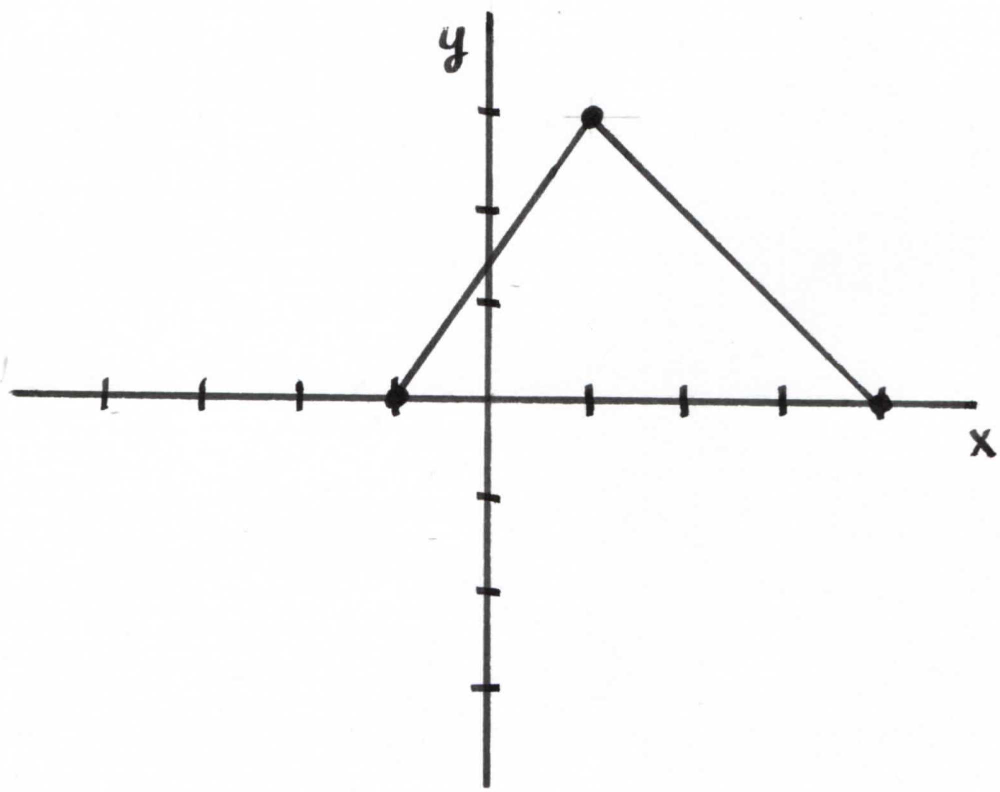
DRAWING 12.14



DRAWING 12.15

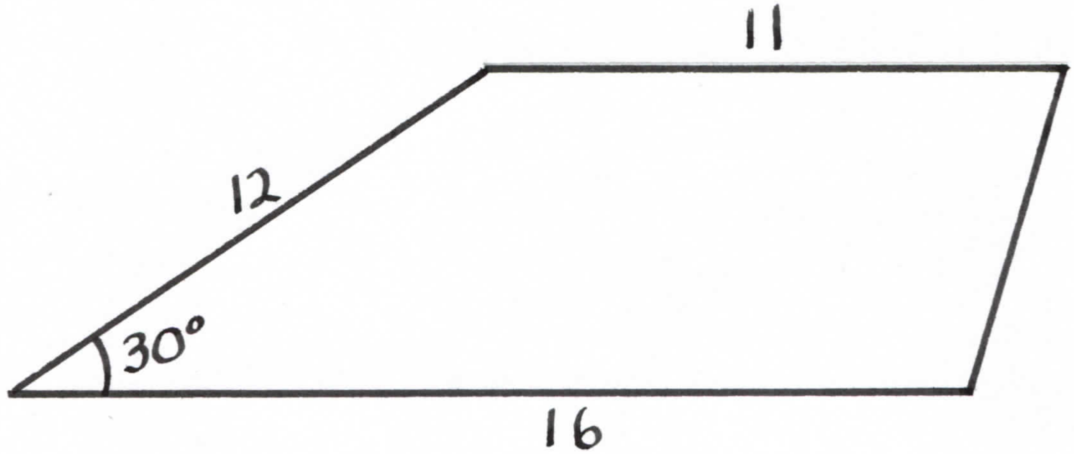


DRAWING 12.16

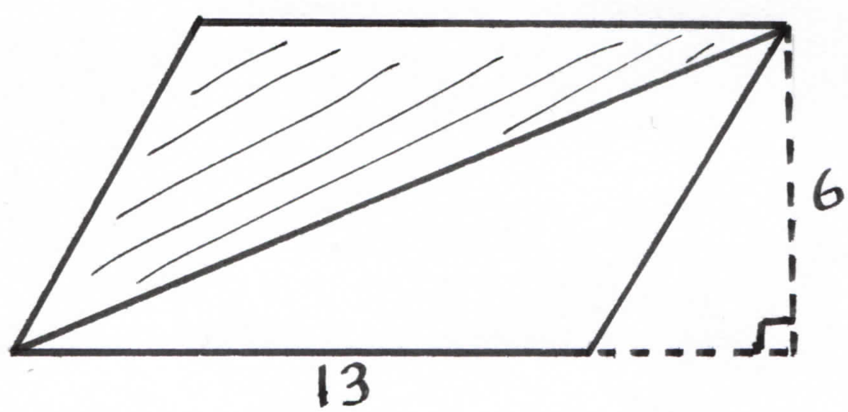


DRAWINGS 12.17

(a)



(b)



CHAPTER XIII: Construction with Straight-Edge and Compass.

A **straight-edge** is a ruler without the markings. A **compass** is a device with two ends, one of which can be held at a fixed point (traditionally, a **sharp end** jabbed into paper), while the other end makes a mark or marks a fixed distance away from the original point.

The straight-edge is used to draw lines or line segments. In particular, given two points P and Q, the straight-edge can be used to draw the line segment between P and Q.

The "fixed distance" of the compass setting can be adjusted. A compass can draw a circle; the fixed point is the center of the circle, the fixed distance is the radius of the circle, and the compass will then draw the circle by marking *all* points that fixed distance away from the center.

The compass can reproduce distances. More precisely, given two points P and Q, the compass can be set so that one end is at P, the other end at Q. Given any third point, call it R, the compass can then make a mark at a point S whose distance to R equals the distance between P and Q: $\|\overrightarrow{RS}\| = \|\overrightarrow{PQ}\|$.

Throughout this chapter, **constructing**, **drawing**, or **bisecting**, means using only a straight-edge and compass.

The classical Greeks were very interested in drawing or constructing figures or angles. This section will present some of these constructions. In all constructions, the reader is encouraged to try original ideas before looking at ours.

One very natural figure that we'd like to construct is a **regular polygon**, meaning a polygon with sides of equal length and interior angles of equal measure. It is interesting that not all regular polygons are **constructible**, that is, able to be constructed. A regular polygon of three sides (called an **equilateral triangle**), of four sides (called a **square**), of five sides (a regular **pentagon**), of six sides (a regular **hexagon**), or of eight sides (a regular **octagon**) is constructible, but a regular polygon of seven sides (a regular **septagon**) is not. We do not wish to explain why; explanation requires knowledge of the branch of mathematics known as *abstract algebra*.

We would like to describe our constructions primarily with drawings, thus we need to begin with a brief dictionary of drawings.

Using the straight-edge to draw the line segment between two given points will be drawn as in DRAWINGS 13.1 at the end of this chapter.

Given two points P and Q, making the fixed distance of the compass setting equal to $\|\overrightarrow{PQ}\|$ will be drawn as in DRAWINGS 13.2 at the end of this chapter; notice that the arc drawn at Q is a piece of a circle of radius $\|\overrightarrow{PQ}\|$ centered at P; P is where the sharp end of the compass is placed, the other end makes an arc that includes the point Q.

If, in addition to DRAWINGS 13.2, we want to make an arc of points that same fixed distance away from a third point R, it will be drawn as in DRAWINGS 13.3 at the end of this chapter, with the sharp end of the compass at R, the other end describing an arc, with the same radius $\|\overrightarrow{PQ}\|$.

Construction 13.1. Draw an equilateral triangle (see HWVII.5) with sides of a specified length. See DRAWINGS 13.4 at the end of this chapter.

Construction 13.2. Bisect an angle. See DRAWINGS 13.5 at the end of this chapter.

Construction 13.3. Bisect a line segment. See DRAWINGS 13.6 at the end of this chapter.

Construction 13.4. Given a point on a line, draw a line thru that point perpendicular to the original line.

This is a special case of Construction 13.2, for the angle of measure π. See DRAWING 13.7 at the end of this chapter.

Construction 13.5. Given a line and a point not on that line, draw a line thru that point perpendicular to the original line.

Recall that we have an algebraic construction in Theorem 4.11. See DRAWINGS 13.8 at the end of this chapter.

Construction 13.6. Given a line and a point not on that line, draw a line thru that point parallel to the original line.

This begins with Construction 13.5, then apply Construction 13.4 to the original point. See DRAWINGS 13.9 at the end of this chapter.

Construction 13.7. Draw a square, with sides of a specified length.

This will primarily be Construction 13.4, while using the compass to reproduce the specified length. See DRAWINGS 13.10 at the end of this chapter.

Remarks 13.8. (a) For $n = 3, 4, 5, \ldots$, constructing a regular n-gon is equivalent to constructing an angle of measure $\frac{2\pi}{n}$, which in turn is equivalent to constructing an isosceles triangle, with angle between two sides of equal length measuring $\frac{2\pi}{n}$.

To believe these equivalences, we need a fact that we do not wish to prove: inside any regular polygon is a unique point, called the **center** of the polygon, that is equidistant from each vertex.

The assertion of faith in the previous paragraph implies that any regular polygon may be inscribed (see DRAWINGS 13.11(a) at the end of this chapter) in a circle centered at the polygon's center with radius equal to that common distance from center to vertex. By drawing all lines from center to vertices, we obtain n (the number of vertices of the polygon) congruent (by SAS) isosceles triangles as described in the first line of this Remark.

Conversely, given n such congruent isosceles triangles, they may be pasted together to form a regular n-gon. See DRAWINGS 13.11(a) at the end of this chapter.

For example, we could have drawn a square by drawing a circle, then a diameter of the circle, then another diameter perpendicular to the first diameter (use Construction 13.4). The points where the diameters hit the circle become the vertices of a square, drawn in red in DRAWINGS 13.11(b) at the end of this chapter; notice that we have simultaneously drawn four congruent isosceles right triangles; more precisely, we have angles of measure $\frac{\pi}{2} = \frac{2\pi}{4}$ at the center, between sides of equal length.

(b) For $n = 3, 4, 5, \ldots$, angles of measure $\frac{2\pi}{n}$ appear elsewhere in a regular n-gon and suggest a possible general method for constructing a regular n-gon.

Corollary 3.10 asserts that the sum of the measures of the interior angles in an n-gon is $(n-2)\pi$. It then follows that each interior angle in a *regular* n-gon measures $\left(\frac{(n-2)}{n}\right)\pi$, thus each exterior angle (see Definitions 2.16) measures

$$\pi - \left(\frac{(n-2)}{n}\right)\pi = \frac{2\pi}{n}.$$

See DRAWINGS 13.11(c) at the end of this chapter, of each vertex in a regular n-gon.

If angles of measure $\left(\frac{(n-2)}{n}\right)\pi$ can be constructed (equivalent to angles of measure $\frac{2\pi}{n}$ being constructible), then this suggests the following strategy for constructing a regular n-gon, with sides of a specified length, call it c.

Begin with a line segment of length c, labeled Q_1Q_2 in DRAWINGS 13.11(d) at the end of this chapter. At the point Q_2, draw a line segment of length c, call it Q_2Q_3, that makes an angle of measure $\left(\frac{(n-2)}{n}\right)\pi$ with the original line segment Q_1Q_2. Continue adding on line segments of length c that make an angle of $\left(\frac{(n-2)}{n}\right)\pi$ with the previous line segment, until your string of line segments meets at Q_1. See DRAWINGS 13.11(d) at the end of this chapter.

For this construction to work, we must verify that the n line segments constructed in the previous paragraph will terminate at the original point Q_1. This is equivalent to the vectors between consecutive vertices, as in Definitions 2.3, adding up to the zero vector $\vec{0}$. That is, in DRAWINGS 13.11(e) at the end of this chapter, we need

$$\left(\overrightarrow{S_0}+\overrightarrow{S_1}+\overrightarrow{S_2}+\cdots+\overrightarrow{S_{n-1}}\right)=\overrightarrow{0}. \quad (*)$$

We will now find it very convenient to think of vectors as complex numbers, as in 1.16. From Theorem 2.13, for arbitrary complex z representing a vector, multiplying by $e^{\frac{2\pi i}{n}}$ rotates z by $\frac{2\pi}{n}$ counterclockwise; that is, produces an exterior angle of measure $\frac{2\pi}{n}$, as in DRAWINGS 13.11(f) at the end of this chapter.

In DRAWINGS 13.11(e) at the end of this chapter, representing $\overrightarrow{S_0}$ by the complex (real) number c, we thus have

$$\overrightarrow{S_1}=e^{\frac{2\pi i}{n}}c, \overrightarrow{S_2}=e^{2\frac{2\pi i}{n}}c, \overrightarrow{S_3}=e^{3\frac{2\pi i}{n}}c, \ldots, \overrightarrow{S_{n-1}}=e^{(n-1)\frac{2\pi i}{n}}c,$$

so that (see (*) above and in DRAWINGS 13.11(e) at the end of this chapter), by HWI.5,

$$\left(\overrightarrow{S_0}+\overrightarrow{S_1}+\overrightarrow{S_2}+\cdots+\overrightarrow{S_{n-1}}\right)=\left(c+e^{\frac{2\pi i}{n}}c+e^{2\frac{2\pi i}{n}}c+e^{3\frac{2\pi i}{n}}c+\cdots+e^{(n-1)\frac{2\pi i}{n}}c\right)$$

$$=c\left(1+e^{\frac{2\pi i}{n}}+\left(e^{\frac{2\pi i}{n}}\right)^2+\left(e^{\frac{2\pi i}{n}}\right)^3\cdots+\left(e^{\frac{2\pi i}{n}}\right)^{(n-1)}\right)=c\left[\frac{1-\left(e^{\frac{2\pi i}{n}}\right)^n}{1-\left(e^{\frac{2\pi i}{n}}\right)}\right]=c\left[\frac{1-e^{2\pi i}}{1-\left(e^{\frac{2\pi i}{n}}\right)}\right]$$

$$=c\left[\frac{1-1}{1-\left(e^{\frac{2\pi i}{n}}\right)}\right]=0,$$

proving (*), as needed for our construction.

For example, we could construct a square by careening around making right-angle turns; see DRAWINGS 13.11(g) at the end of this chapter.

Since

$$\left(c+e^{\frac{\pi}{4}i}c+e^{2\frac{\pi}{4}i}c+e^{3\frac{\pi}{4}i}c\right)=(c+ci-c-ci)=0,$$

the line segments fit together to make a square.

Similar cancellations can be calculated directly for applying this construction to a regular hexagon ($n=6$) and a regular octagon ($n=8$). For variety, and to avoid constructing the same angle many times, we will use different constructions for the regular hexagon and octagon.

Finally, we should reiterate that the construction of a regular n-gon in Remarks 13.8(b) presupposes that the angle of measure $\frac{2\pi}{n}$ is constructible.

Construction 13.9. Draw a regular hexagon, with sides of a specified length.

Begin by drawing a circle whose radius is that specified length. Starting at any point on that circle, use the compass, as in Construction 13.1, to construct a sequence of contiguous equilateral triangles sharing a vertex at the center of the circle, each with sides whose lengths are equal to the specified length. See DRAWINGS 13.12 at the end of this chapter.

Notice that we have simultaneously shown that the angle of measure $\frac{\pi}{3} = \frac{2\pi}{6}$ is constructible; see Remarks 13.8(a).

Construction 13.10. Draw a regular octagon.

As with Construction 13.9, begin with a circle. Draw a line segment thru a diameter of the circle. Bisect angles (Construction 13.2) twice, to get a sequence of points on your circle; make those points the vertices of your octagon. See DRAWINGS 13.13(a) at the end of this chapter.

To get a regular octagon with sides of a specified length, we need to specify the radius of the circle that our regular octagon construction begins with.

Begin with a line segment of the specified length. By Constructions 13.2 and 13.4, we may draw rays beginning at the endpoints of the original line segment, each making an angle of measure $\frac{3\pi}{4}$ with said line segment. Use Construction 13.2 to bisect those $\frac{3\pi}{4}$ angles. Extend those lines performing the last bisection until they intersect; each of those line segments just constructed will have length equal to the desired radius. See DRAWINGS 13.13(b) at the end of this chapter.

Although it does not facilitate our construction, we can't help mentioning that the Law of Cosines implies that, if c is the length of a side and r is the radius of the circle drawn to get our octagon, then

$$r = \left(\sqrt{\left(1 + \frac{1}{\sqrt{2}}\right)}\right) c.$$

See the second-to-last page of DRAWINGS 13.13(b) at the end of this chapter.

Construction 13.11. Draw an isosceles right triangle, with legs of a specified length.

If we had drawn a square, as in Construction 13.7, we could get the desired right triangle twice, by drawing a diagonal of the square.

But if we don't have a square at hand, here's a simpler construction of the right triangle.

Use the compass to draw one leg, of the specified length. Use Construction 13.4 to draw a line perpendicular to that leg, at one end of the leg; use the compass to make that perpendicular line the specified length. That constructs both legs of the triangle; the hypotenuse follows inevitably by connecting the two unshared endpoints of the legs. See DRAWINGS 13.14 at the end of this chapter.

Construction 13.12. Draw a right triangle with interior angles measuring $\frac{\pi}{6}$ and $\frac{\pi}{3}$, and shorter leg of a specified length.

Begin by using the compass to draw an equilateral triangle whose sides have length equal to twice the specified length. Use Construction 13.5 to draw a perpendicular line from a vertex to the opposite side (or, keep the mark for the length of the original side, which will be half the length of the side of the equilateral triangle). Either side of that perpendicular line will be the desired triangle. See DRAWINGS 13.15 at the end of this chapter.

Construction 13.13. Given three points in the plane that are not on a line, draw the circle through those points and draw the center of said circle.

Call the points P_1, P_2, and P_3. Use Construction 13.3 to draw the perpendicular bisector of the line segment P_1P_2, call it ℓ_1, then the perpendicular bisector of the line segment P_1P_3, call it ℓ_2. Let C be the intersection of ℓ_1 and ℓ_2. C is equidistant from P_1, P_2, and P_3, thus, by setting our compass at the distance from C to P_1, we may draw a circle of that radius centered at C, that will go through P_1, P_2, and P_3.

See DRAWINGS 13.16 at the end of this chapter.

REASONS WHY THE CONSTRUCTIONS WORK

Construction 13.1. By Theorem 7.10 or HWVII.5, it is sufficient to draw a triangle whose sides are of equal length; the equality of the measures of the interior angles follows automatically.

Constructions 13.2–5. Proposition 11.5.

Construction 13.6. In the second page of DRAWINGS 13.9 at the end of this chapter, denote by ℓ_1 the line perpendicular to both ℓ' and ℓ. By Corollary 5.3, with $\ell_2 \equiv \ell_1, \ell_3 \equiv \ell', \ell_4 \equiv \ell$, it follows that ℓ' is parallel to ℓ.

Construction 13.7. See the second page of DRAWINGS 13.10 at the end of this chapter. By Corollary 5.3, with $\ell_1 \equiv \ell_2 \equiv \overrightarrow{PQ}, \ell_3 \equiv \overrightarrow{RP}, \ell_4 \equiv \overrightarrow{SQ}$, or Proposition 4.18, we have $\overrightarrow{RP}$ and $\overrightarrow{SQ}$ parallel. By Proposition 3.5, $RSQP$ is a parallelogram; by Proposition 3.3, $\|\overrightarrow{RS}\| = \|\overrightarrow{PQ}\|$. By Corollary 5.3 again, the interior angles at R and S are right angles.

Construction 13.9. As with Construction 13.1, the triangles we constructed with sides of equal specified lengths also have equal measures, namely $\frac{\pi}{3}$. Since $2\pi = 6(\frac{\pi}{3})$, we may paste together six of those congruent equilateral triangles to form the desired hexagon, as in the last page of DRAWINGS 13.12 at the end of this chapter.

Construction 13.10. Now we have eight congruent (by SAS) isosceles triangles, each with an angle of measure $\frac{\pi}{4}$ between sides of equal length. Since $8(\frac{\pi}{4}) = 2\pi$, we may paste those triangles together to form a regular octagon, as in the last page of DRAWINGS 13.13(a) at the end of this chapter.

Construction 13.11. This is prior constructions.

Construction 13.12. Prior constructions take us to the top of the last page of DRAWINGS 13.15 at the end of this chapter. The orthogonality of $\overrightarrow{SQ}$, in the last drawing of the last page of DRAWINGS 13.15 at the end of this chapter follows from Theorem 5.10.

Construction 13.13. Construction 13.3, especially the last page of DRAWINGS 13.6 at the end of this chapter, tell us how to draw ℓ_1 and ℓ_2. The Pythagorean theorem tells us that every point on ℓ_1 is equidistant from P_1 and P_2, while every point on ℓ_2 is equidistant from P_1 and P_3. Thus the point C is equidistant from P_1, P_2, and P_3, so that a circle centered at C goes through those three points.

HOMEWORK

In all problems, "construct" means "construct with straight edge and compass." Assume, for all problems, that we are given a line segment of length one.

HWXIII.1. Construct a line segment of length $\sqrt{2}$.

HWXIII.2. Construct an arc of length $\frac{\pi}{2}$.

HWXIII.3. Construct a line segment of length $\frac{\sqrt{3}}{2}$.

HWXIII.4. Construct a trapezoid of area 2.5.

HWXIII.5. Construct a triangle of area $\frac{1}{2}$.

HWXIII.6. Construct a right triangle of area 4.5.

HWXIII.7. Construct a sector of a disc of area $\frac{\pi}{8}$.

HWXIII.8. Construct a disc of area 9π.

HWXIII.9. We have already mentioned that a polygon is *constructible* if it can be constructed. Similarly, a length is called **constructible** if a line segment of that length can be constructed; an angle measure is called **constructible** if an angle of that measure can be constructed.

Throughout this problem, you may assume the following without proof:

The sum of two constructible lengths or measures is constructible;

The positive difference of two constructible lengths or measures is constructible;

The product of two constructible lengths or measures is constructible;

The quotient of two constructible lengths or measures is constructible;

Tbe square root of a constructible length or measure is constructible.

A regular septagon (seven-sided regular polygon) is not constructible.

Use these facts, Remarks 13.8(a), and Constructions 13.1–7 and 13.9–13 to answer the following. Assume we are given a line segment of length one.

(a) Show that an angle of positive measure $\theta < \frac{\pi}{2}$ is constructible if and only if a line segment of length $\cos\theta$ is constructible.

(b) Find a positive $\theta < \frac{\pi}{2}$ such that an angle of measure θ is not constructible.

(c) Find a positive s for which a line segment of length s is not constructible.

HOMEWORK ANSWERS

Throughout, we use Constructions 13.1 through 13.13.

HWXIII.1. See DRAWINGS 13.17 at the end of this chapter.

HWXIII.2. See DRAWINGS 13.18 at the end of this chapter.

HWXIII.3. See DRAWINGS 13.19 at the end of this chapter.

HWXIII.4. See DRAWINGS 13.20 at the end of this chapter.

HWXIII.5. See DRAWINGS 13.21 at the end of this chapter.

HWXIII.6. See DRAWINGS 13.22 at the end of this chapter.

HWXIII.7. See DRAWINGS 13.23 at the end of this chapter.

HWXIII.8. See DRAWINGS 13.24 at the end of this chapter.

HWXIII.9.

(a) See DRAWINGS 13.25 at the end of this chapter.

(b) $\theta \equiv \frac{2\pi}{7}$ is not constructible, by Remarks 13.8(a) and the fact that a regular septagon is not constructible.

(c) $s \equiv \cos(\frac{2\pi}{7})$ is not a constructible length, by (a) and (b).

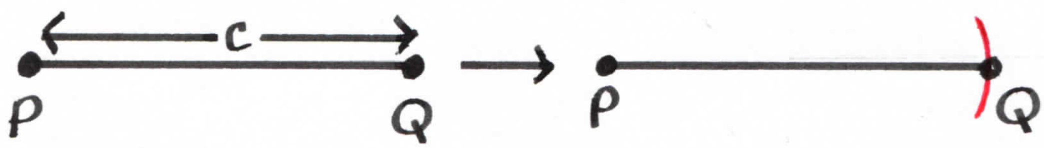
DRAWINGS 13.1

DRAWINGS 13.2

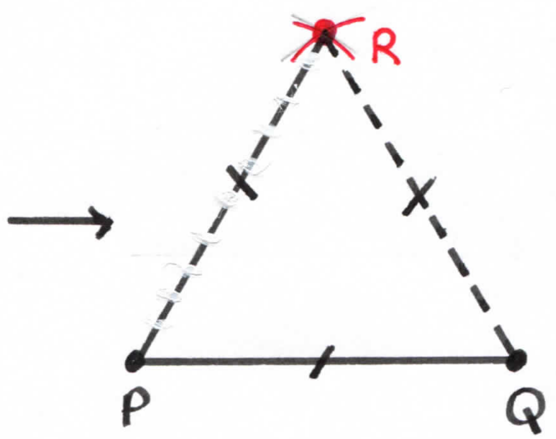
DRAWINGS 13.3

DRAWINGS 13.4

Given: $c = \|\overrightarrow{PQ}\|$ specified

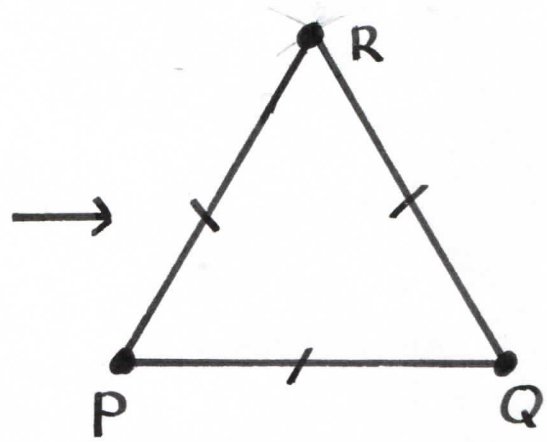


(compass setting $c = \|\overrightarrow{PQ}\|$)



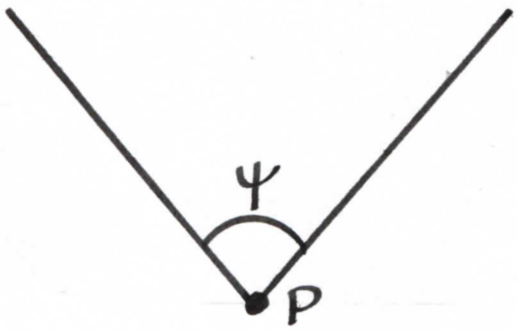
(both arcs have radius $c = \|\overrightarrow{PQ}\|$:

$\|\overrightarrow{PQ}\| = \|\overrightarrow{PR}\| = \|\overrightarrow{QR}\| = c$)

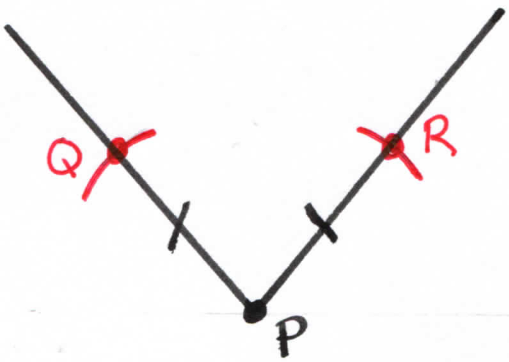


DRAWINGS 13.5

Given:

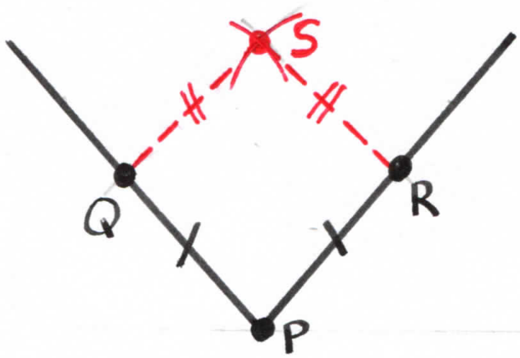


→



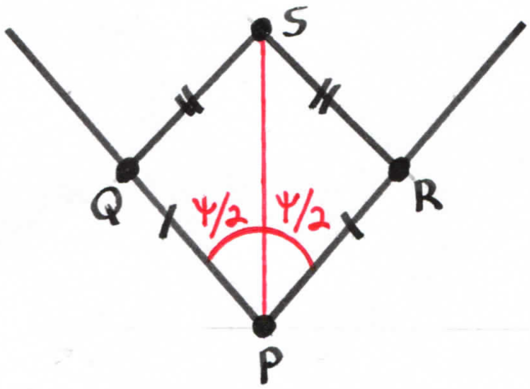
($\|\overrightarrow{PQ}\| = \|\overrightarrow{PR}\|$)

→



($\|\overrightarrow{QS}\| = \|\overrightarrow{RS}\|$, might not equal $\|\overrightarrow{PQ}\|$)

→

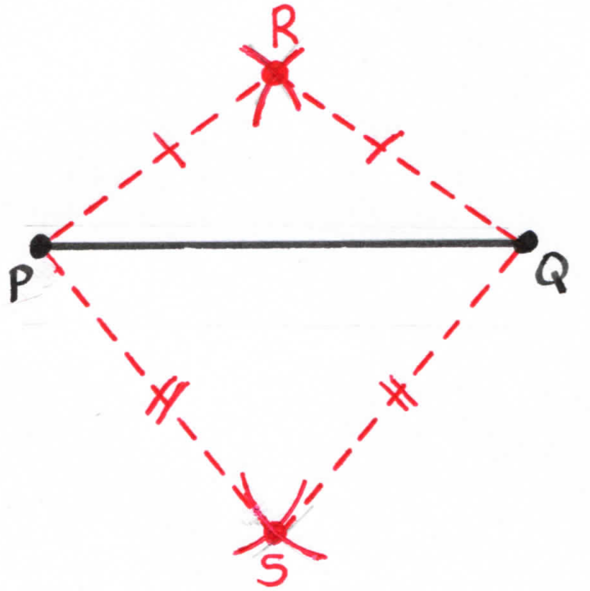


($\overrightarrow{SP}$ bisects angle between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$)

DRAWINGS 13.6

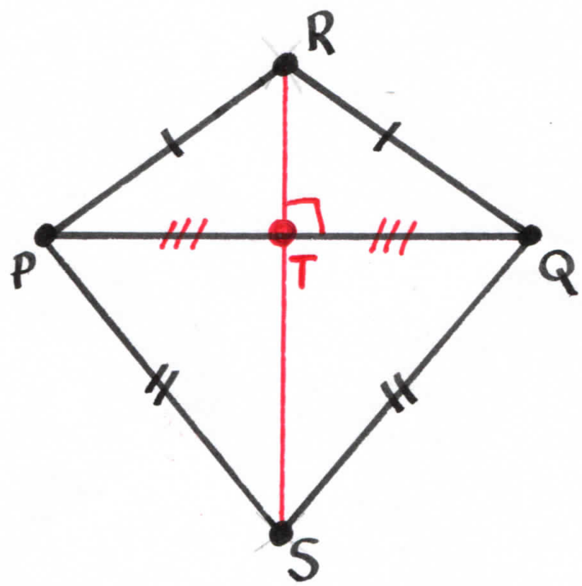
Given:

$\longrightarrow$



($\|\overrightarrow{PR}\| = \|\overrightarrow{QR}\|$;
$\|\overrightarrow{PS}\| = \|\overrightarrow{QS}\|$,
might not equal $\|\overrightarrow{PR}\|$)

$\longrightarrow$

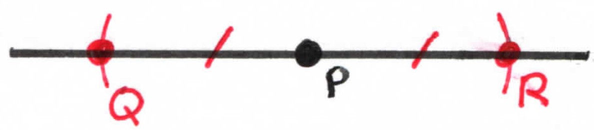


($\overrightarrow{RS}$ is the $\perp$ bisector of $\overrightarrow{PQ}$:
$\|\overrightarrow{PT}\| = \|\overrightarrow{TQ}\|$, $\overrightarrow{PT} \perp \overrightarrow{TR}$)

DRAWINGS 13.7
(special case of DRAWINGS 13.5)

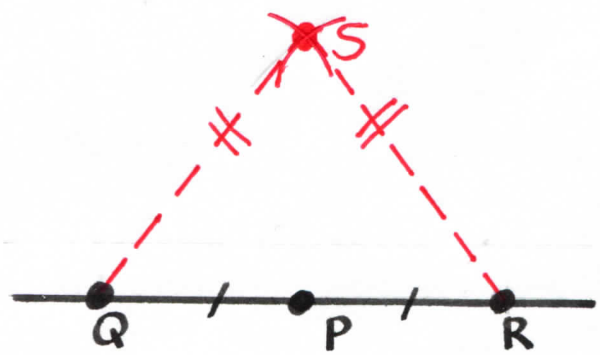
Given:

→



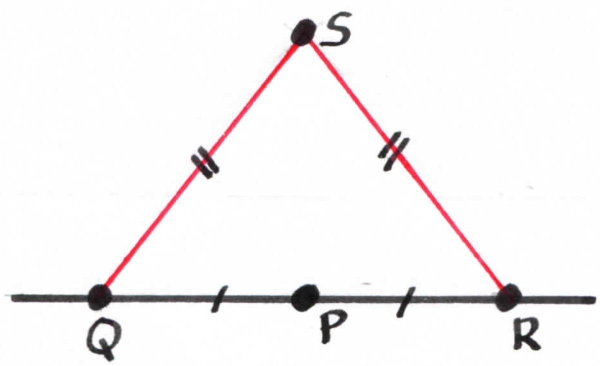
$\|\overrightarrow{QP}\| = \|\overrightarrow{PR}\|$

→



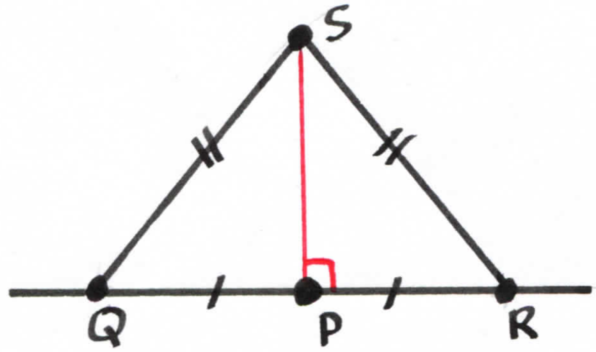
Compass setting same for two arcs:
$\|\overrightarrow{QS}\| = \|\overrightarrow{RS}\|$

→



$\|\overrightarrow{QS}\| = \|\overrightarrow{RS}\|$,
$\|\overrightarrow{QP}\| = \|\overrightarrow{PR}\|$

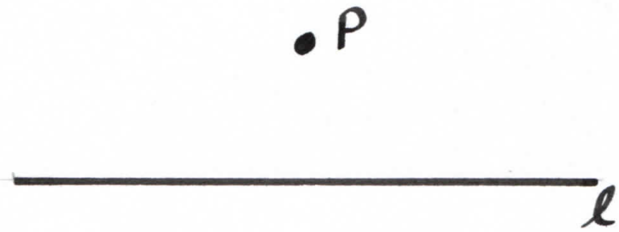
→



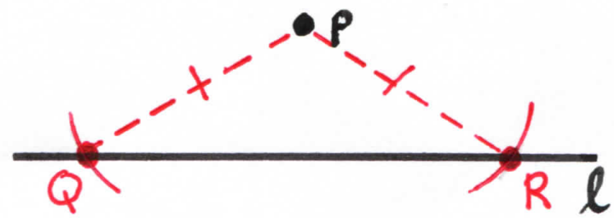
$\overrightarrow{SP}$ is $\perp$ to $\overrightarrow{QR}$

DRAWINGS 13.8

Given:

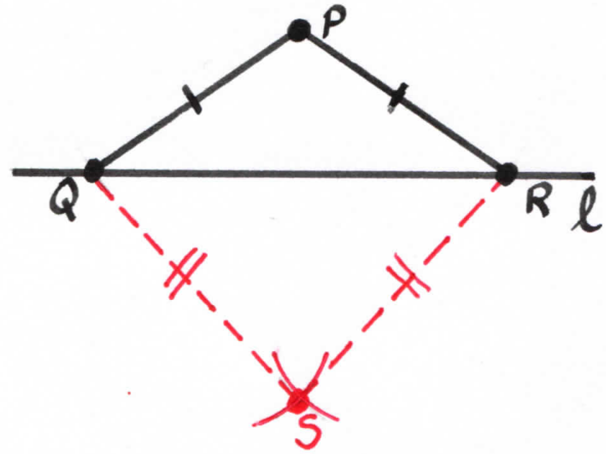


→



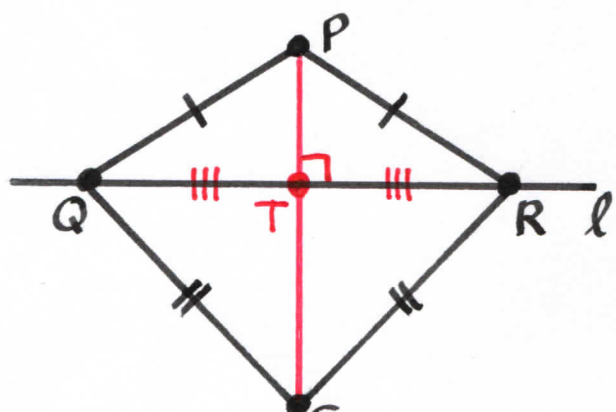
Same compass setting on both arcs: $\|\overrightarrow{QP}\| = \|\overrightarrow{PR}\|$

→



Same compass setting on both arcs passing through S: $\|\overrightarrow{QS}\| = \|\overrightarrow{SR}\|$; $\|\overrightarrow{QS}\|$ might not equal $\|\overrightarrow{QP}\|$

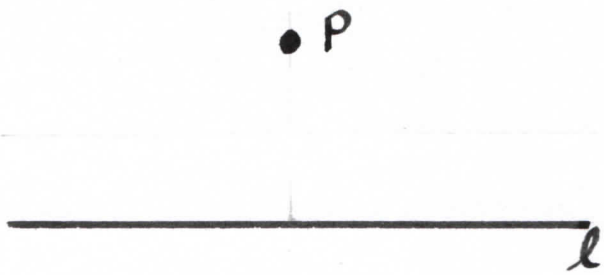
→



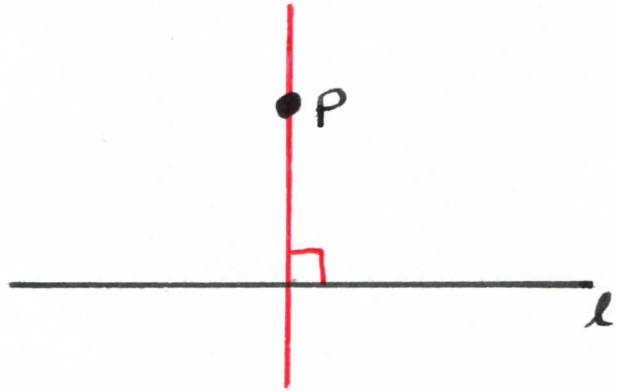
$\overrightarrow{PS}$ is $\perp$ to ℓ, and $\|\overrightarrow{QT}\| = \|\overrightarrow{TR}\|$: $\overrightarrow{PS}$ is $\perp$ bisector of $\overrightarrow{QR}$

DRAWINGS 13.9

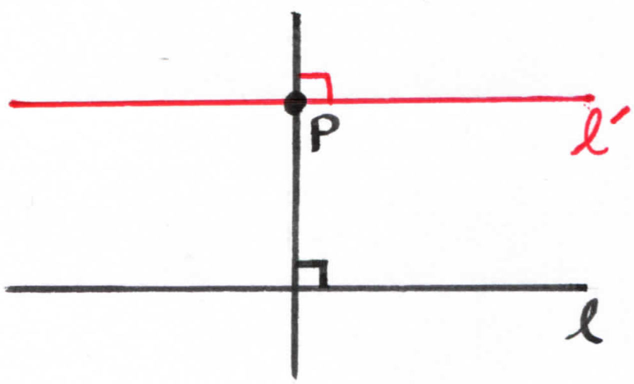
Given:



→ ... (Construction 13.5, DRAWINGS 13.8)

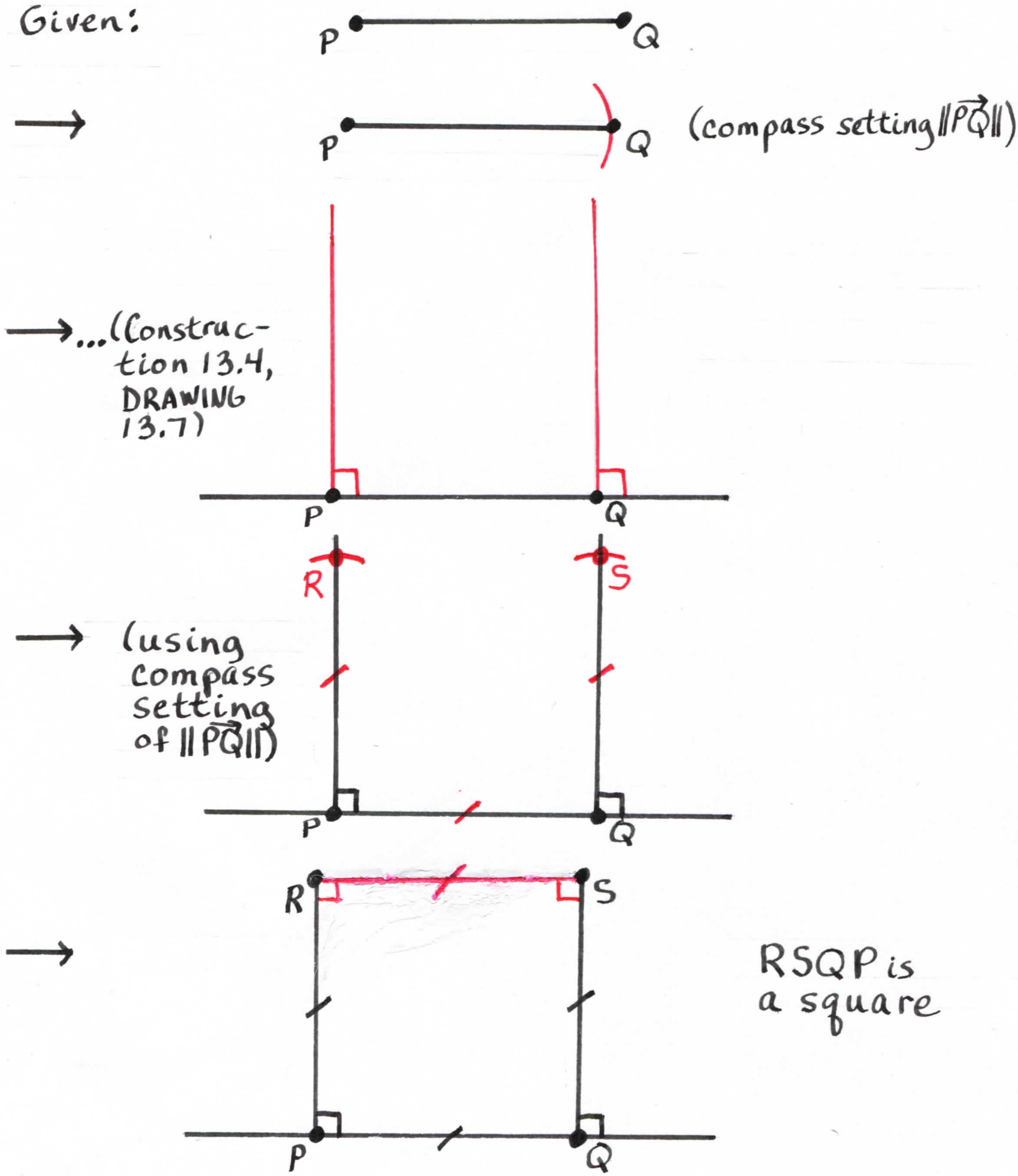


→ ... (Construction 13.4, DRAWINGS 13.7)

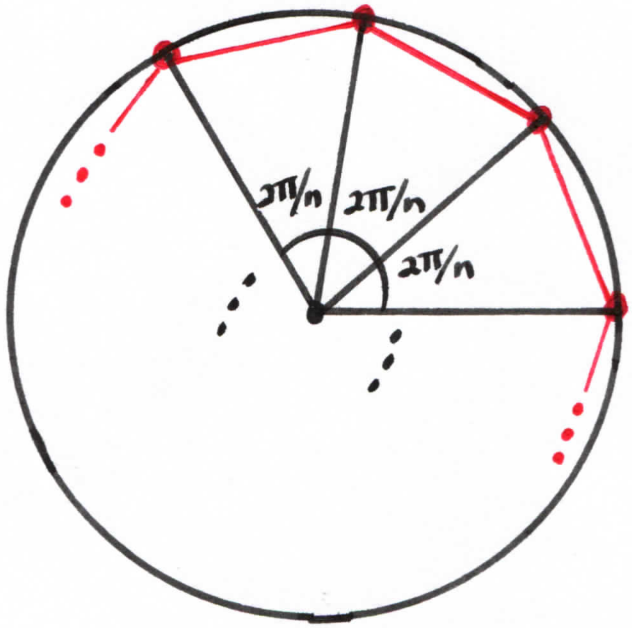


l' is parallel to l

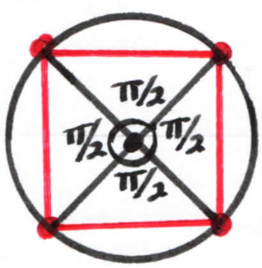
DRAWINGS 13.10



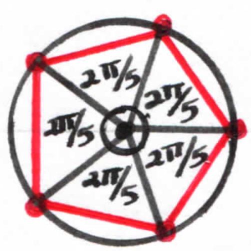
DRAWINGS 13.11(a)



n=3

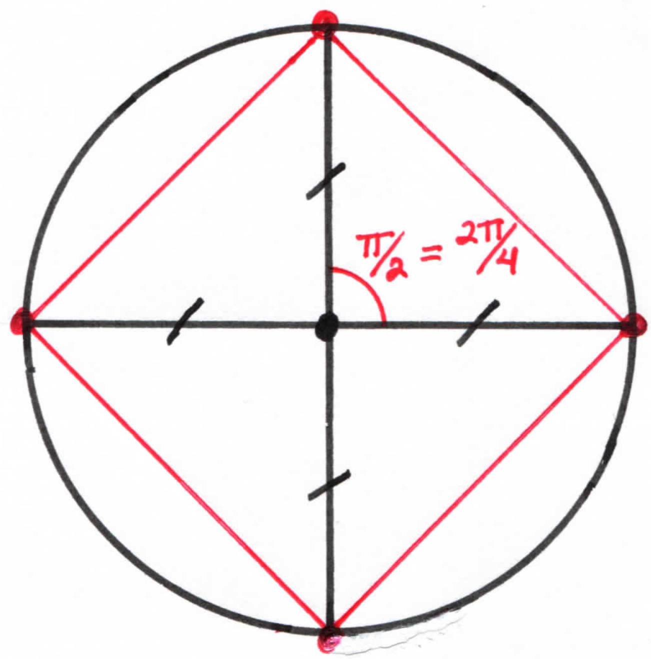


n=4



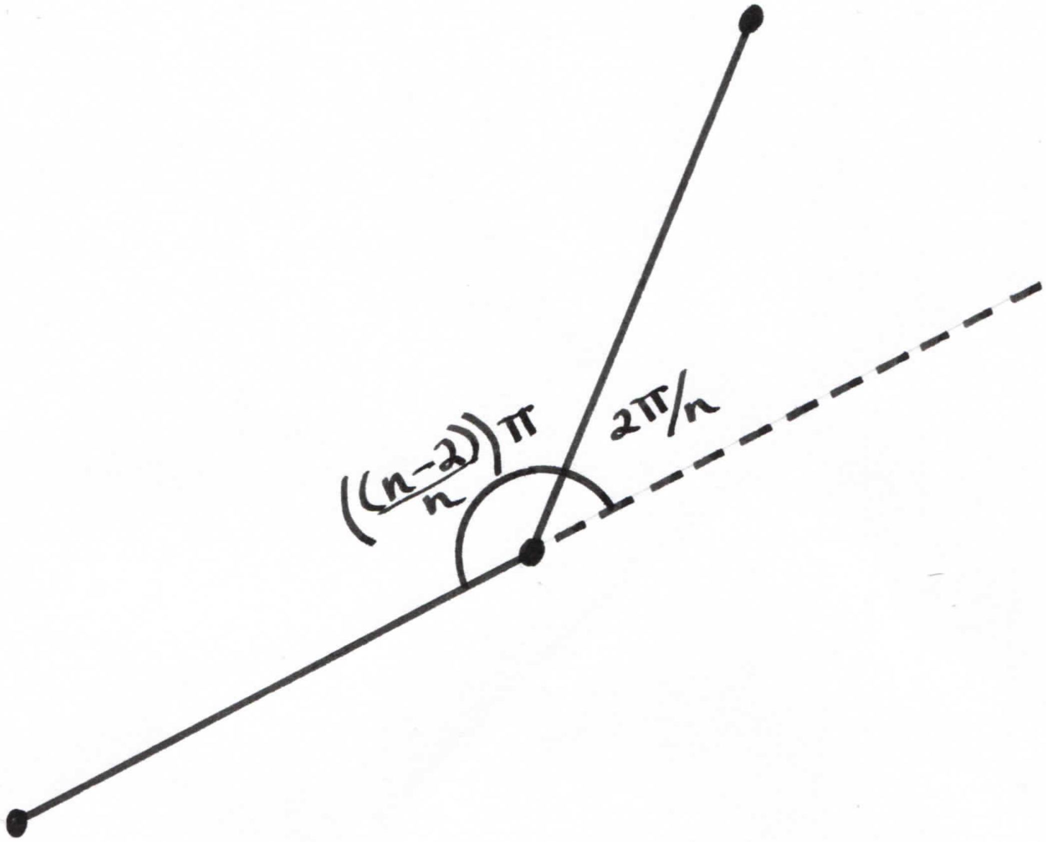
n=5

DRAWINGS 13.11(b)



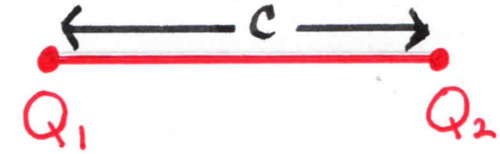
Square is drawn in red.

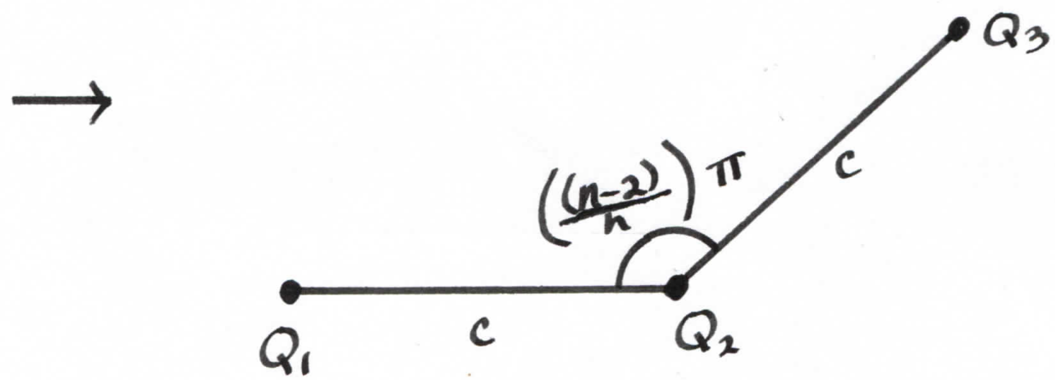
DRAWINGS 13.11(c)

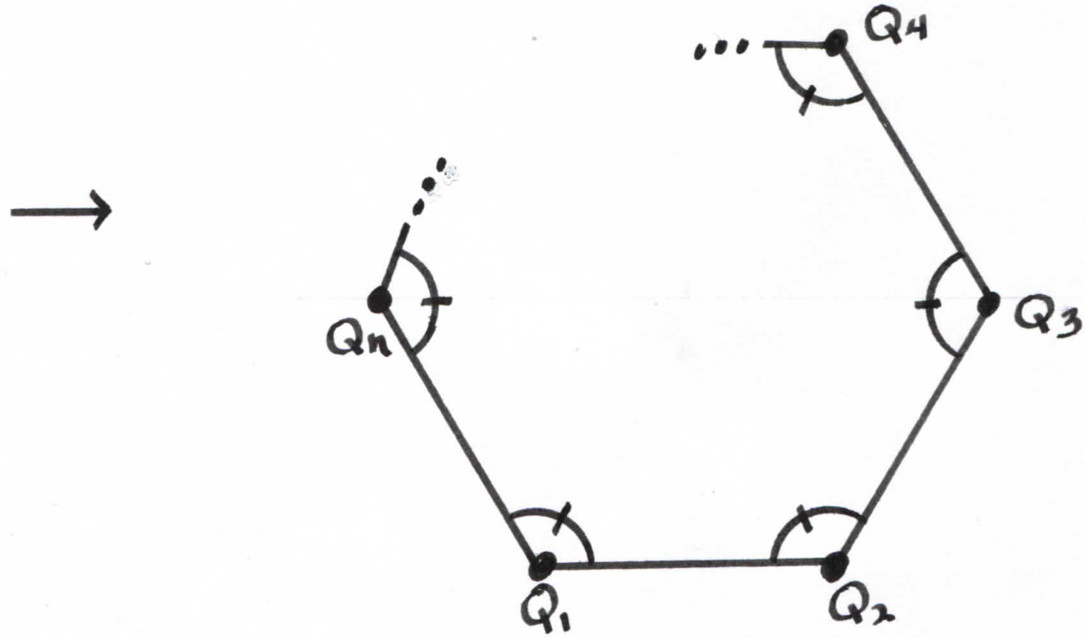


Dots are vertices of regular n-gon.

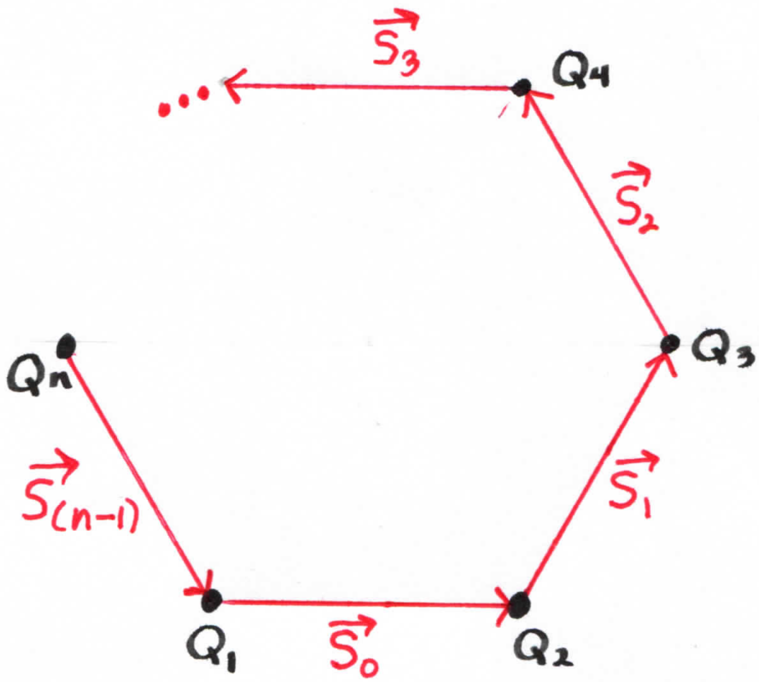
DRAWINGS 13.11(d)





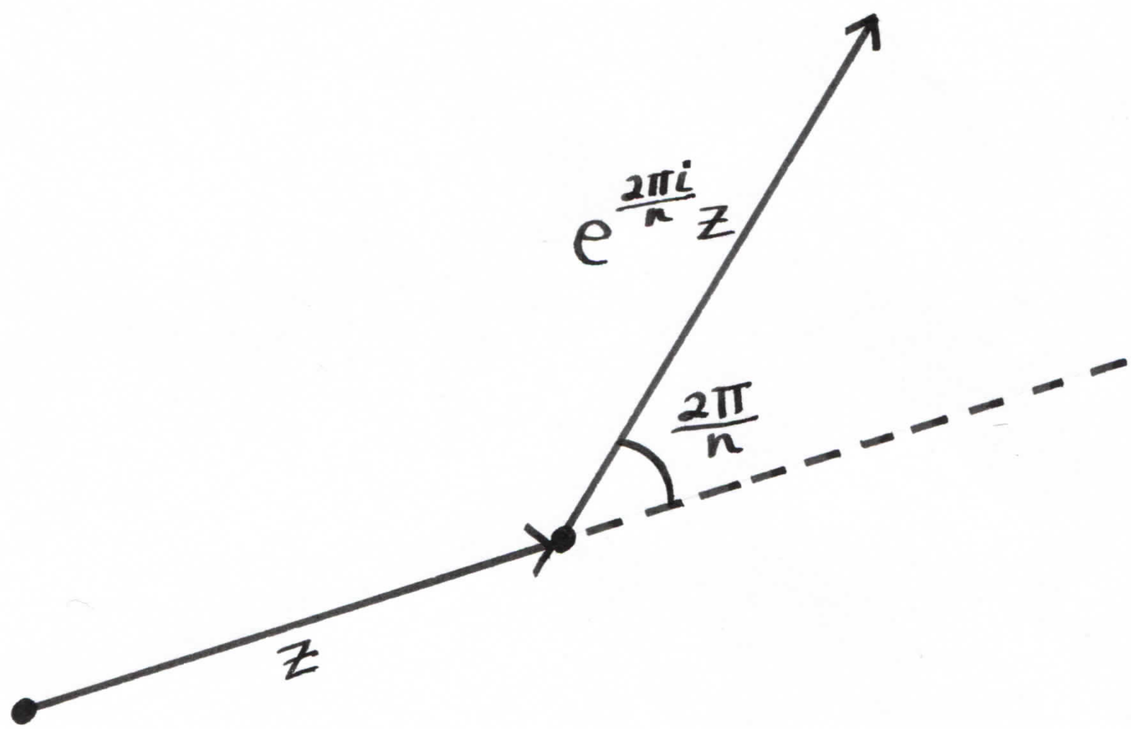


DRAWINGS 13.11 (e)

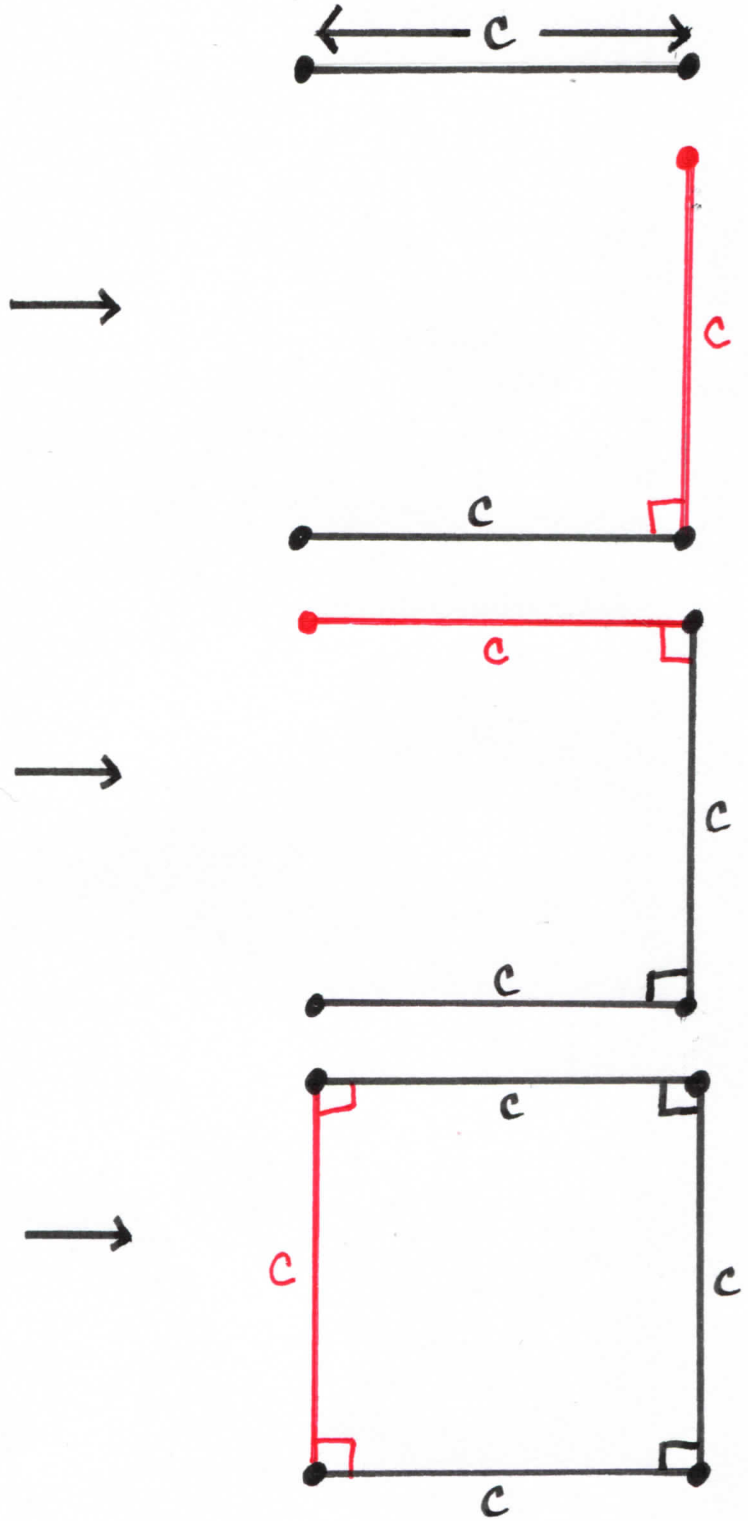


$$\left(\vec{S}_0 + \vec{S}_1 + \vec{S}_2 + \cdots + \vec{S}_{(n-1)}\right) = \vec{0} \ ? \qquad (*)$$

DRAWINGS 13.11(f)



DRAWINGS 13.11 (g)



DRAWINGS 13.12

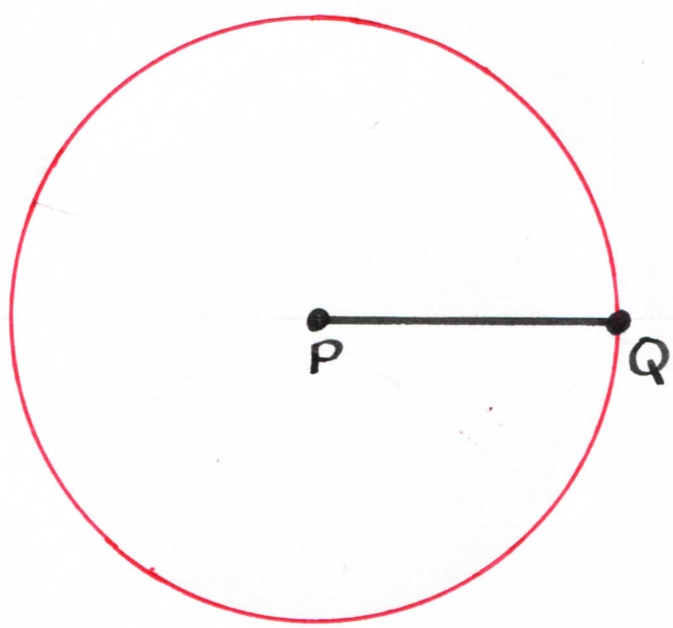
Given:

$\|\overrightarrow{PQ}\|$ = specified length

→

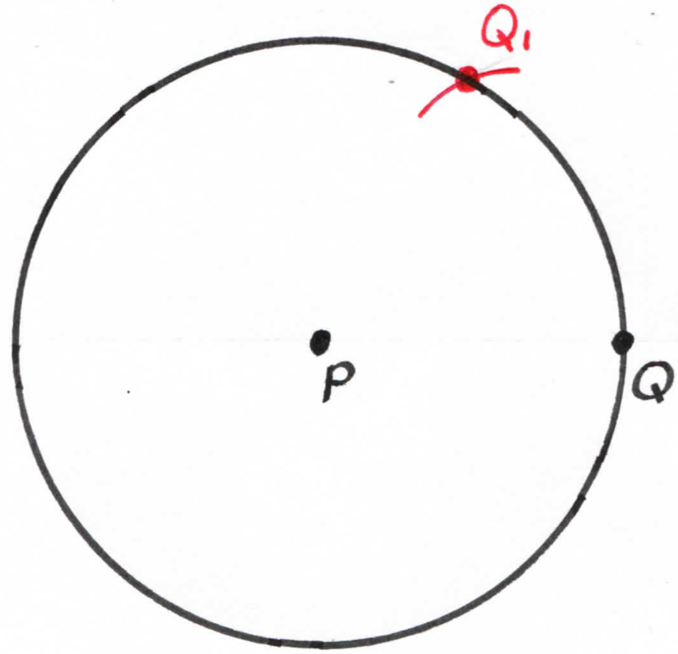
compass setting $\|\overrightarrow{PQ}\|$

→



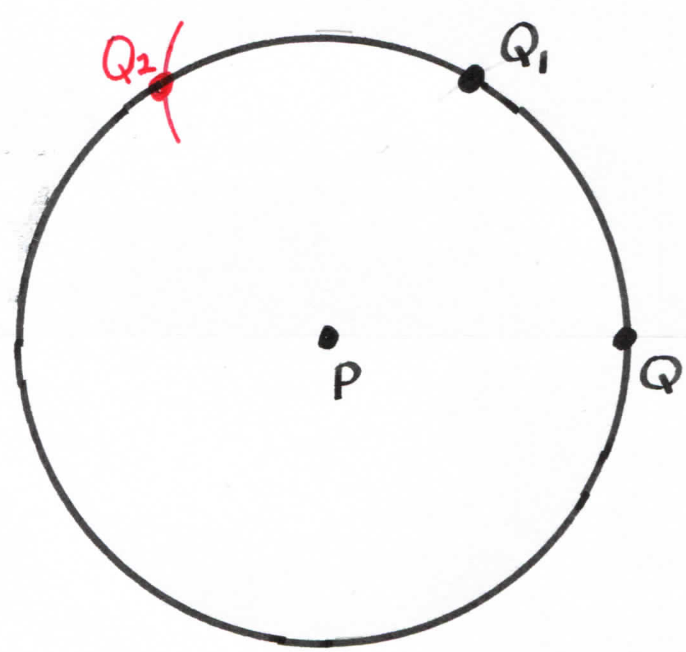
circle, radius $\|\overrightarrow{PQ}\|$

DRAWINGS 13.12 (continued)



Compass setting $\|\overrightarrow{PQ}\|$;

$\|\overrightarrow{Q_1Q}\| = \|\overrightarrow{PQ}\|$
$= \|\overrightarrow{PQ_1}\|$

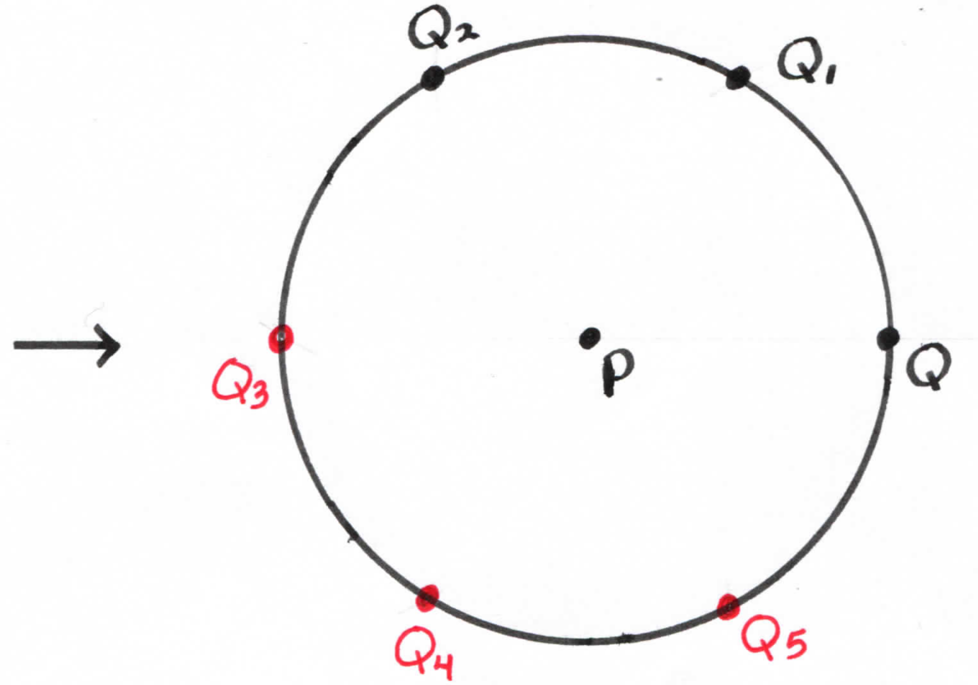


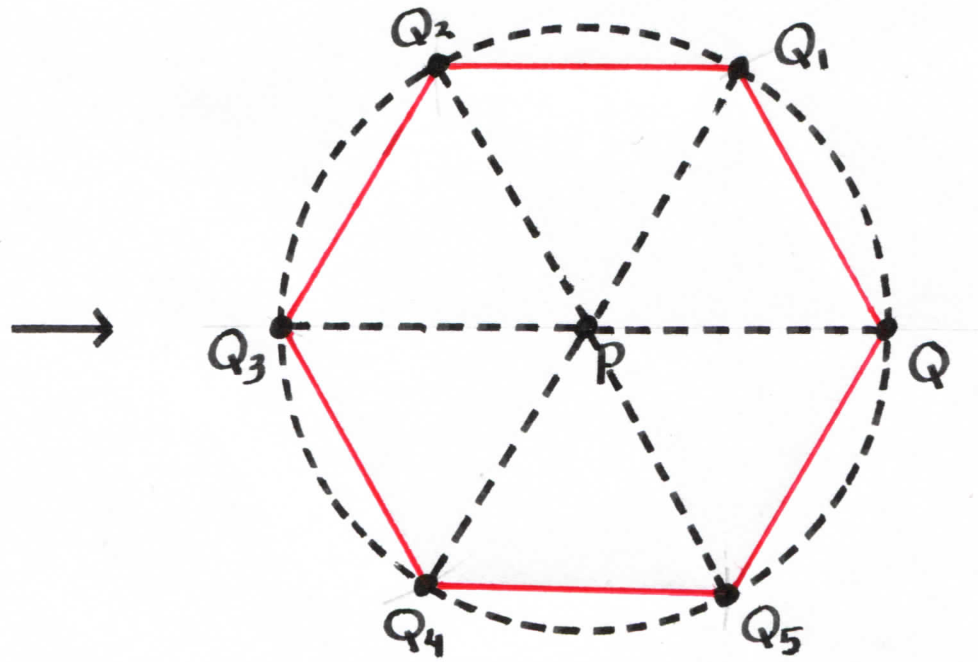
Same compass setting:

$\|\overrightarrow{Q_2Q_1}\| = \|\overrightarrow{Q_1Q}\|$
$= \|\overrightarrow{PQ}\|$
$= \|\overrightarrow{PQ_2}\|$

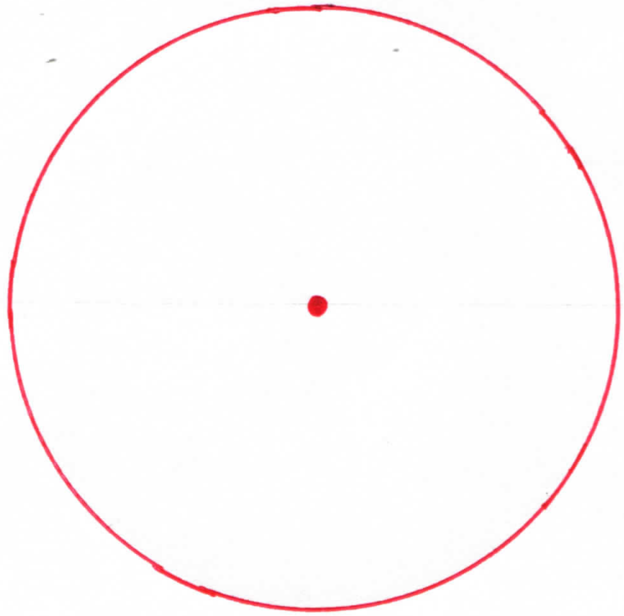
... etc.

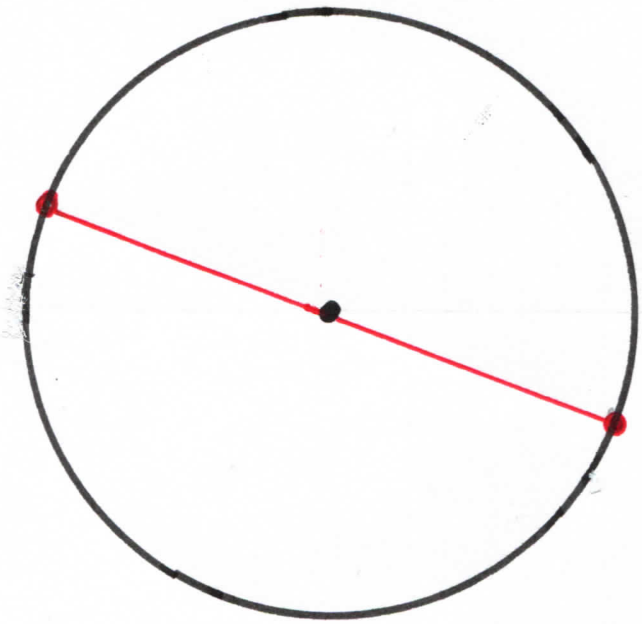
DRAWINGS 13.12 (continued)





DRAWINGS 13.13(a)

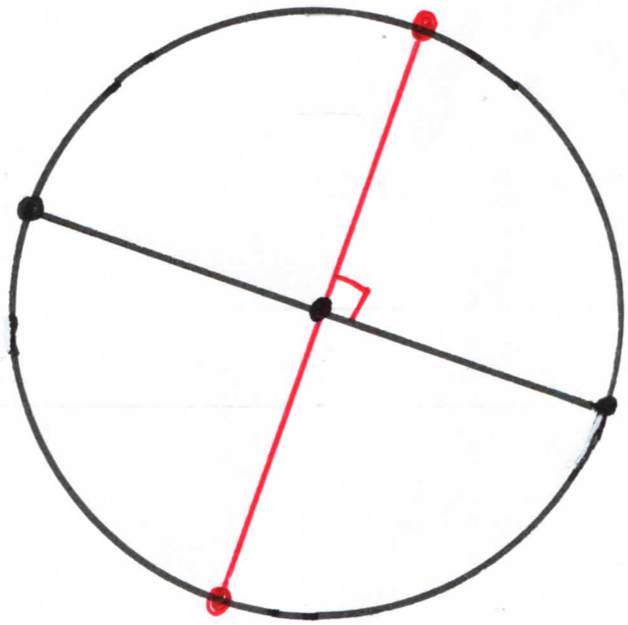




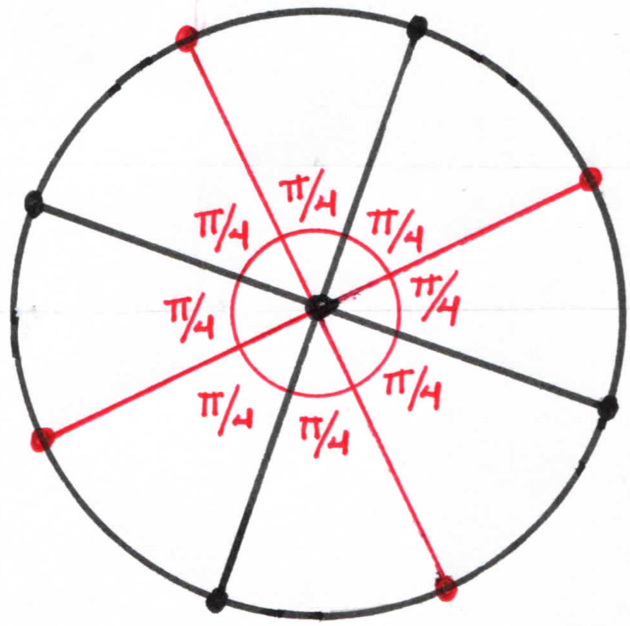
Draw any diameter.

DRAWINGS 13.13(a) (continued)

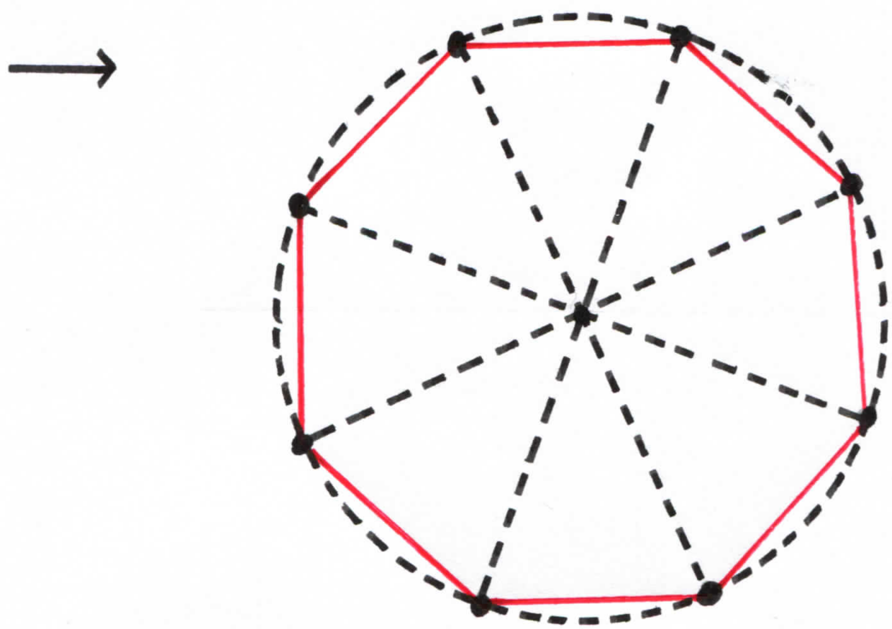
→ (Construction 13.2 and DRAWINGS 13.5)



→ (Construction 13.2 and DRAWINGS 13.5)

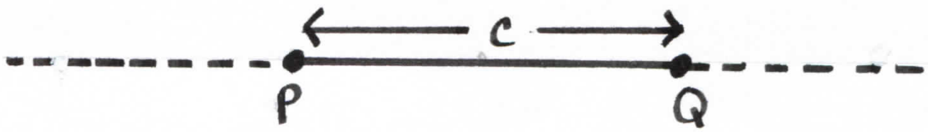


DRAWINGS 13.13(a) (continued)

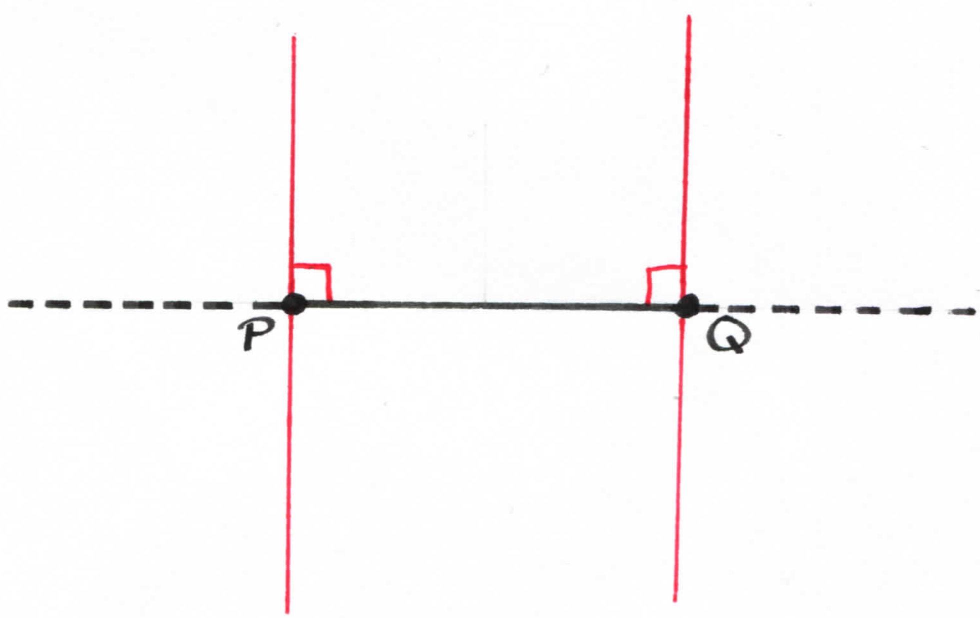


DRAWINGS 13.13(b)

Line segment PQ of specified length $c = \|\overrightarrow{PQ}\|$:

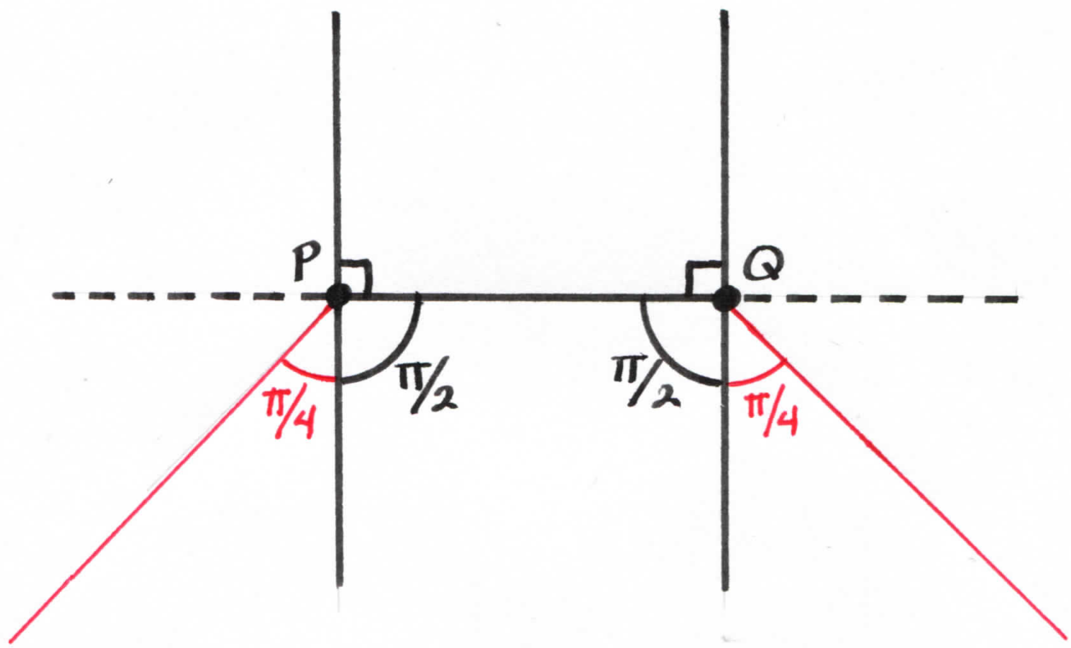


Draw lines $\perp$ to PQ at P and Q (Construction 13.4).



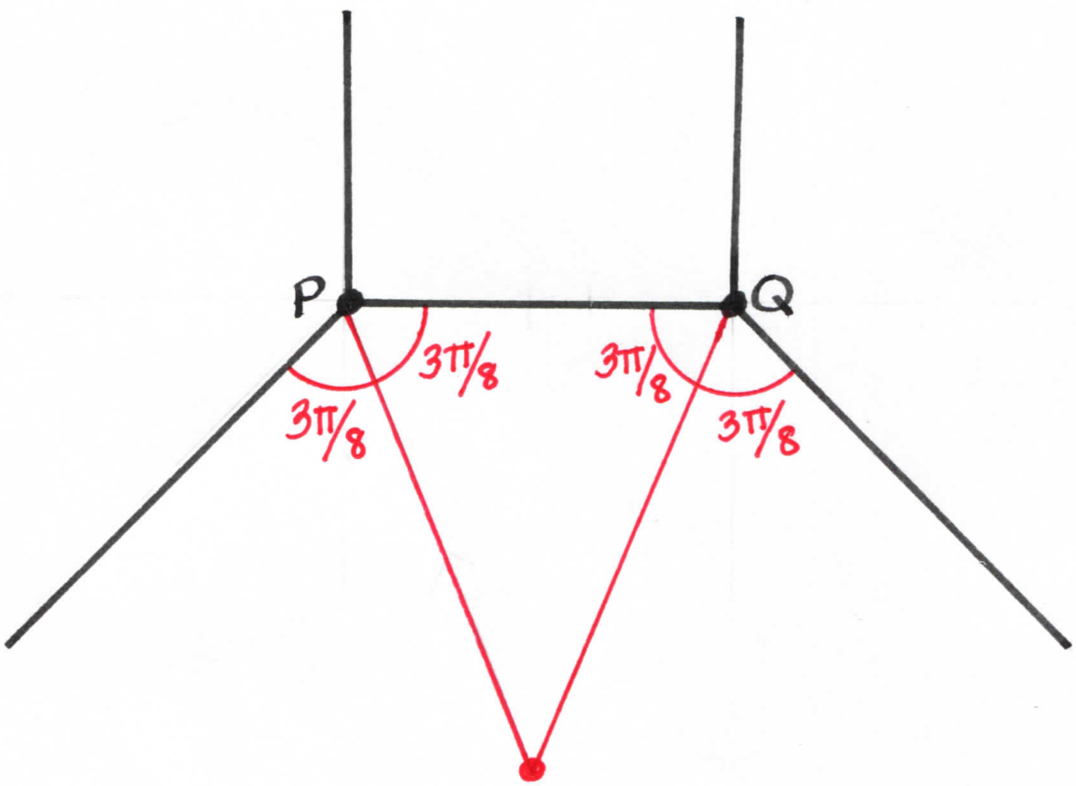
DRAWINGS 13.13(b) (continued)

Bisect right angles, using Construction 13.2.

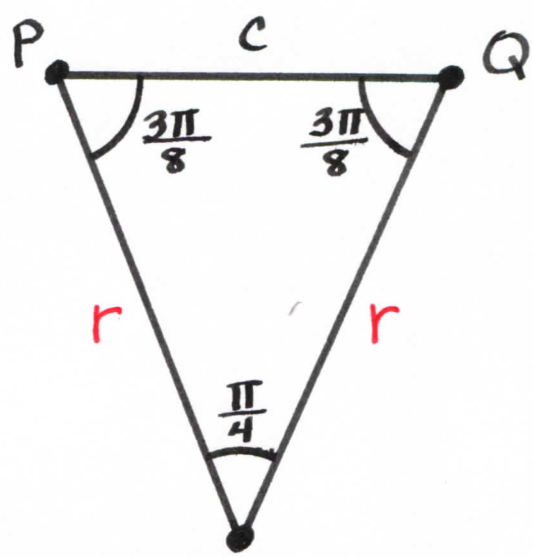


DRAWINGS 13.13 (b) (continued)

Bisect the angles of measure $\frac{3\pi}{4}$.

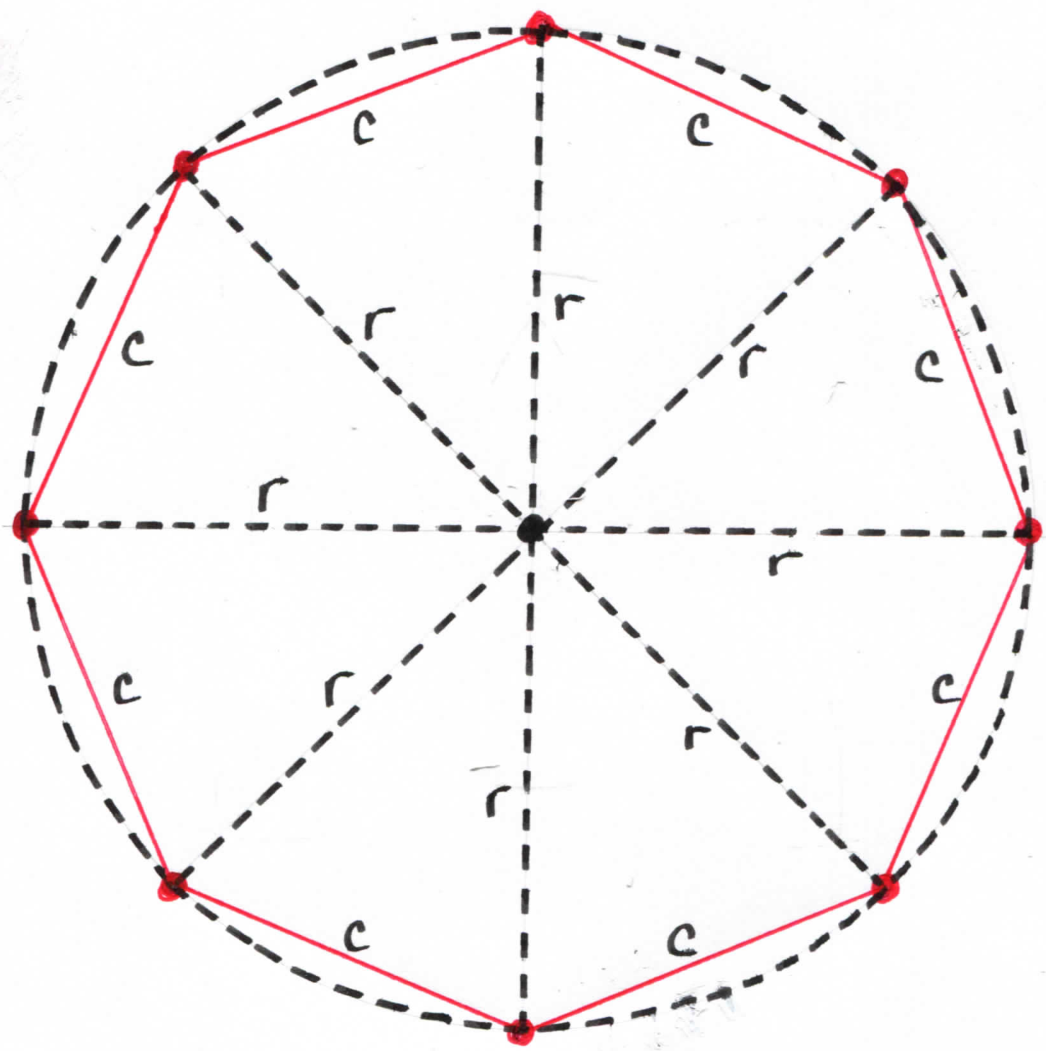


DRAWINGS 13.13(b) (continued)



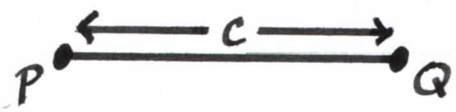
Length labelled r is radius of circle that begins construction of regular octagon.

DRAWINGS 13.13 (b) (continued)



DRAWINGS 13.14

Given:

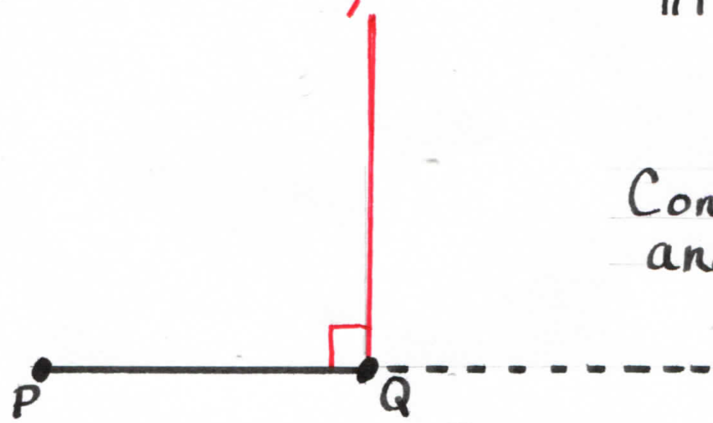


$c = \|\overrightarrow{PQ}\|$ specified

→

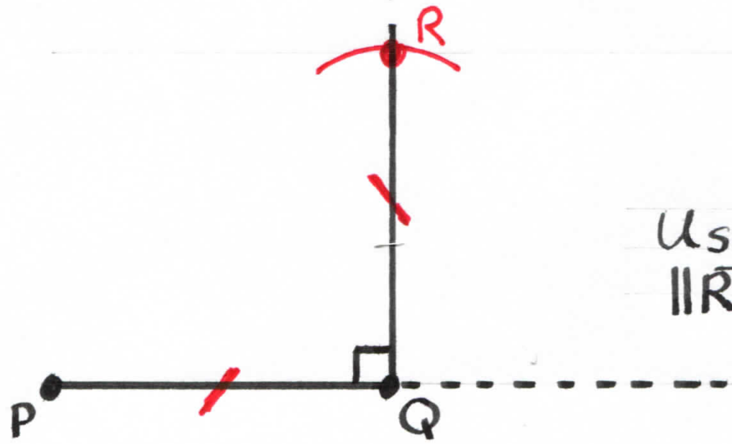
Compass setting
$\|\overrightarrow{PQ}\| = c$

→



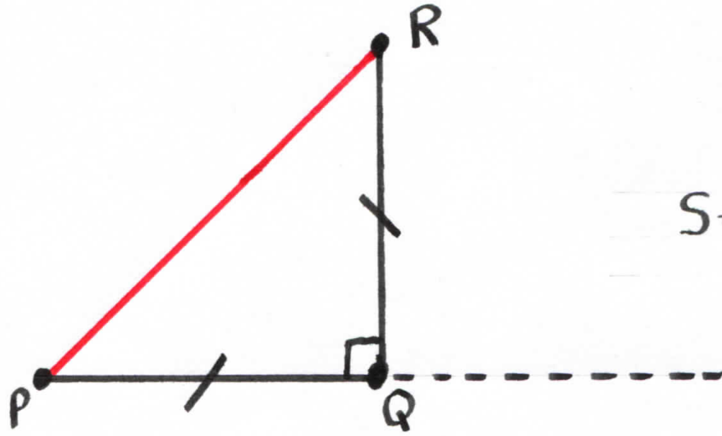
Construction 13.4
and DRAWINGS 13.7

→



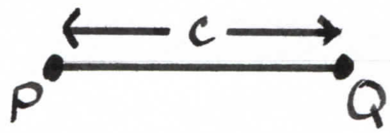
Use compass:
$\|\overrightarrow{RQ}\| = \|\overrightarrow{PQ}\|$

→

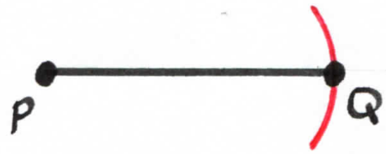


Straight-edge

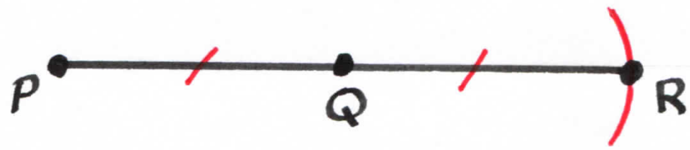
DRAWINGS 13.15

Given: 

$c = \|\overrightarrow{PQ}\|$ specified

$\longrightarrow$ 

Set compass at $\|\overrightarrow{PQ}\| = c$.

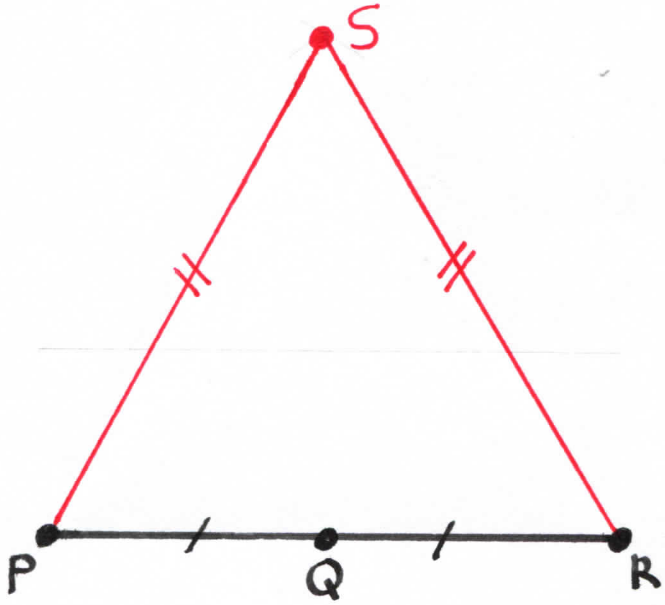
$\longrightarrow$ 

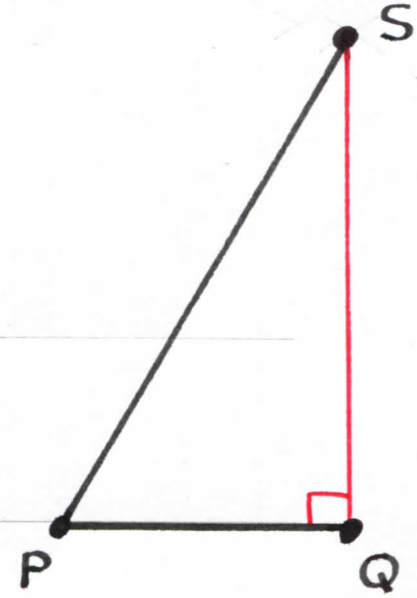
Extend line; make $\|\overrightarrow{QR}\| = \|\overrightarrow{PQ}\|$.

$\longrightarrow$ 

Set compass at $\|\overrightarrow{PR}\|$.

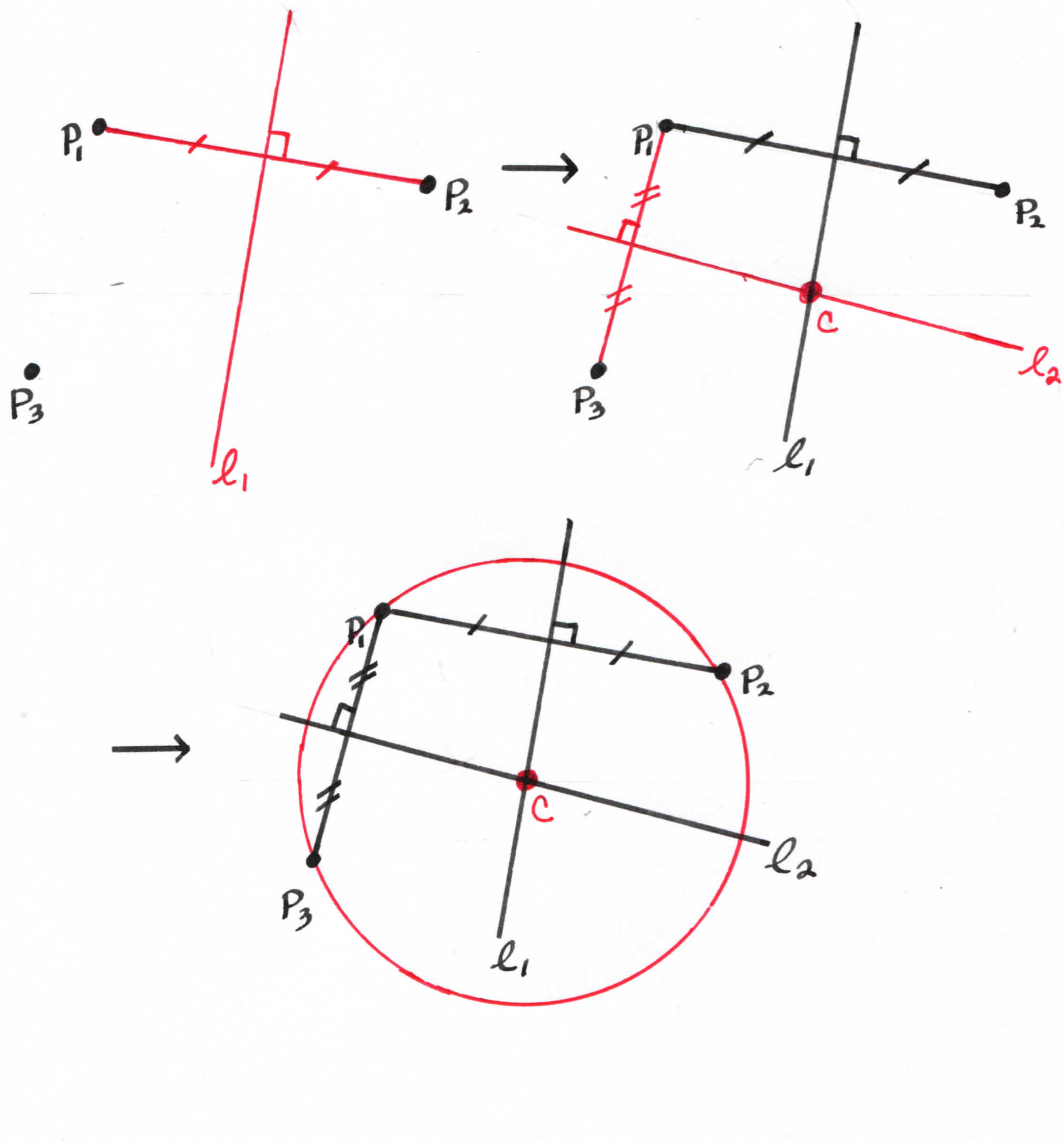
$\longrightarrow$ (... Construction 13.1 and DRAWINGS 13.4)



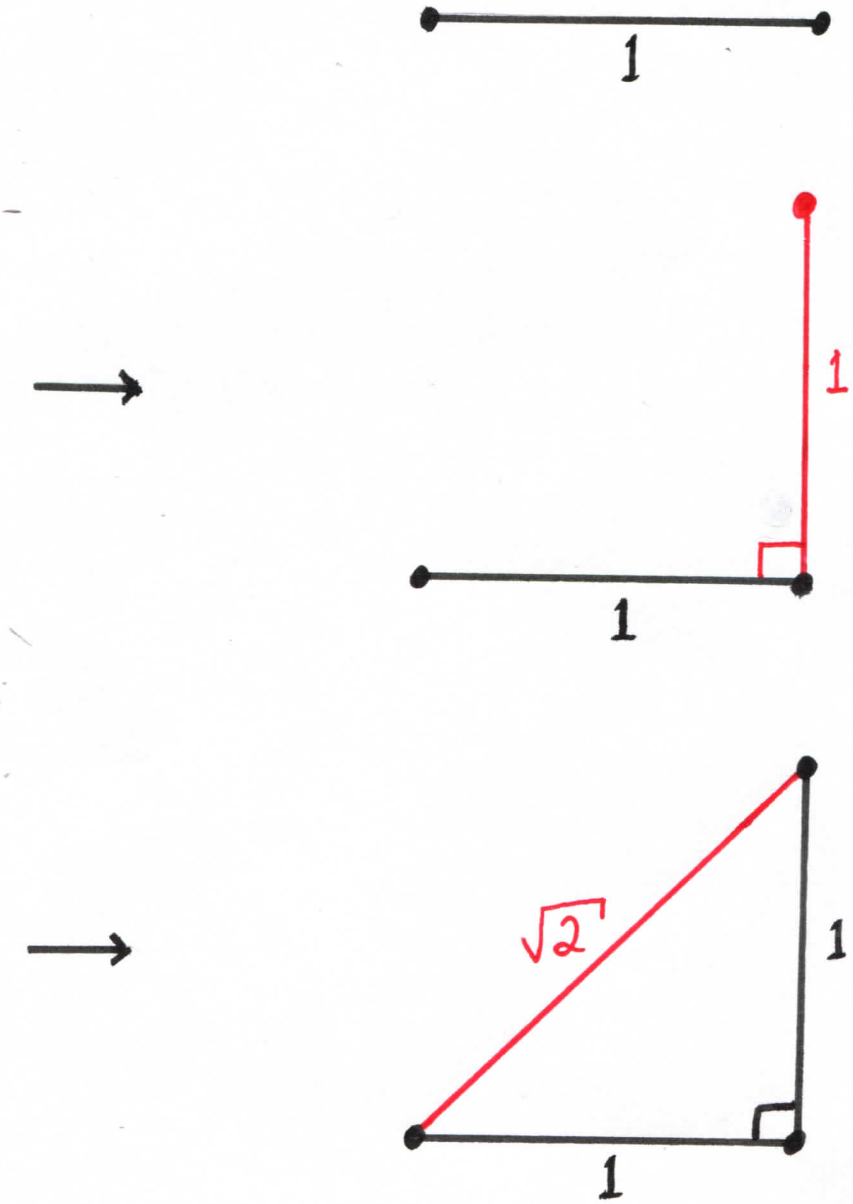
$\longrightarrow$


$$\|\overrightarrow{SR}\| = \|\overrightarrow{SP}\| = \|\overrightarrow{PR}\| = 2\|\overrightarrow{PQ}\|$$

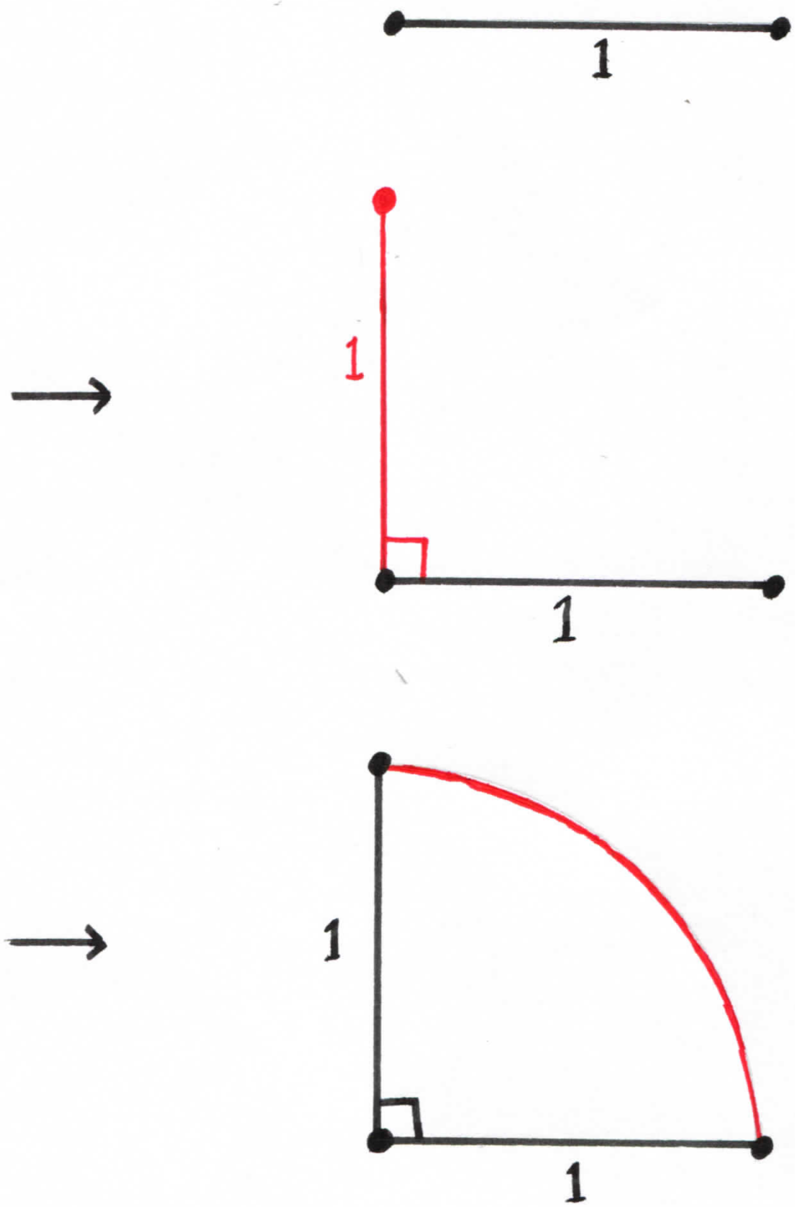
DRAWINGS 13.16



DRAWINGS 13.17



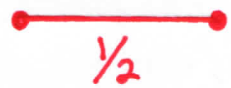
DRAWINGS 13.18



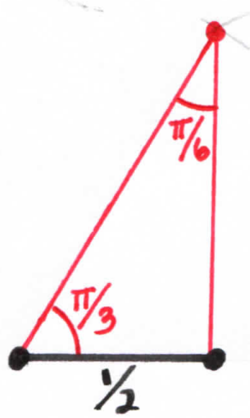
Compass,
set at length 1.

DRAWINGS 13.19

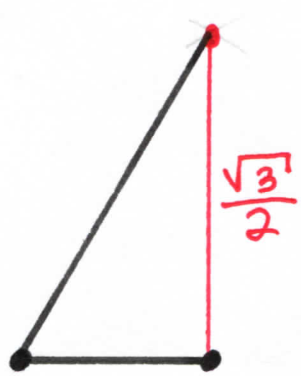
$\longrightarrow$



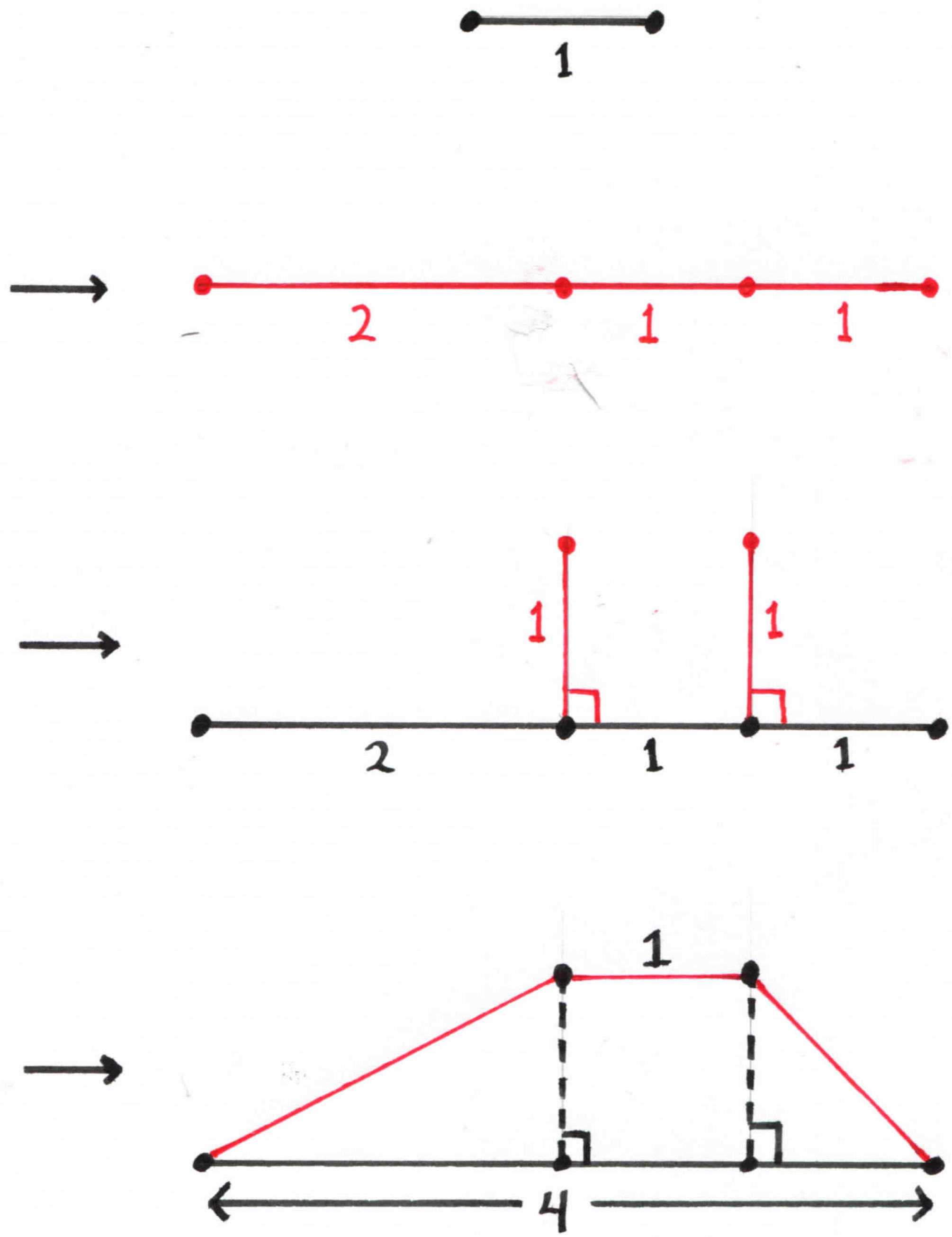
$\longrightarrow$ (Construction 13.12)



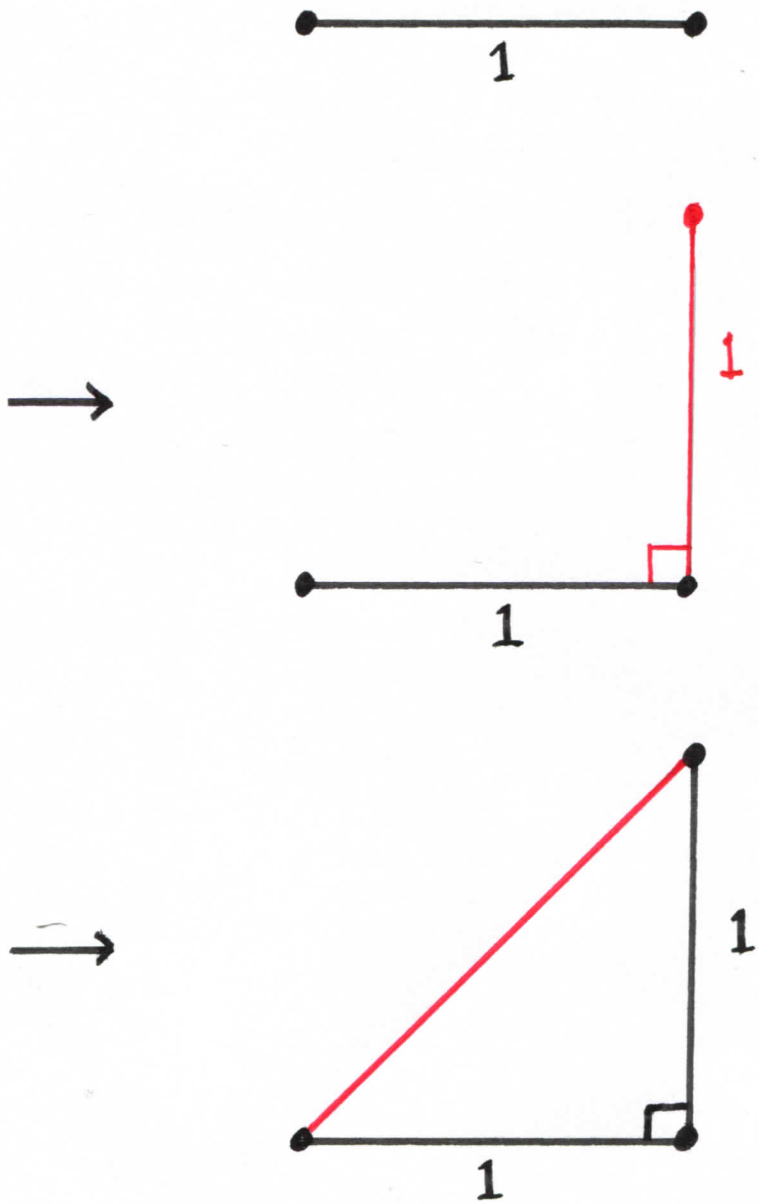
$\longrightarrow$



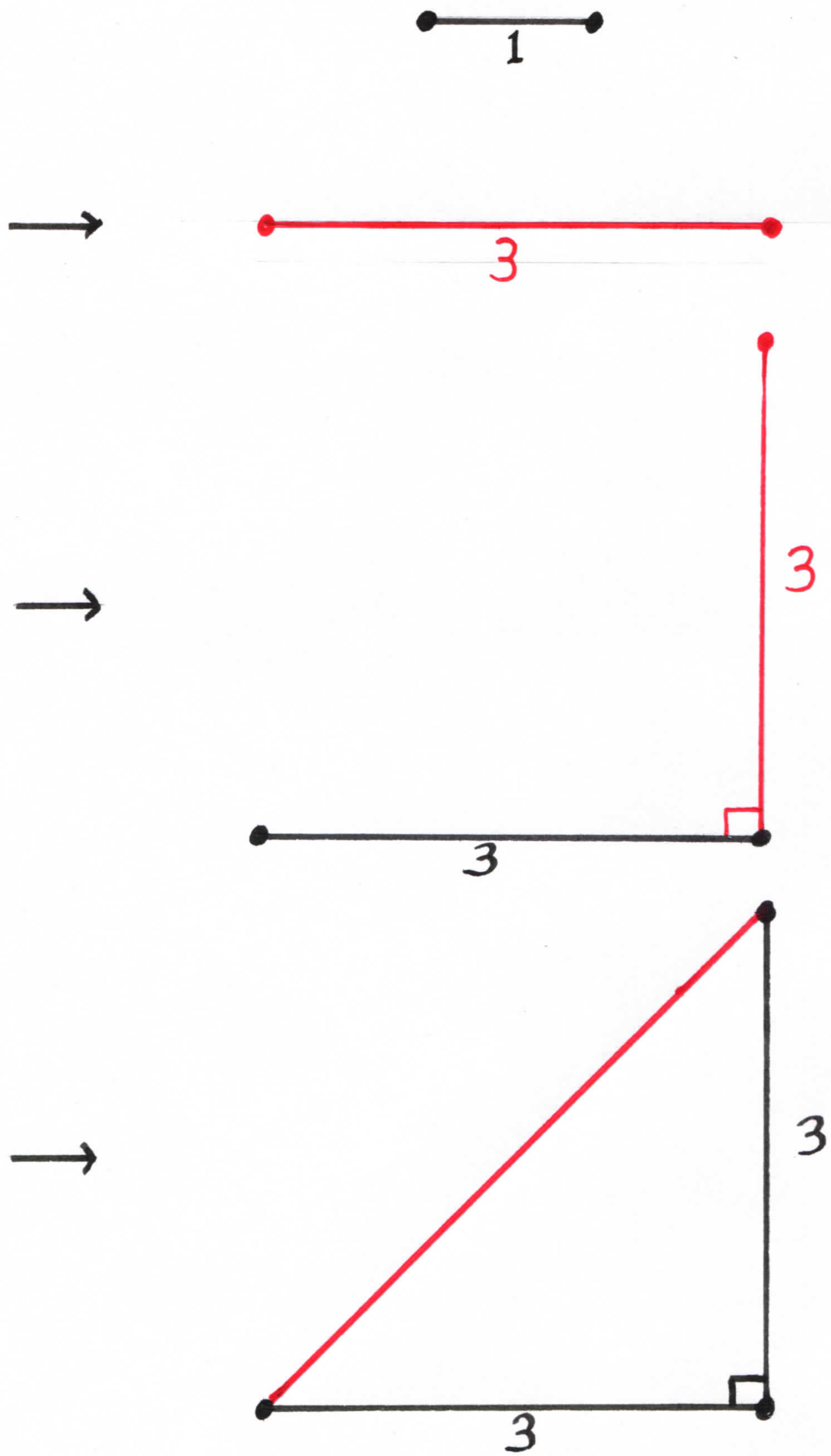
DRAWINGS 13.20



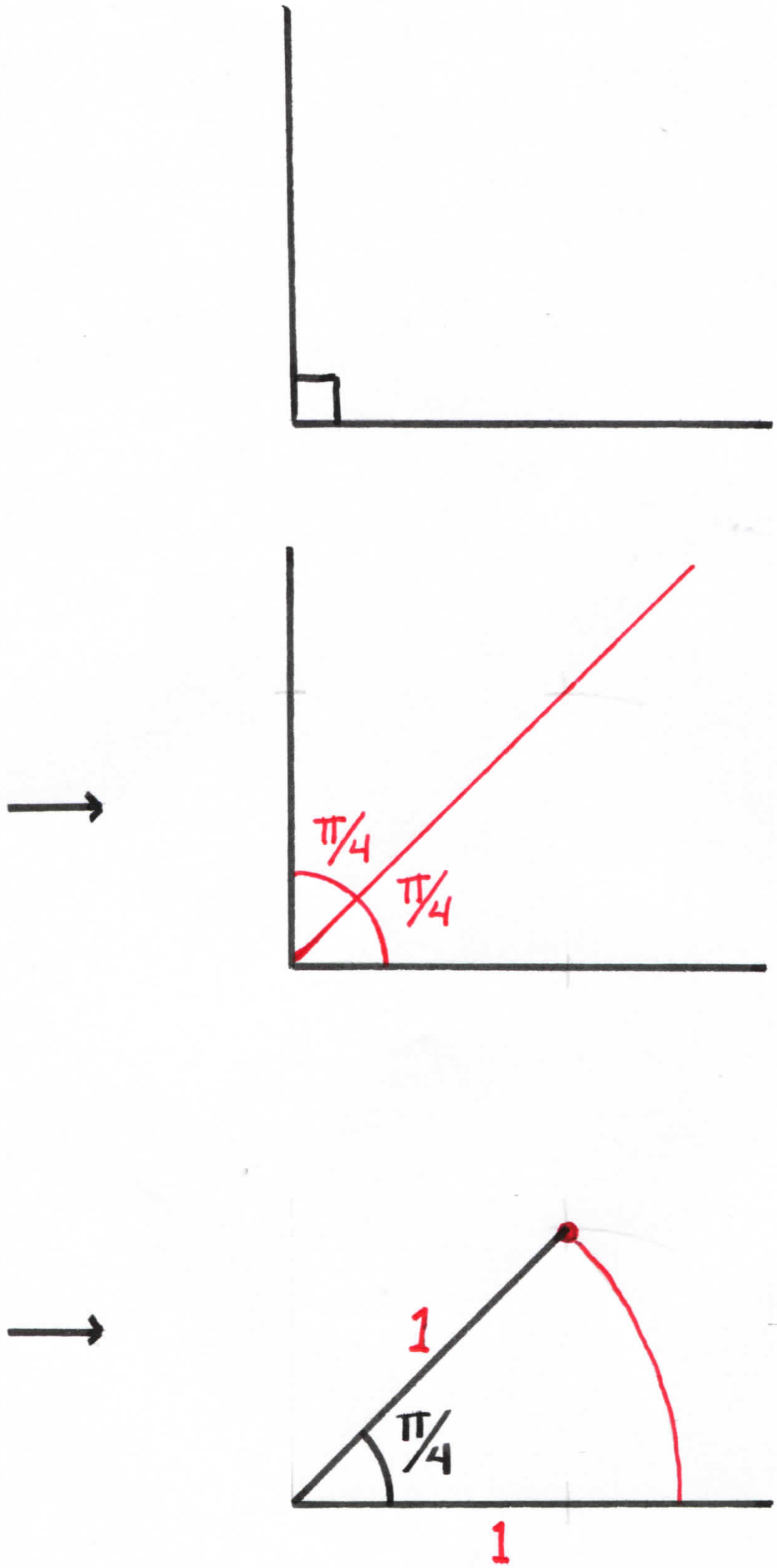
DRAWINGS 13.21



DRAWINGS 13.22

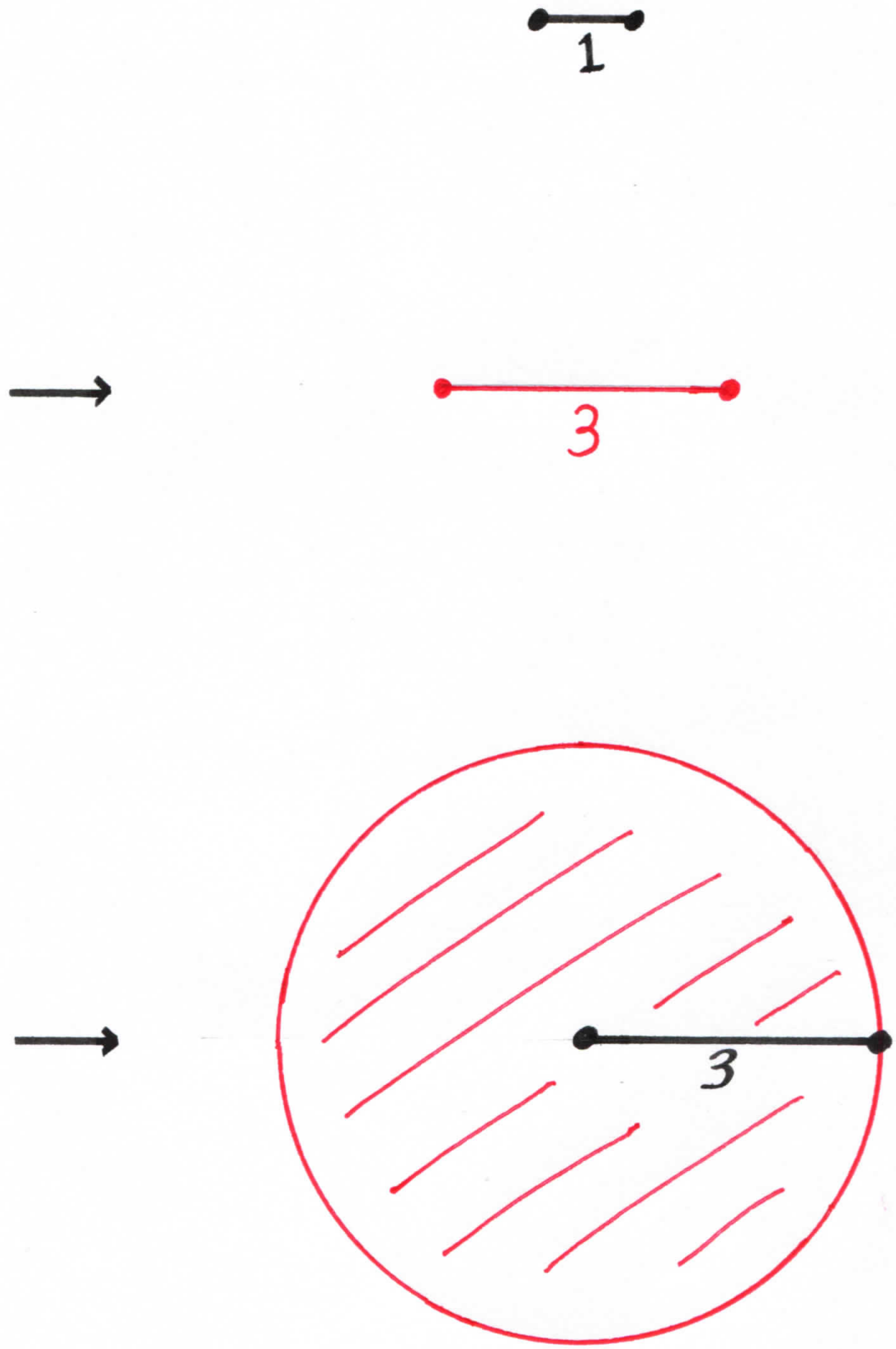


DRAWINGS 13.23



Area of sector is $\pi/8$.

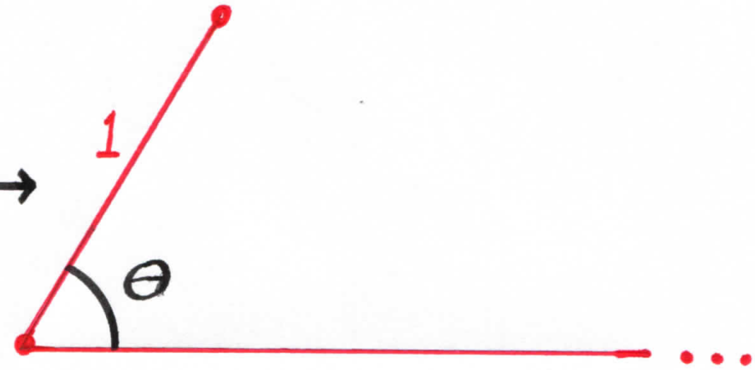
DRAWINGS 13.24



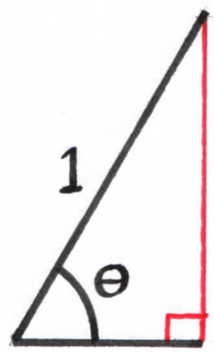
Disc of radius 3; area $= 9\pi$.

DRAWINGS 13.25

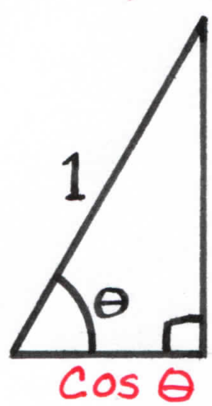
θ constructible $\longrightarrow$



$\longrightarrow$



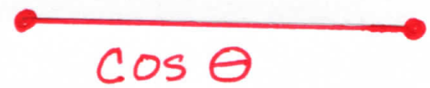
$\longrightarrow$



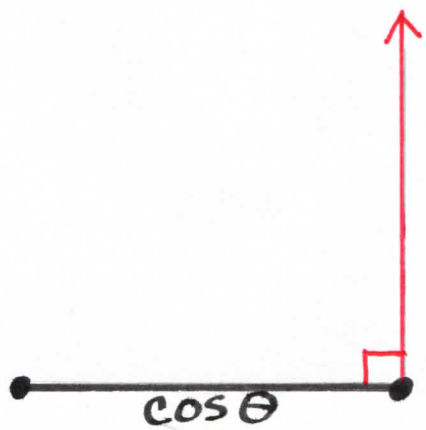
$\longrightarrow$ $\cos\theta$ is constructible

DRAWINGS 13.25 (continued)

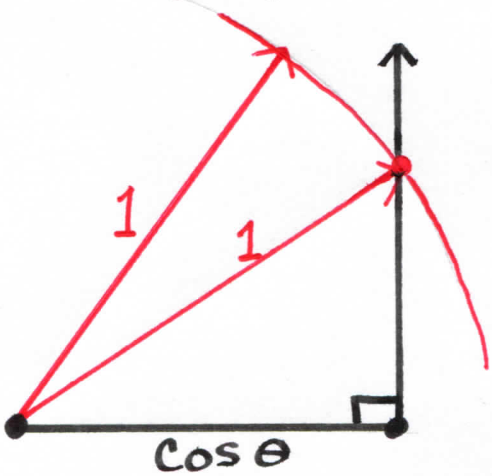
$\cos\theta$ constructible $\rightarrow$



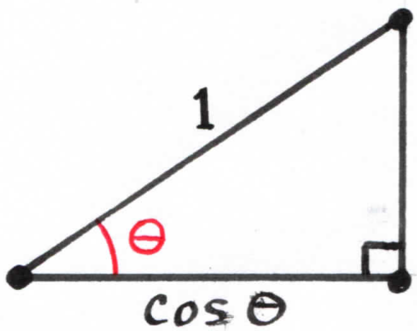
$\rightarrow$



$\rightarrow$



$\rightarrow$



θ constructed

CHAPTER XIV: More Geometry and Trigonometry Problems.

Examples 14.1–14.18 will be preceded by a listing of relevant geometry and trigonometry results, with their original numbering, and will be followed by solutions.

Corollary 2.14. (b) Suppose $\vec{a}$ and $\vec{b}$ point in opposite directions. Then π equals the measure of both the clockwise and counterclockwise angles from $\vec{a}$ to $\vec{b}$. See DRAWINGS 2.17 at the end of Chapter II.

Proposition 2.19. Consider an arc and sector as in Definitions 2.18.

(a) The measure of the angle between the two lines in the boundary of the sector is $(\theta_2 - \theta_1)$.

(b) The length of the arc is $R(\theta_2 - \theta_1)$.

(c) The perimeter of the sector is $2R + R(\theta_2 - \theta_1)$.

Proposition 3.3. In a parallelogram, opposite sides have equal length.

Proposition 3.6. Suppose ℓ_1, ℓ_2, and ℓ_3 are lines, with ℓ_1 and ℓ_2 parallel, ℓ_3 intersecting both ℓ_1 and ℓ_2 and angles of measure $\theta_j, j = 1, 2, \ldots, 8$ as drawn in DRAWING 3.4 at the end of Chapter III.

Then $\theta_1 + \theta_2 = \pi, \theta_1 = \theta_3 = \theta_5 = \theta_7$, and $\theta_2 = \theta_4 = \theta_6 = \theta_8$.

Proposition 3.8. In a parallelogram, opposite interior angles are of equal measure and the measures of adjacent interior angles add up to π.

Corollary 3.10. For $n = 3, 4, 5, \ldots$, the sum of the measures of the interior angles in an n-gon is $(n-2)\pi$.

Proposition 3.13. The diagonals of a parallelogram bisect each other.

HWIII.2. Prove that, in the drawing of a diagonal of a parallelogram in DRAWING 3.21 (see end of Chapter III), $\theta_1 = \theta_4$ and $\theta_2 = \theta_3$.

Proposition 4.15. Suppose $\vec{a}$ and $\vec{b}$ are two nontrivial vectors. Then the following are equivalent.

(a) $\vec{a}$ and $\vec{b}$ are orthogonal.

(b) (Pythagorean theorem) $\|\vec{a} + \vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2$.

(c) $\|\vec{a} + s\vec{b}\| \geq \|\vec{a}\|$ for all real s.

(d) The measure of the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{2}$.

Corollary 5.3. Suppose $\ell_1, \ell_2, \ell_3, \ell_4$ are lines, ℓ_1 and ℓ_2 are parallel, and both ℓ_3 and ℓ_4 are perpendicular to ℓ_2 (see DRAWING 5.4(a) at the end of Chapter V). Then both ℓ_3 and ℓ_4 are perpendicular to ℓ_1, ℓ_3 is parallel to ℓ_4, the line segments from ℓ_1 to ℓ_2 are of equal length, and the line segments from ℓ_3 to ℓ_4 are of equal length. See DRAWING 5.4(b) at the end of Chapter V.

Proposition 5.5. In any parallelogram, the diagonals are perpendicular if and only if the parallelogram is a rhombus.

Proposition 5.6. In a parallelogram, the diagonals are of equal length if and only if the parallelogram is a rectangle.

Proposition 5.7. In a parallelogram, the sum of the squares of the lengths of the sides equals the sum of the squares of the lengths of the diagonals.

Some Properties 6.3. From staring at the picture of $(\cos\theta, \sin\theta)$ in DRAWING 6.1 at the end of Chapter VI and using the symmetry of the unit circle, the following properties seem believable, for any real θ (see DRAWING 6.3 at the end of Chapter VI). Euler's formula provides straightforward proofs.

(i) $\cos(-\theta) = \cos\theta$.

(ii) $\sin(-\theta) = -\sin\theta$.

(iii) $\cos(\theta + \pi) = -\cos\theta$.

(iv) $\sin(\theta + \pi) = -\sin\theta$.

(v) $(\cos\theta)^2 + (\sin\theta)^2 = 1$.

(vi) $|\sin\theta| \leq 1$ and $|\cos\theta| \leq 1$.

(vii) $\cos(\frac{\pi}{2} + \theta) = -\sin\theta = -\cos(\frac{\pi}{2} - \theta)$ and $\sin(\frac{\pi}{2} + \theta) = \cos\theta = \sin(\frac{\pi}{2} - \theta)$.

(viii) $\cos(\theta + 2k\pi) = \cos\theta$, $\sin(\theta + 2k\pi) = \sin\theta$, for any real θ, integer k (this is called **periodicity** of sine and cosine).

Proposition 6.4. Let θ, ψ be arbitrary real numbers.

(i) $\cos(\theta + \psi) = \cos\theta\cos\psi - \sin\theta\sin\psi$.

(ii) $\sin(\theta + \psi) = \sin\theta\cos\psi + \sin\psi\cos\theta$.

(iii) $(\cos\theta)(\cos\psi) = \frac{1}{2}(\cos(\theta + \psi) + \cos(\theta - \psi))$.

(iv) $(\sin\theta)(\sin\psi) = \frac{1}{2}(\cos(\theta - \psi) - \cos(\theta + \psi))$.

(v) $(\sin\theta)(\cos\psi) = \frac{1}{2}(\sin(\theta + \psi) + \sin(\theta - \psi))$.

(vi) $(\cos\theta)^2 = \frac{1}{2}(1 + \cos(2\theta))$.

(vii) $(\sin\theta)^2 = \frac{1}{2}(1 - \cos(2\theta))$.

(viii) $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$.

(ix) $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$.

Theorem 7.1: Law of Cosines. If a, b, c are the lengths of the sides of a triangle and θ is the angle measure opposite the side of length c, then

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

See DRAWING 7.1 at the end of Chapter VII.

Theorem 7.5. For any right triangle, denote by c the length of the hypotenuse, by a and b the lengths of the legs, and by θ the measure of the angle between the hypotenuse and the side of length a.

Then

$$\cos\theta = \frac{a}{c} \text{ ("adjacent over hypotenuse")} \qquad \sin\theta = \frac{b}{c} \text{ ("opposite over hypotenuse")}$$

and

$$\tan\theta = \frac{b}{a} \text{ ("opposite over adjacent")}.$$

See DRAWING 7.5 at the end of Chapter VII.

Theorem 7.7: Law of Sines. If a triangle has sides of length c_1, c_2, c_3, and, for $j = 1, 2, 3, \theta_j$ is the measure of the angle opposite the side of length c_j, then

$$\frac{c_1}{\sin\theta_1} = \frac{c_2}{\sin\theta_2} = \frac{c_3}{\sin\theta_3}.$$

Theorem 7.10. Let P, Q, R be the three vertices of a triangle (see DRAWINGS 7.10 at the end of Chapter VII). Then the following are equivalent.

(a) $\|\overrightarrow{PR}\| = \|\overrightarrow{QR}\|$.

(b) The orthogonal projection of the vertex R onto the opposite side $\overrightarrow{PQ}$ is the midpoint of $\overrightarrow{PQ}$.

(c) The measure of the interior angle at P equals the measure of the interior angle at Q.

(d) The line segment between R and its orthogonal projection onto $\overrightarrow{PQ}$ bisects the interior angle at R.

Examples 7.11(e)

$\cos(\frac{\pi}{4}) = \sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$.

$\cos(\frac{\pi}{3}) = \sin(\frac{\pi}{6}) = \frac{1}{2}, \cos(\frac{\pi}{6}) = \sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$.

Theorem 7.14. Given any pair of points Q and R on a circle, the central angle formed by them measures twice the measure of the inscribed angle formed by them with a third point P on the circle; that is, $\theta = 2\psi$, in DRAWING 7.14 at the end of Chapter VII.

Theorem 7.15. Suppose the circles in DRAWINGS 7.19 and 7.20 at the end of Chapter VII are both of radius r.

(a) In DRAWING 7.19, $\theta = \frac{1}{2}(\phi + \psi)$.

(b) In DRAWING 7.20, $\theta = \frac{1}{2}(\phi - \psi)$.

Proposition 8.5. If two triangles are similar, then ratios of corresponding lengths of sides are equal. That is, suppose T_1 and T_2 are triangles and f is a composition of (a)–(d) in Definitions 8.1, with $T_2 = f(T_1)$. Further suppose that $\vec{S}_1$ and $\vec{S}_2$ are two sides of T_1. Then

$$\frac{\|\vec{S}_1\|}{\|\vec{S}_2\|} = \frac{\|f(\vec{S}_1)\|}{\|f(\vec{S}_2)\|}.$$

Proposition 9.6. If I is a point and $\vec{a}$ and $\vec{b}$ are vectors, then the area of the triangle with vertices $I, I + \vec{a}, I + \vec{b}$ is $\frac{1}{2}|\det \begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix}|$.

Theorem 10.1. Suppose T_1 and T_2 are triangles, T_1 has sides S_1, S_2, and S_3, T_2 has sides S_4, S_5, and S_6, and, for $j = 1, 2, 3, 4, 5, 6$, θ_j is the measure of the angle opposite S_j. See DRAWING 10.1 at the end of Chapter X.

(a) Any one of the following conditions (SAS, SSS, AAS) implies that T_1 and T_2 are congruent.

SAS. $\|S_1\| = \|S_4\|, \|S_2\| = \|S_5\|$ and $\theta_3 = \theta_6$. See DRAWING 10.2 at the end of Chapter X.

Informally, agreement on two sides and the angle between the two sides implies congruence.

SSS. $\|S_1\| = \|S_4\|, \|S_2\| = \|S_5\|$, and $\|S_3\| = \|S_6\|$. See DRAWING 10.3 at the end of Chapter X.

Informally, agreement on all sides implies congruence.

AAS. $\|S_1\| = \|S_4\|, \theta_1 = \theta_4$, and $\theta_2 = \theta_5$. See DRAWING 10.4 at the end of Chapter X.

Informally, agreement on two angles and a side implies congruence.

(b) **AAA or AA.** $\theta_1 = \theta_4, \theta_2 = \theta_5$, and $\theta_3 = \theta_6$ (the last equality follows automatically from the first two, by Proposition 3.8) implies that T_1 and T_2 are similar. See DRAWING 10.5 at the end of Chapter X.

Informally, agreement of all angles implies similarity.

Proposition 11.4. A diagonal in a parallelogram bisects opposite interior angles if and only if the parallelogram is a rhombus (Definitions 5.4).

Proposition 12.1. The area of either triangle formed by drawing a diagonal in a parallelogram, as in DRAWING 12.1 at the end of Chapter XII, is half the area of the parallelogram.

Theorem 12.3. The area of a trapezoid equals the height times the average of the lengths of the two parallel sides; that is,

$$\frac{1}{2}h(b_1 + b_2),$$

where h is the distance between the parallel sides of lengths b_1 and b_2, as in DRAWING 12.2 at the end of Chapter XII.

Theorem 12.4. If $0 < \psi \leq 2\pi$ and a closed sector of a disc of radius R is determined by an arc of length $R\psi$, then the sector has area $\frac{1}{2}R^2\psi$. See DRAWING 12.8 at the end of Chapter XII.

Theorem 12.8. As in Definitions 12.7, $H > 0, r > 0$, and Ω is a subset of $\mathbf{R}^2$.

(a) The volume of the cylinder of height H and radius r is $\pi r^2 H$.

(b) The volume of the generalized cylinder of height H and base Ω is (area of Ω)H.

(c) The volume of the cone of height H and radius r is $\frac{\pi}{3}r^2 H$.

(d) The volume of the generalized cone of height H and base Ω is $\frac{1}{3}$(area of Ω)H.

(e) The volume of a ball of radius r is $\frac{4}{3}\pi r^3$.

PROBLEMS

Make decimal approximations with a calculator when necessary to avoid trig or inverse trig functions in your answer.

Examples 14.1. In each of the circles in DRAWINGS 14.1 at the end of this chapter, find the measure of the angle θ. C will always be the center of the circle.

Examples 14.2. Get exact expressions (no calculator or other decimal expansions) for the following.

(a) $\sin(\frac{5\pi}{12})$.

(b) $\sin(\frac{7\pi}{12})$.

(c) $\cos(-\frac{\pi}{3})$.

Example 14.3. Find interior angles, where possible, in DRAWINGS 14.2 at the end of this chapter. Do *not assume* the sides AE and BD are parallel.

Example 14.4. SAME as in Example 14.3, except assume AE and BD are parallel.

Example 14.5. Find x in DRAWING 14.3 at the end of this chapter. The quadrilateral is a parallelogram.

Examples 14.6. Find the volumes of each of the following.

(a) A cylinder whose base has radius 10, height is 6.

(b) A cone whose base has radius 10, height is 6.

(c) A pyramid with a square base (as in Egypt) whose side lengths are 700 feet and height is 600 feet.

(d) A generalized cylinder whose base has area 100 meters squared, with a height of 20 meters.

(e) A ball of radius 18 inches.

Example 14.7. Get the area of a disc whose circumference is 12π.

Example 14.8. Suppose all the interior angles of a six-sided polygon (called a *hexagon*) have equal measure. Find the measure of each interior angle.

Example 14.9. Find x in DRAWING 14.4 at the end of this chapter.

Example 14.10. In DRAWING 14.5 at the end of this chapter, C is the center of a circle and ℓ is a tangent to the circle at P. Get the radius of the circle.

Example 14.11. In DRAWING 14.6 at the end of this chapter, ℓ_1 and ℓ_2 are parallel. Find all angle measures in DRAWING 14.6.

Examples 14.12. In each of the following drawings of parallelograms, fill in lengths of sides and measures of angles where possible.

(a) DRAWING 14.7 at the end of this chapter.

(b) DRAWING 14.8 at the end of this chapter.

(c) DRAWING 14.9 at the end of this chapter.

(d) DRAWING 14.10 at the end of this chapter.

(e) DRAWING 14.11 at the end of this chapter.

(f) DRAWING 14.12 at the end of this chapter.

(g) DRAWING 14.13 at the end of this chapter.

(h) DRAWING 14.14 at the end of this chapter.

(i) DRAWING 14.15 at the end of this chapter.

(j) DRAWING 14.16 at the end of this chapter.

Example 14.13. When I am five feet away from a ten-foot tall lamppost, my shadow extends four feet further away from the lamppost. How tall am I? Assume I and the lamppost are both perpendicular to the ground.

Example 14.14. In DRAWING 14.17 at the end of this chapter, the two horizontal lines are parallel. Find x in DRAWING 14.17.

Examples 14.15. In each of the following, get the area and perimeter. Any curve that is not a line is an arc of a circle. Any point that appears to be the center of a circle is the center of a circle.

(a) DRAWING 14.18 at the end of this chapter.

(b) DRAWING 14.19 at the end of this chapter.

(c) DRAWING 14.20 (parallelogram) at the end of this chapter.

(d) DRAWING 14.21 (trapezoid) at the end of this chapter.

Examples 14.16. Fill in lengths of sides and measures of angles, where possible.

(a) DRAWING 14.22 at the end of this chapter.

(b) DRAWING 14.23 at the end of this chapter.

(c) DRAWING 14.24 at the end of this chapter.

(d) DRAWING 14.25 at the end of this chapter.

(e) DRAWING 14.26 at the end of this chapter.

(f) DRAWING 14.27 at the end of this chapter.

(g) DRAWING 14.28 at the end of this chapter.

(h) DRAWING 14.29 at the end of this chapter.

Examples 14.17. In DRAWING 14.30 at the end of this chapter, get $\cos\theta, \sin\theta, \tan\theta, \cos(2\theta), \sin(2\theta), \cos(\frac{1}{2}\theta), \sin(\frac{1}{2}\theta)$ and a decimal approximation of θ. Also get the area and perimeter of the triangle.

Example 14.18. Fill in lengths of sides and measures of angles, where possible, in DRAWING 14.31 at the end of this chapter, where the lines ℓ_1 and ℓ_2 are parallel.

Example 14.19. In DRAWING 14.66 at the end of this chapter, prove that (a)–(e) below are equivalent.

(a) $\frac{\|\overrightarrow{DE}\|}{\|\overrightarrow{DA}\|} = \frac{\|\overrightarrow{DC}\|}{\|\overrightarrow{DB}\|}$.

(b) $\frac{\|\overrightarrow{DE}\|}{\|\overrightarrow{EA}\|} = \frac{\|\overrightarrow{DC}\|}{\|\overrightarrow{CB}\|}$.

(c) The triangles DEC and DAB are similar.

(d) $\overrightarrow{EC}$ is parallel to $\overrightarrow{AB}$.

(e) $\frac{\|\overrightarrow{EC}\|}{\|\overrightarrow{AB}\|} = \frac{\|\overrightarrow{DE}\|}{\|\overrightarrow{DA}\|} = \frac{\|\overrightarrow{DC}\|}{\|\overrightarrow{DB}\|}$.

Note that HWIII.4 is a special case of (a) $\to$ [(d) and (e)], with $\frac{\|\overrightarrow{DE}\|}{\|\overrightarrow{DA}\|} = \frac{1}{2} = \frac{\|\overrightarrow{DC}\|}{\|\overrightarrow{DB}\|}$.

We recommend the vector methods of Chapter III, as in HWIII.4, for (a) $\to$ [(d) and (e)].

SOLUTIONS

Examples 14.1. (a) This is 2.19. $\frac{6\pi}{10} = \frac{3\pi}{5}$ radians.

(b) This is 7.14. $\frac{1}{2}\left(\frac{3\pi}{5}\right) = \frac{3\pi}{10}$ radians.

(c) This is 7.15(a). $\frac{1}{2}\left(\frac{6\pi+2\pi}{10}\right) = \frac{2\pi}{5}$ radians.

(d) This is 7.15(b). $\frac{1}{2}\left(\frac{6\pi-2\pi}{10}\right) = \frac{\pi}{5}$ radians.

Examples 14.2. This is 7.11(e), 6.3 and 6.4. See DRAWINGS 14.32 at the end of this chapter.

(a) $\sin(\frac{5\pi}{12}) = \sin(\frac{\pi}{4} + \frac{\pi}{6}) = \sin(\frac{\pi}{4})\cos(\frac{\pi}{6}) + \sin(\frac{\pi}{6})\cos(\frac{\pi}{4}) = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)$

OR we could calculate

$$\sin(\frac{5\pi}{12}) = \sqrt{\frac{1}{2}(1-\cos(\frac{5\pi}{6}))} = \sqrt{\frac{1}{2}(1+\cos(\frac{\pi}{6}))} = \sqrt{\frac{1}{2}(1+\frac{\sqrt{3}}{2})}.$$

(b) $\sin(\frac{7\pi}{12}) = \sin(\frac{\pi}{2} + \frac{\pi}{12}) = \sin(\frac{\pi}{2} - \frac{\pi}{12}) = \sin(\frac{5\pi}{12}) = \ldots$ SAME answer as (a).

(c) $\cos(-\frac{\pi}{3}) = \cos(\frac{\pi}{3}) = \frac{1}{2}$.

Example 14.3. This is 2.14 and 3.10. Let θ be the measure of the angle CDB. Then the measure of AED is $(\pi - (\frac{\pi}{4} + \frac{\pi}{6})) = \frac{7\pi}{12}$, the measure at EDB is $(\pi - \theta)$, the measure of DBC is $(\pi - (\theta + \frac{\pi}{6})) = (\frac{5\pi}{6} - \theta)$, and the measure at DBA is $(\pi - (\frac{5\pi}{6} - \theta)) = (\frac{\pi}{6} + \theta)$ See DRAWINGS 14.33 at the end of this chapter.

Example 14.4. This is 2.14 and 3.10, as in Example 14.3, and, in addition, 3.6, which forces θ, from Example 14.3 and DRAWING 14.33, to be $\frac{7\pi}{12}$, so that, replacing θ with $\frac{7\pi}{12}$ throughout Example 14.3, we get DRAWING 14.34 at the end of this chapter.

Example 14.5. This is Propositions 3.3, 3.13, and 5.7. See DRAWING 14.35 at the end of this chapter, then calculate

$$3^2 + 6^2 + 3^2 + 6^2 = (2x)^2 + 8^2 \to x = \sqrt{\frac{13}{2}}.$$

Examples 14.6. This is Theorem 12.8.

(a) $(\pi 10^2)6 = 600\pi$.

(b) $\frac{1}{3}(600\pi) = 200\pi$.

(c) $\frac{1}{3}(700 \text{ feet})^2(600 \text{ feet}) = 98,000,000$ feet cubed.

(d) $(100m^2)(20m) = 2,000$ meters cubed.

(e) $\frac{4}{3}\pi(18 \text{ inches})^3 = 7,776$ inches cubed.

Example 14.7. This is 2.19 and 12.4. Let r be the radius of the disc. Since $2\pi r = 12\pi$, $r = 6$, thus the area is $\frac{1}{2}(6^2)(2\pi) = 36\pi$.

Example 14.8. This is 3.10, with $n = 6$. The sum of the measures of the interior angles is $(6-2)\pi = 4\pi$, so the measure of each angle is $\frac{4\pi}{6}$ radians, or $(\frac{4\pi}{6})(\frac{180}{\pi}) = 120$ degrees.

Example 14.9. By 3.10, $\pi = x + 2x + 3x = 6x$, so $x = \frac{\pi}{6}$.

Example 14.10. By 4.15 and the definition of a tangent line,

$$(8+r)^2 = r^2 + 12^2 \to 64 + 16r + r^2 = r^2 + 144 \to r = 5.$$

See DRAWING 14.36 at the end of this chapter.

Example 14.11. This is 2.14 and 3.6. See DRAWING 14.37 at the end of this chapter for the solution.

Examples 14.12. (a) This is 3.3 and 3.8. See DRAWING 14.38 at the end of this chapter for the solution.

(b) This is 3.3, 4.15 or 5.7, 5.5, 7.5, and 7.11(e). See DRAWING 14.39 at the end of this chapter for the solution.

(c) Same solution as (b).

(d) This is 3.3, 3.13, 4.15, 5.5, 5.6 and 7.10. See DRAWING 14.40 at the end of this chapter for the solution.

(e) This is 3.3 and 5.5. See DRAWING 14.41 at the end of this chapter for the solution.

(f) This is 3.3, 3.13, 4.15, and 5.6. See DRAWING 14.42 at the end of this chapter for the solution. Notice that we have a rectangle, composed of two right triangles, each with hypotenuse 6 and leg 5, hence the other leg is $\sqrt{6^2 - 5^2} = \sqrt{11}$.

(g) This is 3.3 and HWIII.2 or 3.6. See DRAWING 14.43 at the end of this chapter for the solution.

(h) This is 3.3, 3.8, HWIII.2 or 3.6 and 11.4. See DRAWING 14.44 at the end of this chapter for the solution.

(i) This is 3.3, 3.8, HWIII.2 or 3.6, 5.5, 7.5, and 11.4. See DRAWING 14.45 at the end of this chapter for the solution.

(j) This is 3.10, 3.3, 3.13, 4.15, 5.6, 7.5. See DRAWING 14.46 at the end of this chapter for the solution.

Example 14.13. See DRAWING 14.47 at the end of this chapter. By Theorem 10.1(b), $\frac{x}{4} = \frac{10}{9}$, so $x = \frac{40}{9}$ feet.

Example 14.14. This is 3.6 and 10.1(b). See DRAWING 14.48 at the end of this chapter, an enhancement of DRAWING 14.17 at the end of this chapter, from which it follows that triangle ABE is similar to triangle CDE, so that

$$\frac{x}{20} = \frac{10}{5} \to x = 40.$$

Example 14.15. These are primarily 2.19, 12.1, 12.3, and 12.4; 4.15, HWXII.1, 7.11(e), and 7.5 will also be used.

(a) First let's focus on the hidden right triangle; see DRAWING 14.49 at the end of this chapter.

Changing 30 degrees to $\frac{\pi}{6}$ radians, here's an expansion of DRAWING 14.18, in DRAWING 14.50 at the end of this chapter.

Area comes in three pieces, sector plus triangle plus rectangle:

$$\frac{1}{2}4^2\left(\frac{5\pi}{3}\right) + \frac{1}{2}(2\sqrt{3})4 + 6 \times 4 = \frac{40\pi}{3} + 4\sqrt{3} + 24.$$

Perimeter is in two pieces, arclength plus three lengths of sides of a rectangle:

$$4\left(\frac{5\pi}{3}\right) + 6 + 4 + 6 = \frac{20\pi}{3} + 16.$$

(b) As with (a), DRAWING 14.51 at the end of this chapter is a completion of DRAWING 14.19 at the end of this chapter. Let's use it to put pieces together, similarly to part (a).

For area, three pieces, from left to right: sector, triangle inside disc, triangle outside disc:

$$\frac{1}{2}(10^2)\left(\frac{11\pi}{6}\right)+\frac{1}{2}\left(10\cos(\frac{\pi}{12})\right)\left(20\sin(\frac{\pi}{12})\right)+\frac{1}{2}\left(20\sin(\frac{\pi}{12})\right)5$$

$$=50\left(\frac{11\pi}{6}\right)+50\sin(\frac{\pi}{6})+50\sin(\frac{\pi}{12})=50\left[\frac{11\pi}{6}+\frac{1}{2}+\sin(\frac{\pi}{12})\right]\sim 50\left[\frac{11\pi}{6}+0.76\right];$$

the $\sin(\frac{\pi}{6})$ appeared from 6.4(ii).

For perimeter, two pieces, arclength plus two lengths of sides of the triangle outside the disc:

$$10\left(\frac{11\pi}{6}\right)+2\left(\sqrt{25+100\sin^2(\frac{\pi}{12})}\right)\sim\frac{55\pi}{3}+2\left(\sqrt{25+100(0.26)^2}\right).$$

(c) For area, we need height; see DRAWING 14.52 at the end of this chapter.

The area is height times base: $4(\frac{5\sqrt{3}}{2})=10\sqrt{3}$. The perimeter is $(5+4+5+4)=18$.

(d) We'll need both height and base. First, use 7.5 to extend DRAWING 14.21 to DRAWING 14.53 at the end of this chapter.

Corollary 5.3 extends DRAWING 14.53 to DRAWING 14.54 at the end of this chapter.

Finally, the Pythagorean theorem 4.15 extends our picture to DRAWING 14.55 at the end of this chapter, from which we can read off

$$\text{area }=4\left(\frac{1}{2}(10+(4\sqrt{3}+10+3))\right)=8\sqrt{3}+46;\quad \text{perimeter }=4\sqrt{3}+10+3+5+10+8=4\sqrt{3}+36.$$

Examples 14.16. For (a)–(d), we need 2.14, 3.10, 4.15, 7.5, and 7.10.

(a) See DRAWINGS 14.56 at the end of this chapter for the solution.

(b) See DRAWING 14.57 at the end of this chapter for the solutions.

(c) See DRAWINGS 14.58 at the end of this chapter for the solutions.

(d) See DRAWING 14.59 at the end of this chapter for the solutions.

(e) 4.15 and 7.5; see DRAWING 14.60 at the end of this chapter for the solutions.

(f) 7.5; see DRAWING 14.61 at the end of this chapter for the solutions.

(g) Law of Cosines 7.1 and Law of Sines 7.7; see DRAWINGS 14.62 at the end of this chapter for solutions.

(h) 3.10 and 7.10 imply DRAWINGS 14.63 at the end of this chapter.

We could now use Law of Sines 7.7 for the remaining side:

$$\frac{x}{\sin(120)}=\frac{10}{\sin(30)}\to x=\frac{\sqrt{3}}{2}\left(\frac{10}{\frac{1}{2}}\right)=10\sqrt{3};$$

OR the Law of Cosines 7.1:

$$x^2=10^2+10^2-2\times 10\times 10\times\cos(120)=200-200(-\frac{1}{2})=300\to x=10\sqrt{3}.$$

Example 14.17. This is 4.15, 6.3, 6.4, and 7.5. See DRAWING 14.64 at the end of this chapter for solutions.

Example 14.18. This is 5.3. See DRAWING 14.65 at the end of this chapter for the solutions.

Example 14.19. (a) $\iff$ (b). Note that

$$\frac{\|\overrightarrow{DE}\|}{\|\overrightarrow{DA}\|} = \frac{\|\overrightarrow{DE}\|}{\left[\|\overrightarrow{DE}\| + \|\overrightarrow{EA}\|\right]} = \frac{1}{\left[1 + \frac{\|\overrightarrow{EA}\|}{\|\overrightarrow{DE}\|}\right]};$$

similarly,

$$\frac{\|\overrightarrow{DC}\|}{\|\overrightarrow{DB}\|} = \frac{1}{\left[1 + \frac{\|\overrightarrow{CB}\|}{\|\overrightarrow{DC}\|}\right]}.$$

Let $x \equiv \frac{\|\overrightarrow{DE}\|}{\|\overrightarrow{EA}\|}$ and $y \equiv \frac{\|\overrightarrow{DC}\|}{\|\overrightarrow{CB}\|}$.

The equivalence of (a) and (b) now follows from the equivalence of

$\dfrac{1}{1+\frac{1}{x}} = \dfrac{1}{1+\frac{1}{y}}$ (this equality is assertion (a)) and $x = y$ (this equality is assertion (b)),

for positive x and y

(a) $\rightarrow$ (d). Let $\alpha \equiv \frac{\|\overrightarrow{DE}\|}{\|\overrightarrow{DA}\|} = \frac{\|\overrightarrow{DC}\|}{\|\overrightarrow{DB}\|}$.

Then (see DRAWING 14.67 at the end of this chapter)

$$\overrightarrow{EC} = (1-\alpha)\overrightarrow{DA} + \overrightarrow{AB} + (1-\alpha)\overrightarrow{BD} = (1-\alpha)\left[\overrightarrow{BD} + \overrightarrow{DA}\right] + \overrightarrow{AB} = (1-\alpha)\left[-\overrightarrow{AB}\right] + \overrightarrow{AB} = \alpha\overrightarrow{AB},$$

which gives us (d).

This argument also shows that (a) $\rightarrow$ (e), but we prefer to get to (e) via (d) and (c), to get all the equivalences.

(d) $\rightarrow$ (c). Proposition 3.6 implies that the measure of the angle DEC equals the measure of the angle EAB and the measure of the angle DCE equals the measure of the angle CBA (see DRAWING 14.68 at the end of this chapter). This is AAA, equality of angle measures in the triangles DEC and DAB, which, by Theorem 10.1(b), implies similarity.

(c) $\rightarrow$ (e) is Proposition 8.5.

(e) $\rightarrow$ (a) is immediate: (e) appears to be (a), with extra information. □

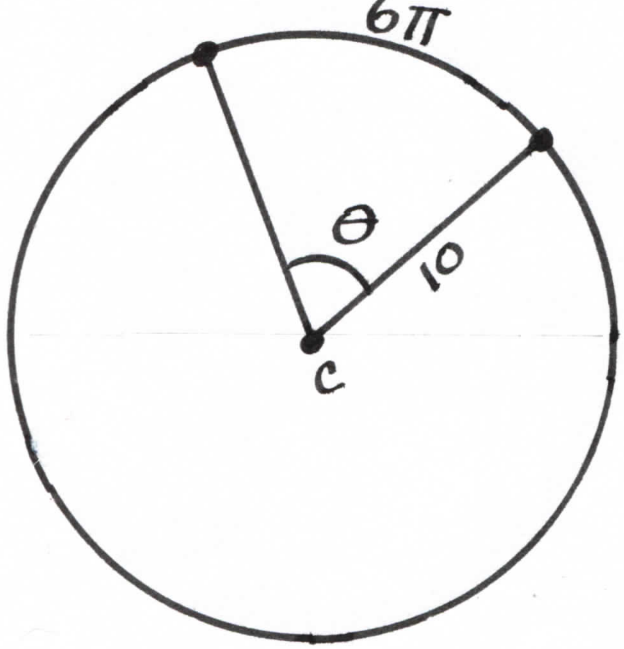
Notice the string of implications coming full circle

$$(a) \rightarrow (b) \rightarrow (a) \rightarrow (d) \rightarrow (c) \rightarrow (e) \rightarrow (a),$$

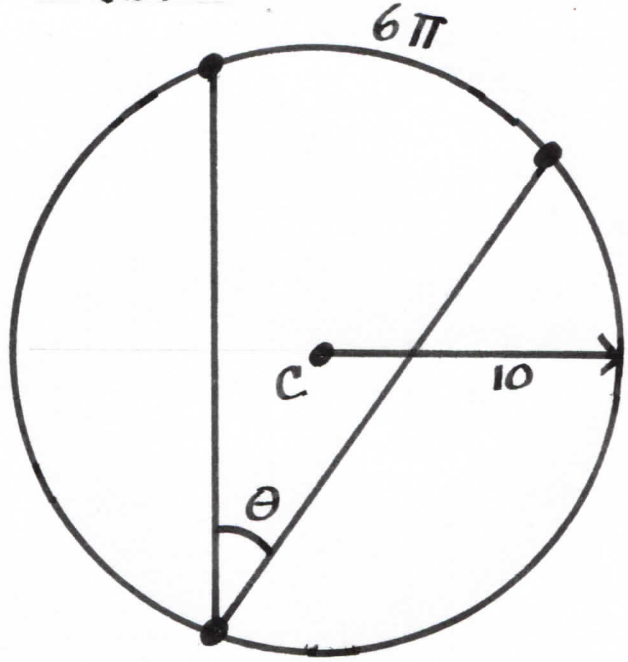
and picking up all the assertions along the way.

DRAWINGS 14.1

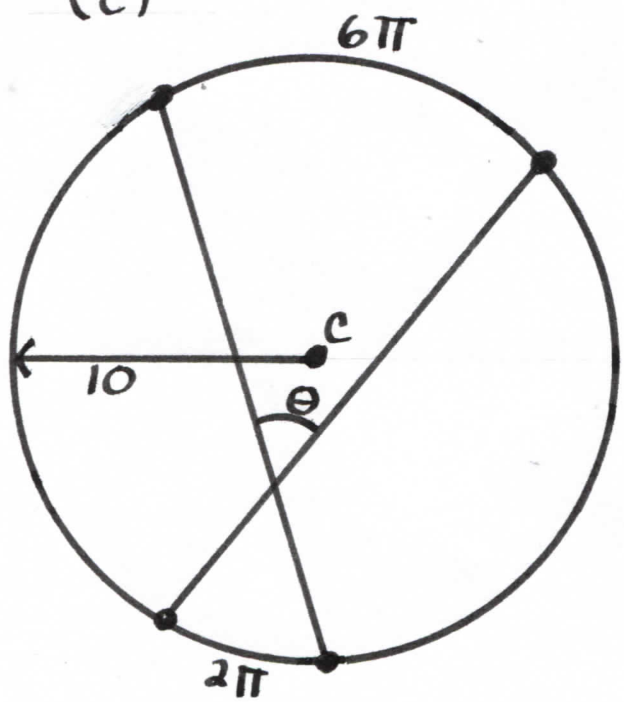
(a)



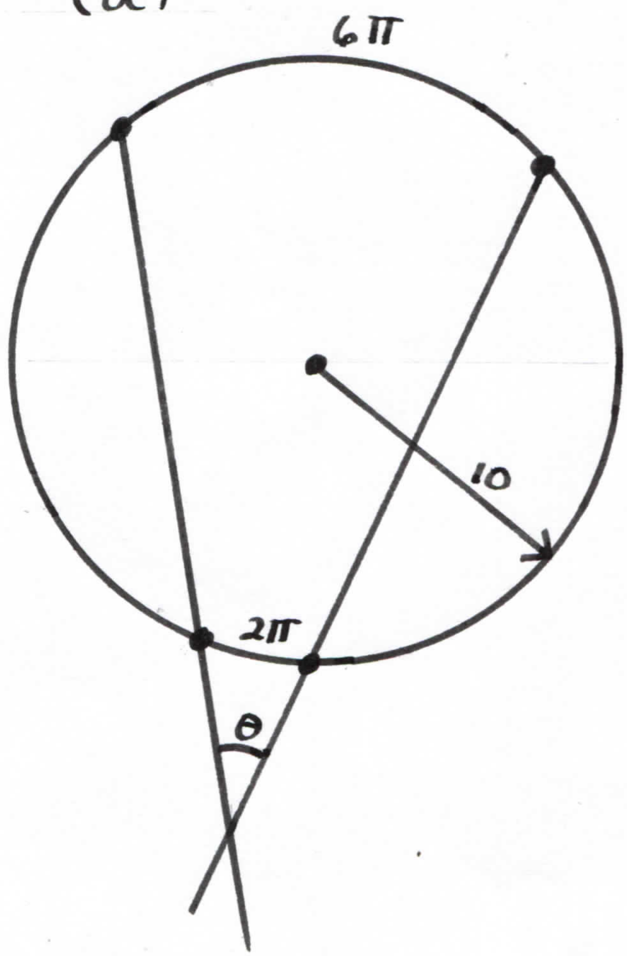
(b)



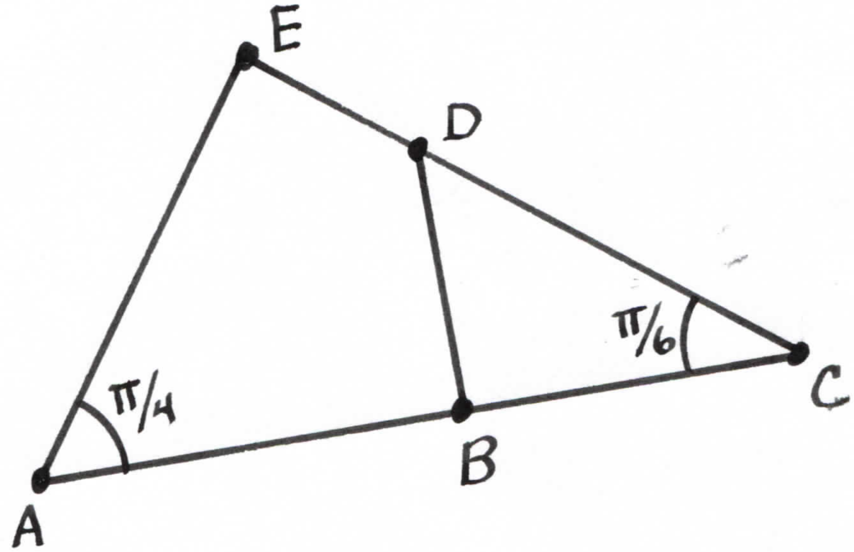
(c)

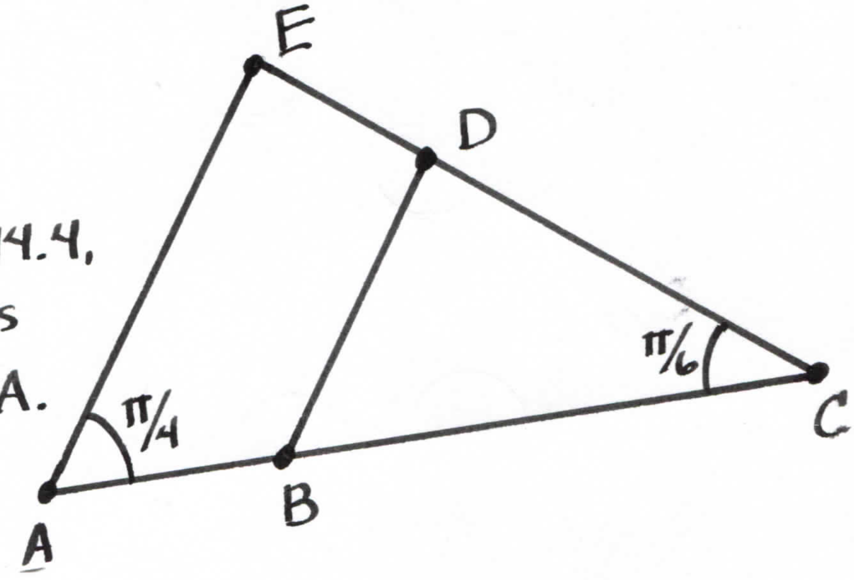


(d)



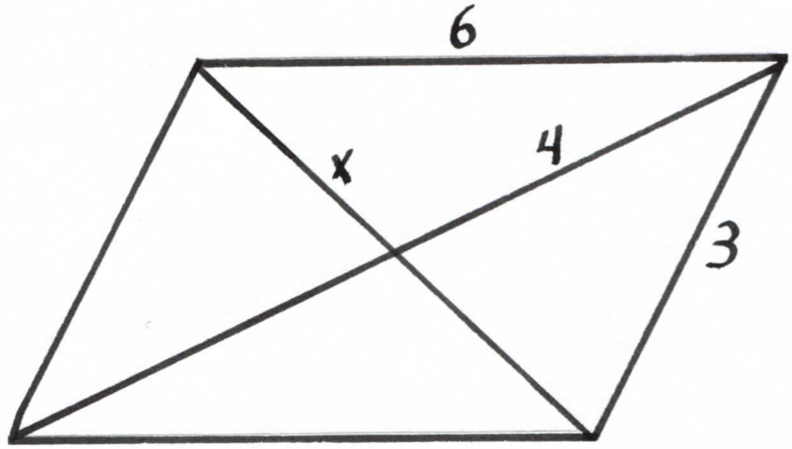
DRAWINGS 14.2



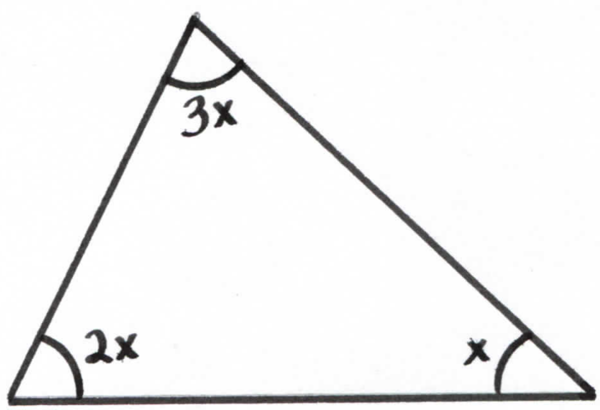


For Example 14.4, assume DB is parallel to EA.

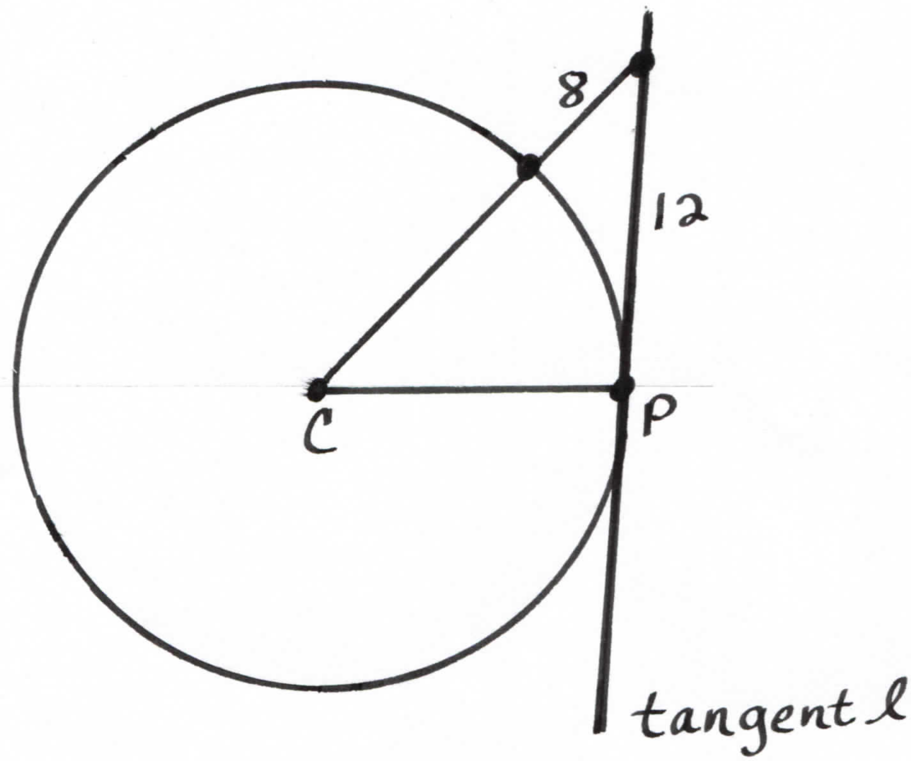
DRAWING 14.3



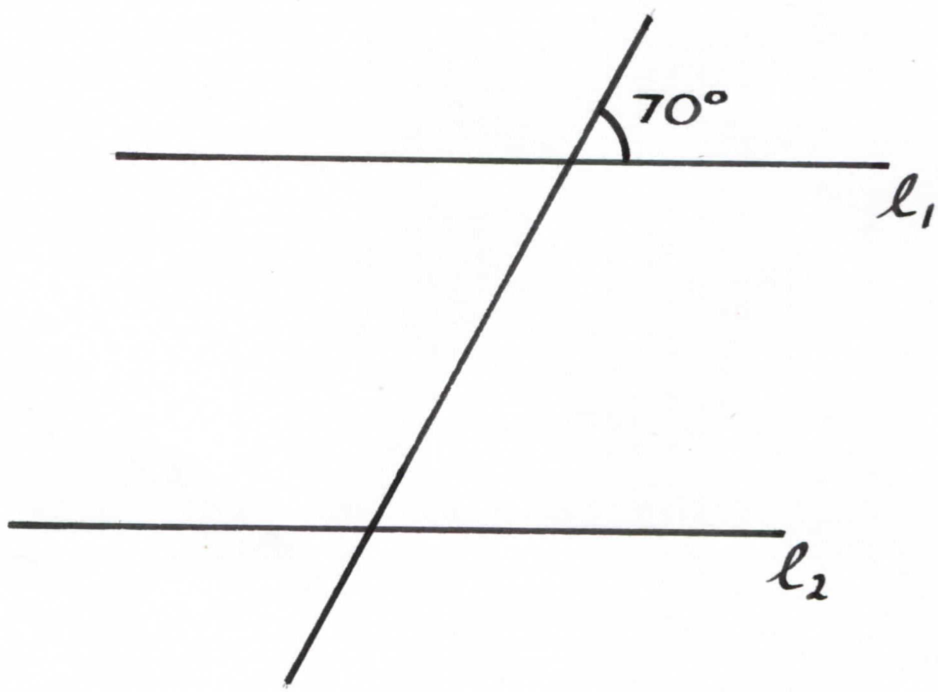
DRAWING 14.4



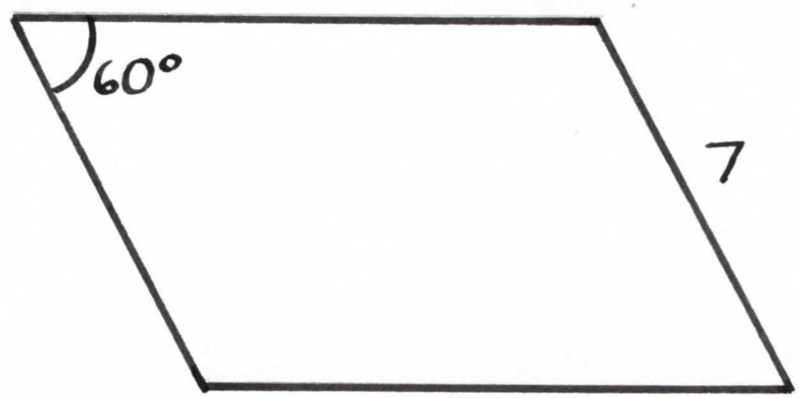
DRAWING 14.5



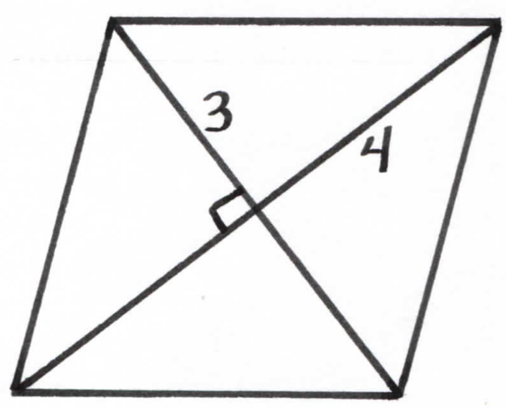
DRAWING 14.6



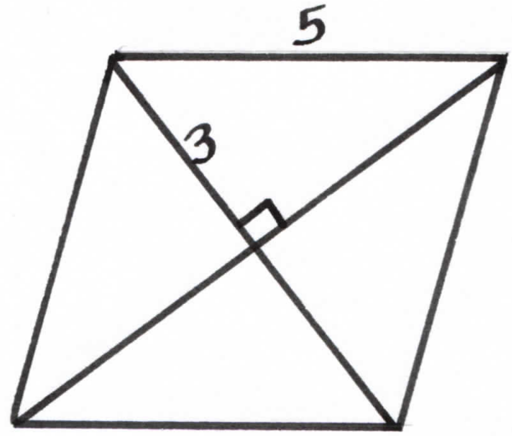
DRAWING 14.7



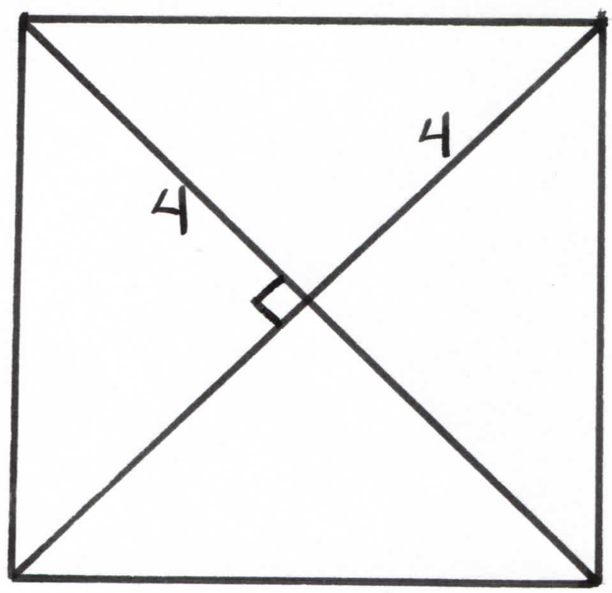
DRAWING 14.8



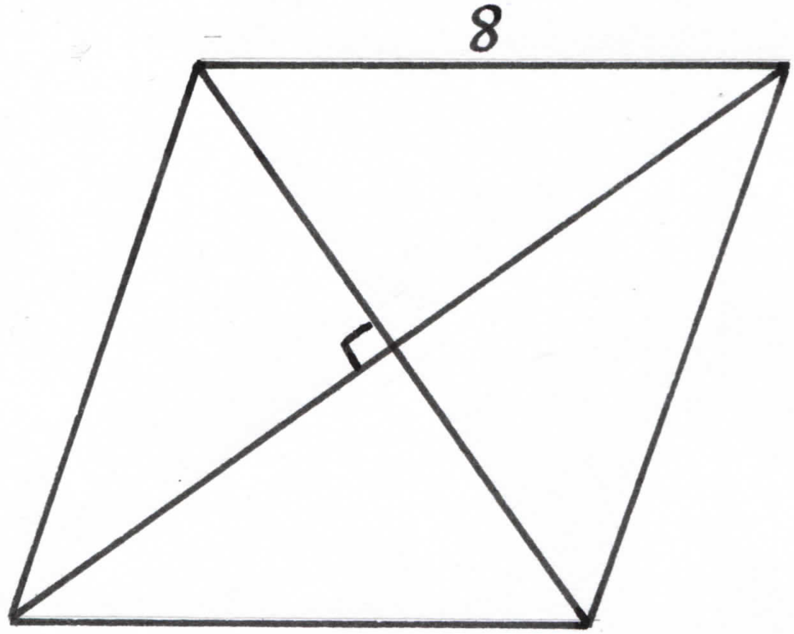
DRAWING 14.9



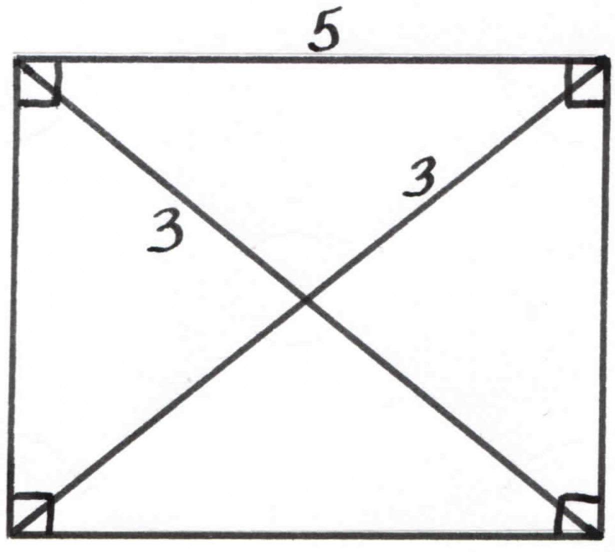
DRAWING 14.10



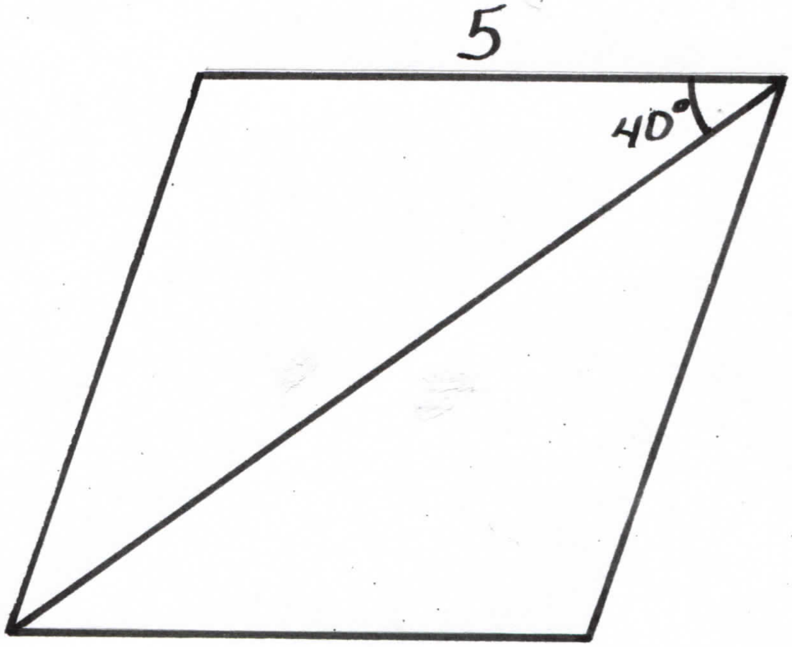
DRAWING 14.11



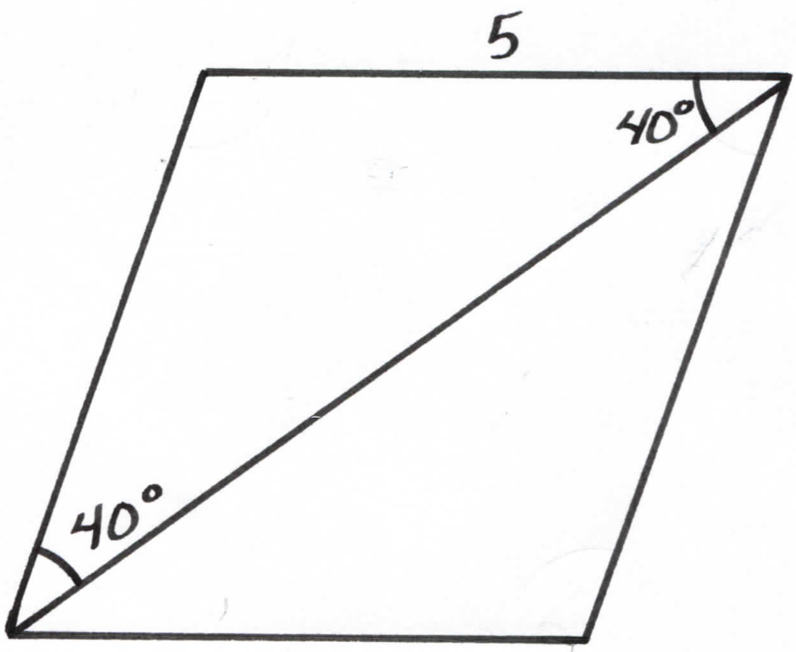
DRAWING 14.12



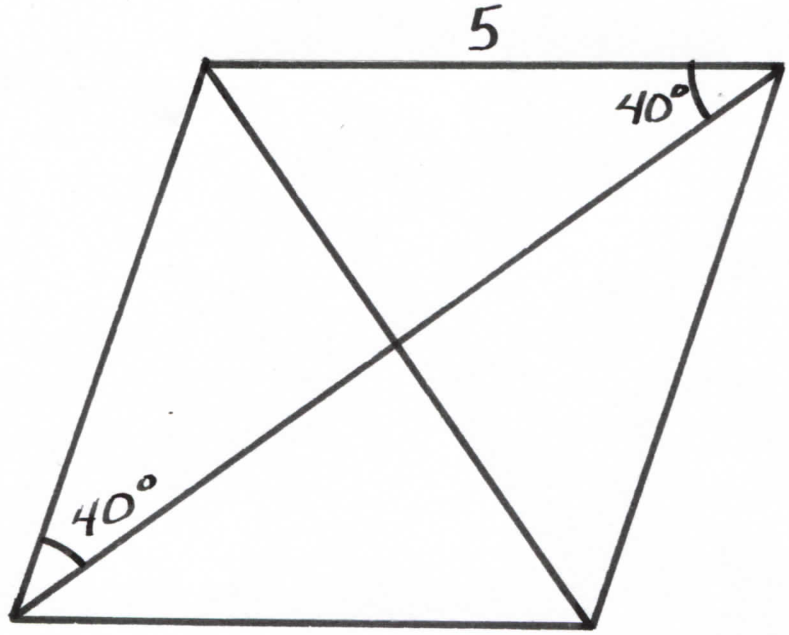
DRAWING 14.13



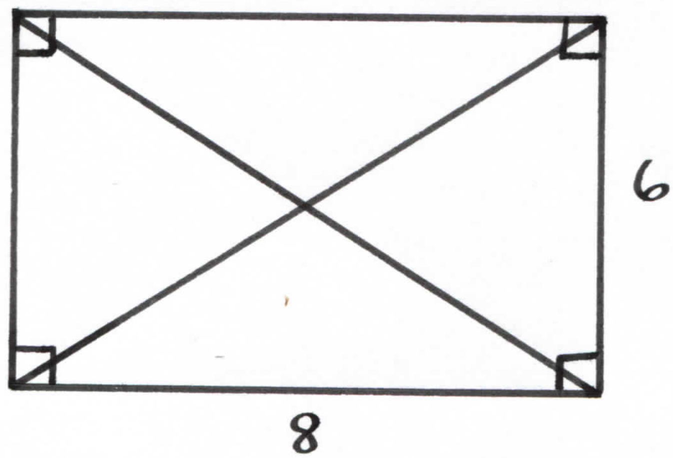
DRAWING 14.14



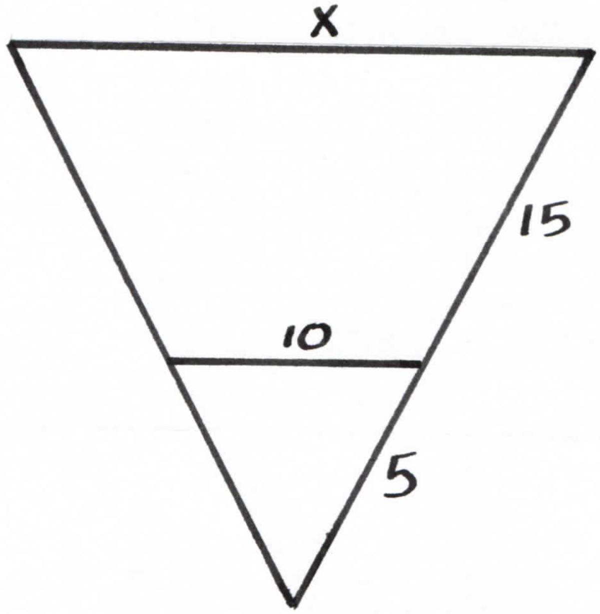
DRAWING 14.15



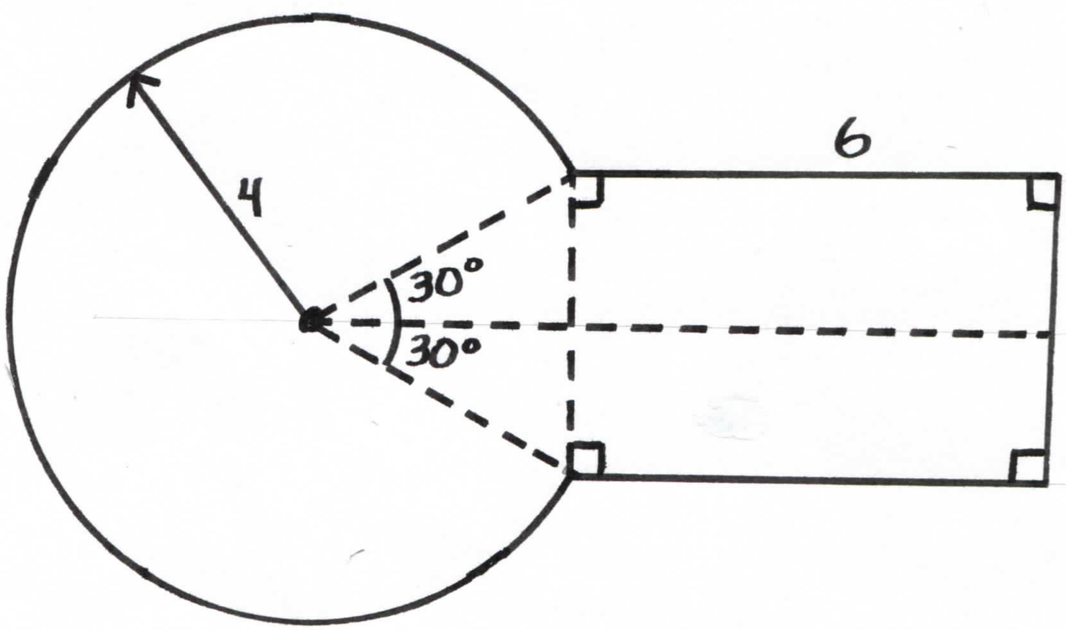
DRAWING 14.16



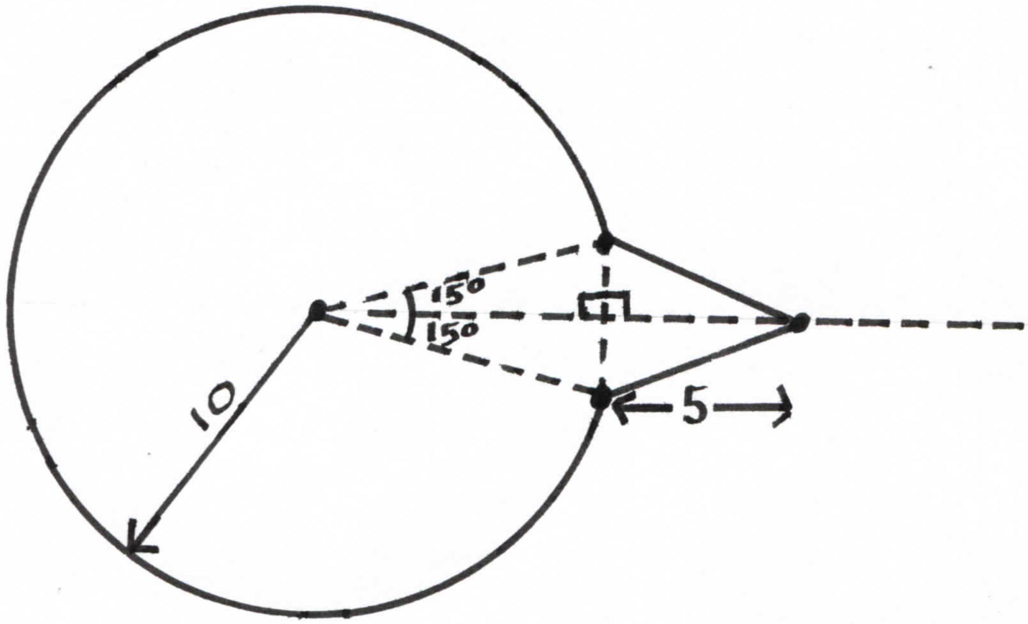
DRAWING 14.17



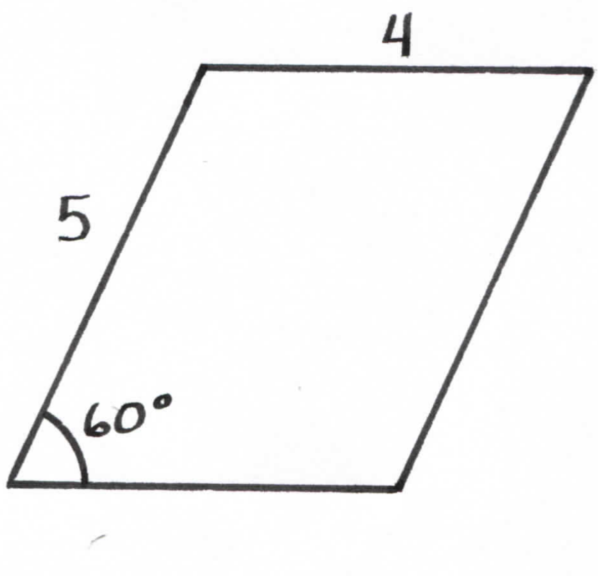
DRAWING 14.18



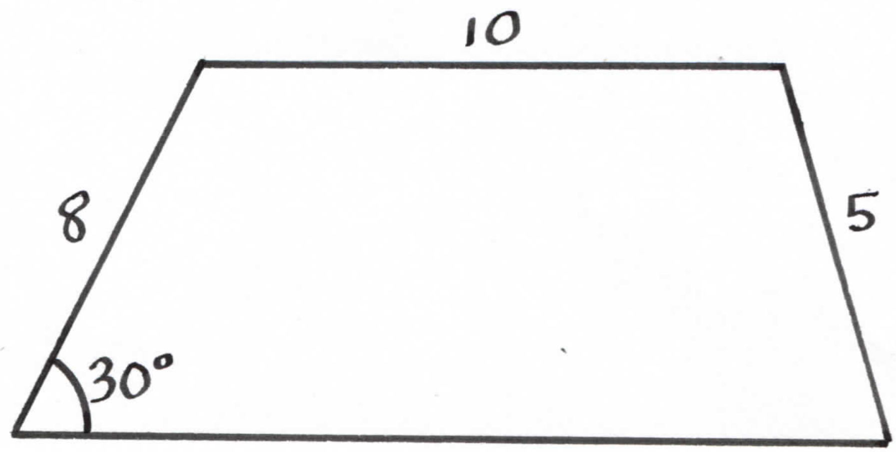
DRAWING 14.19



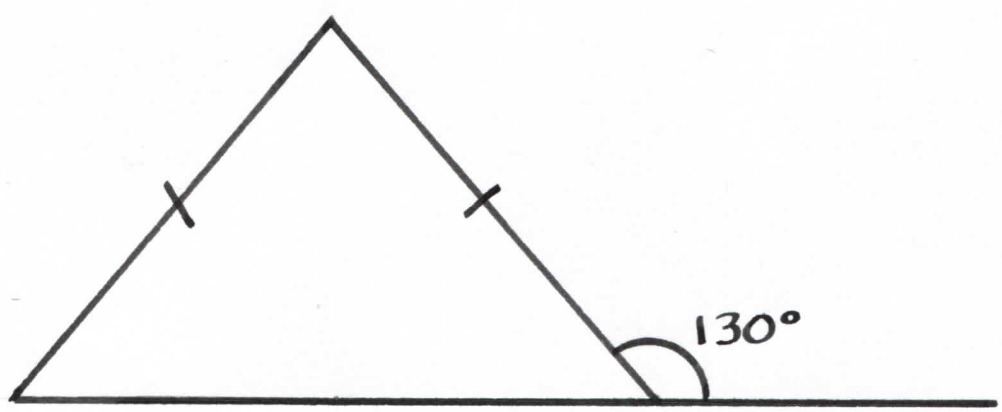
DRAWING 14.20



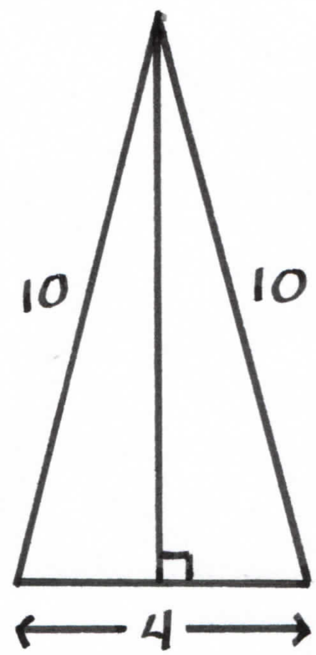
DRAWING 14.21



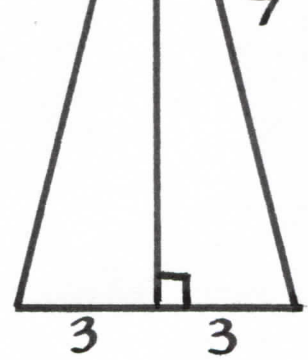
DRAWING 14.22



DRAWING 14.23



DRAWING 14.24



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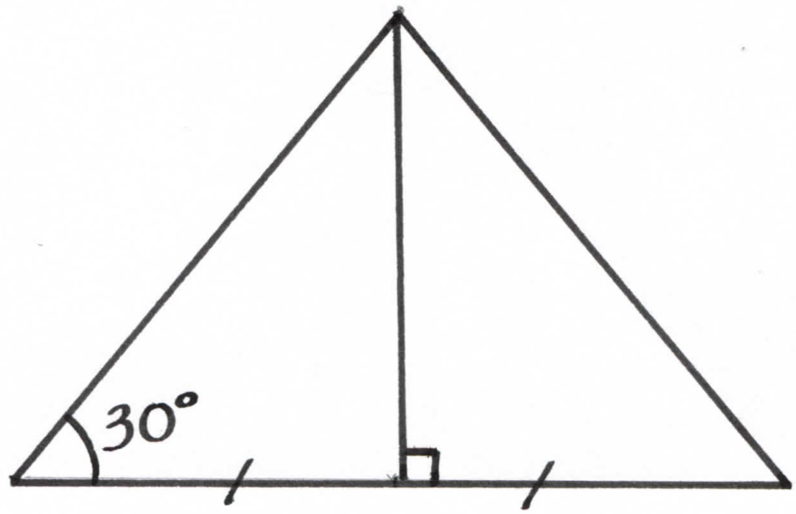
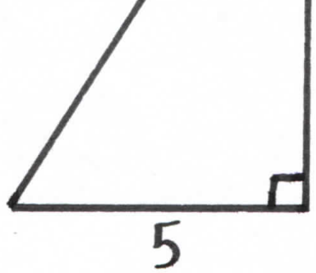
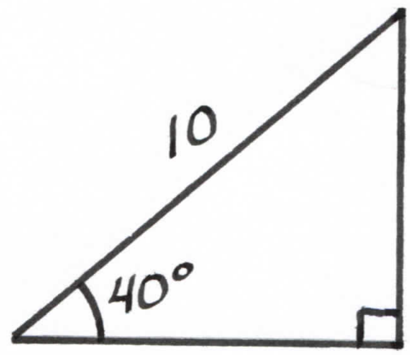


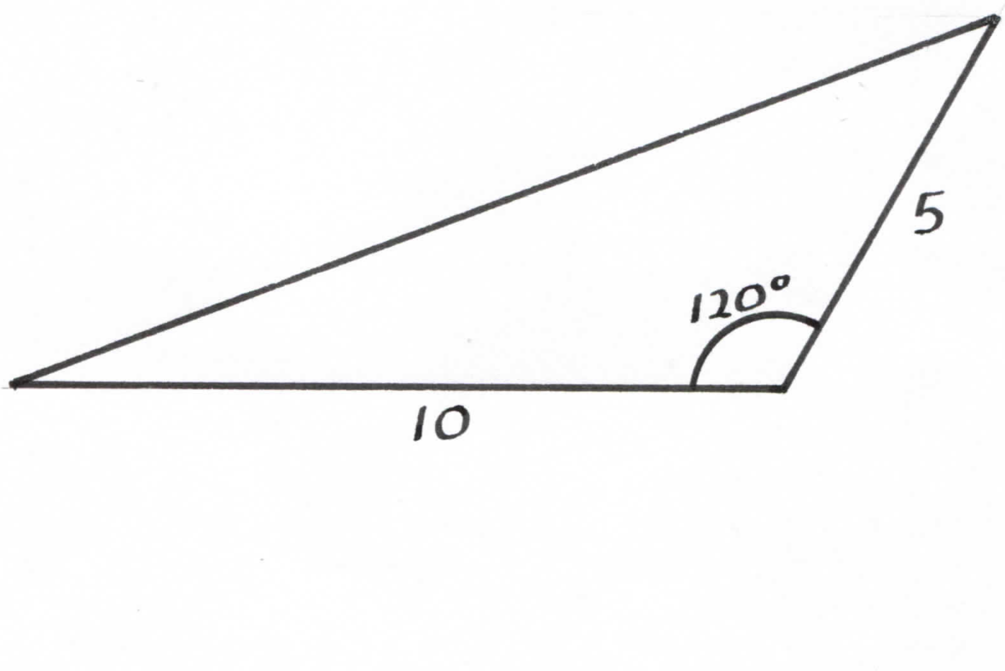
DRAWING 14.26



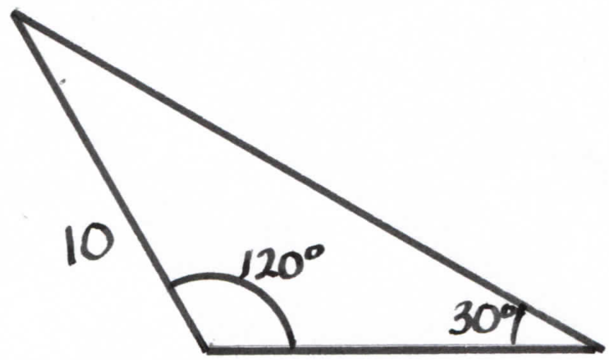
DRAWING 14.27



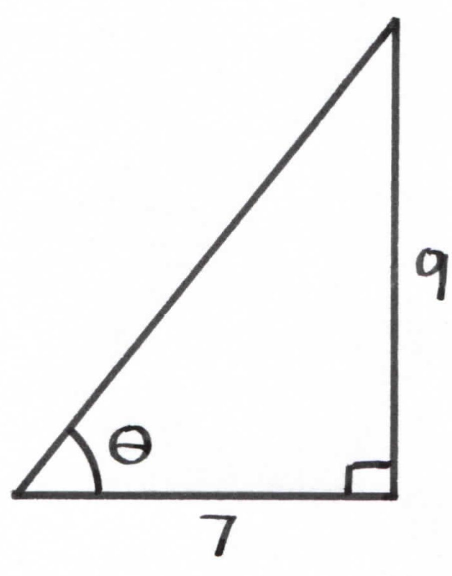
DRAWING 14.28



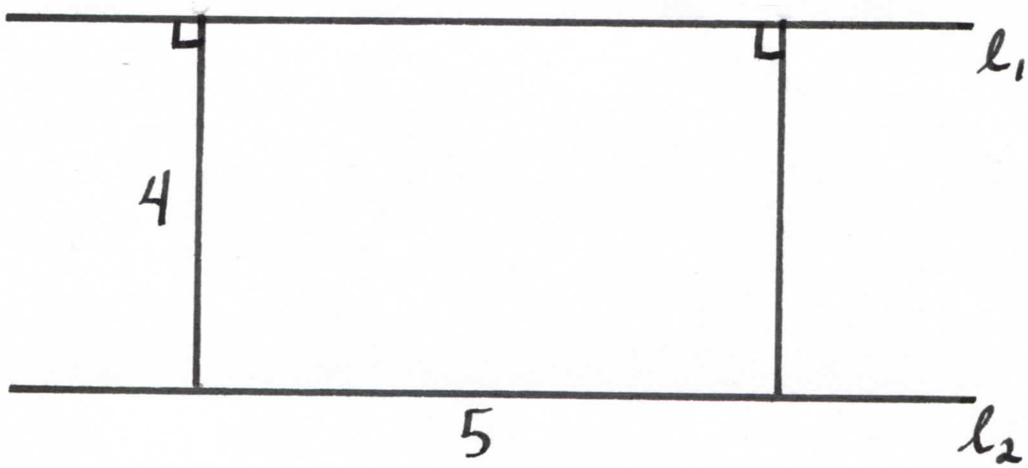
DRAWING 14.29



DRAWING 14.30

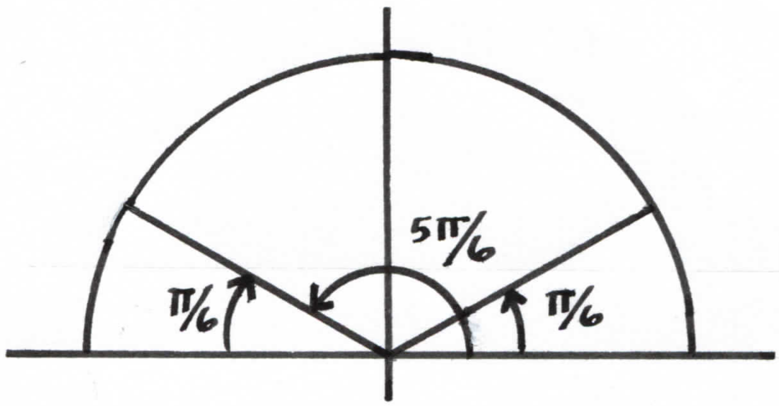


DRAWING 14.31

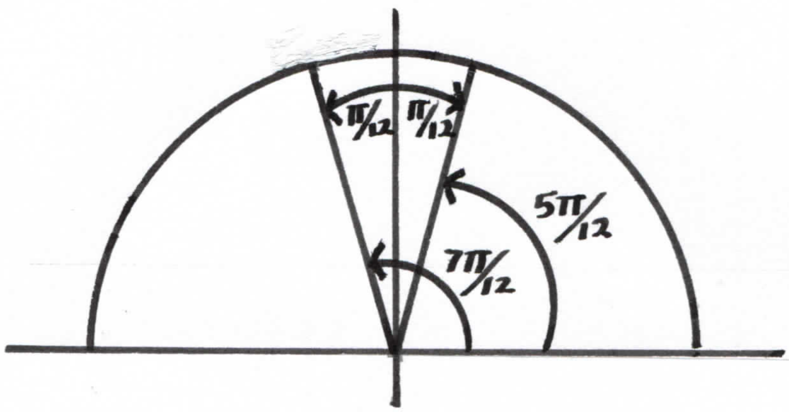


DRAWINGS 14.32

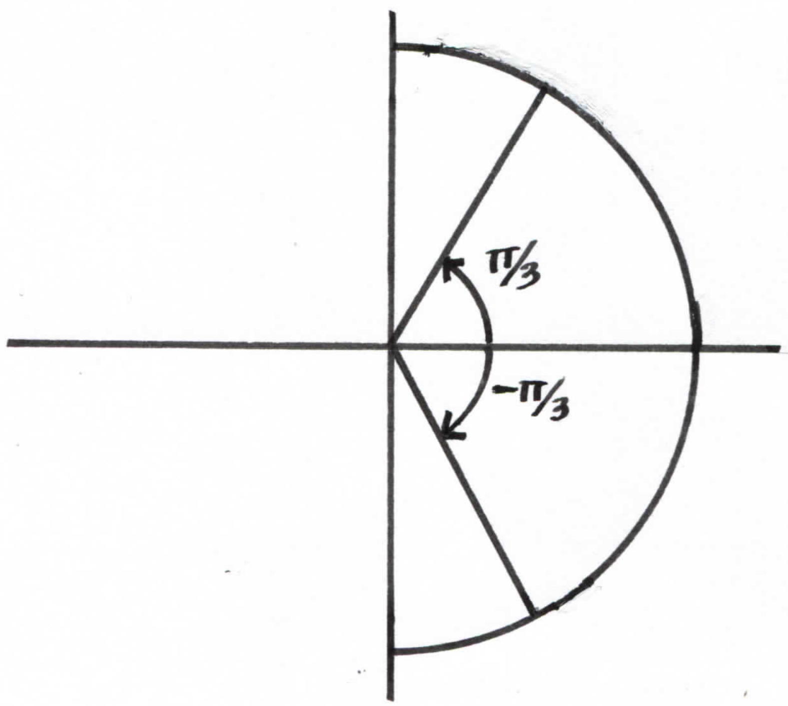
(a)



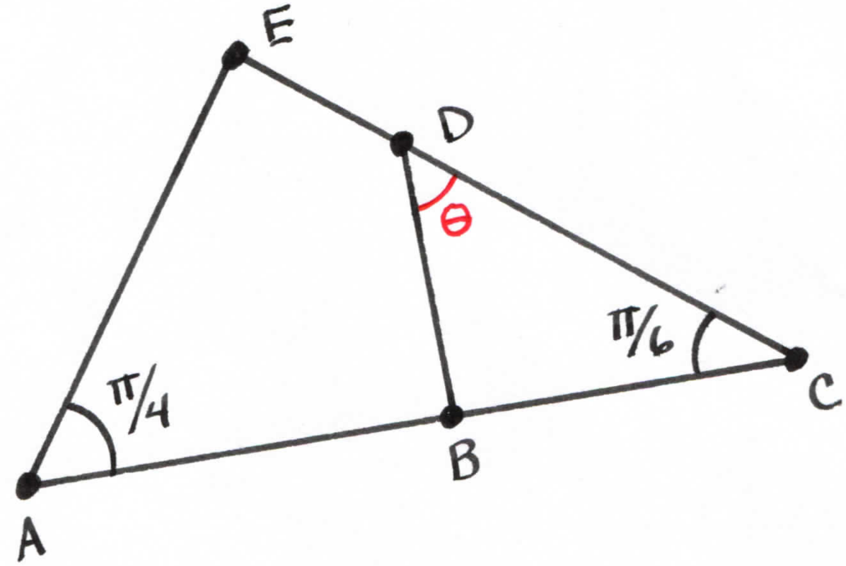
(b)



(c)



DRAWINGS 14.33



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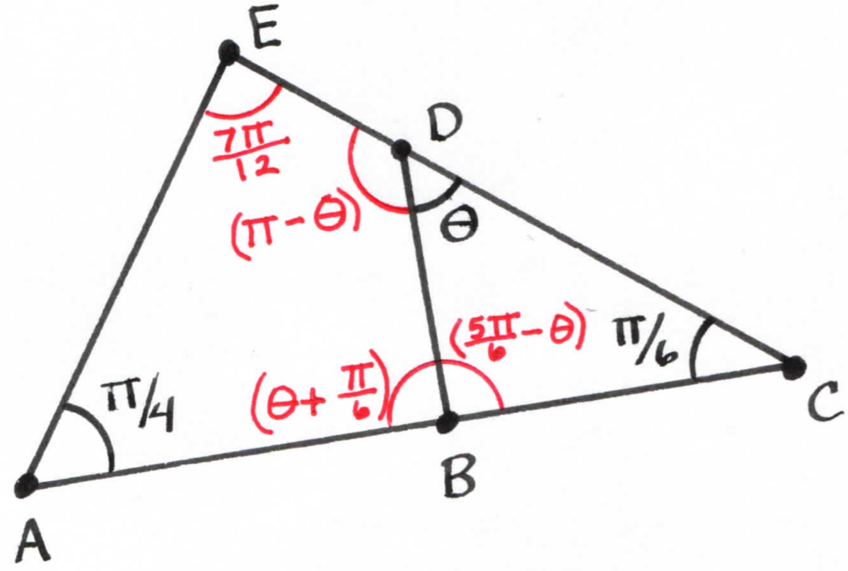
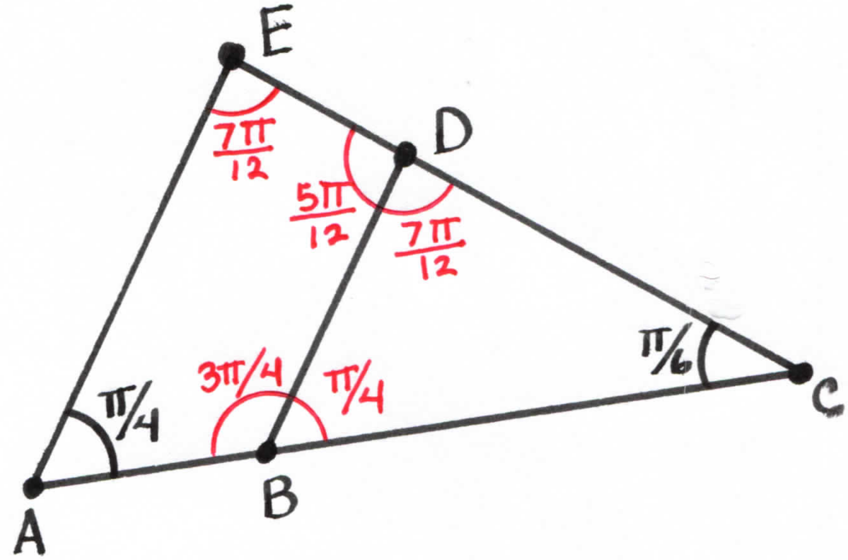
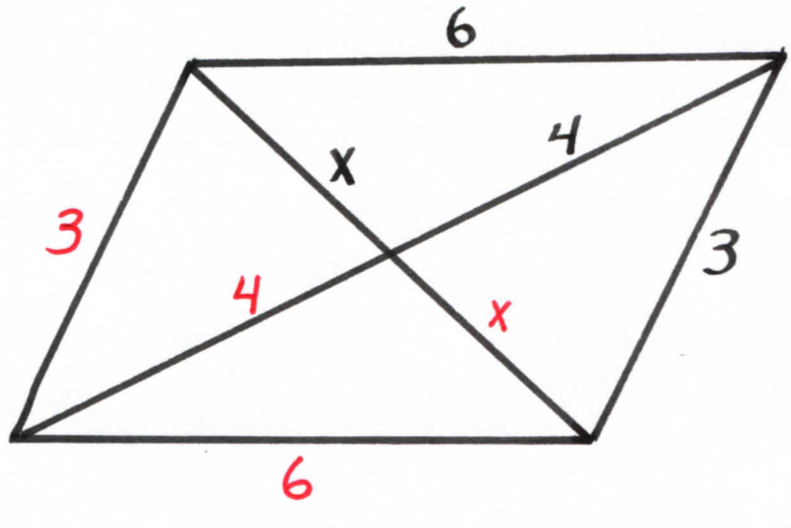


DRAWING 14.34

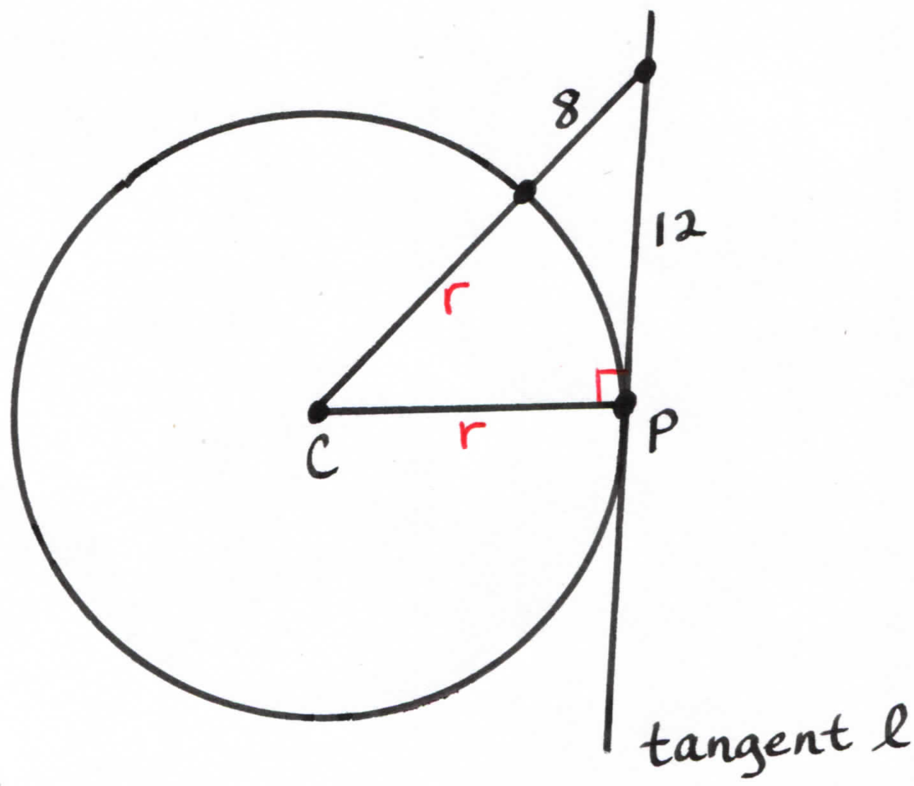


(from DRAWINGS 14.33 and Proposition 3.6)

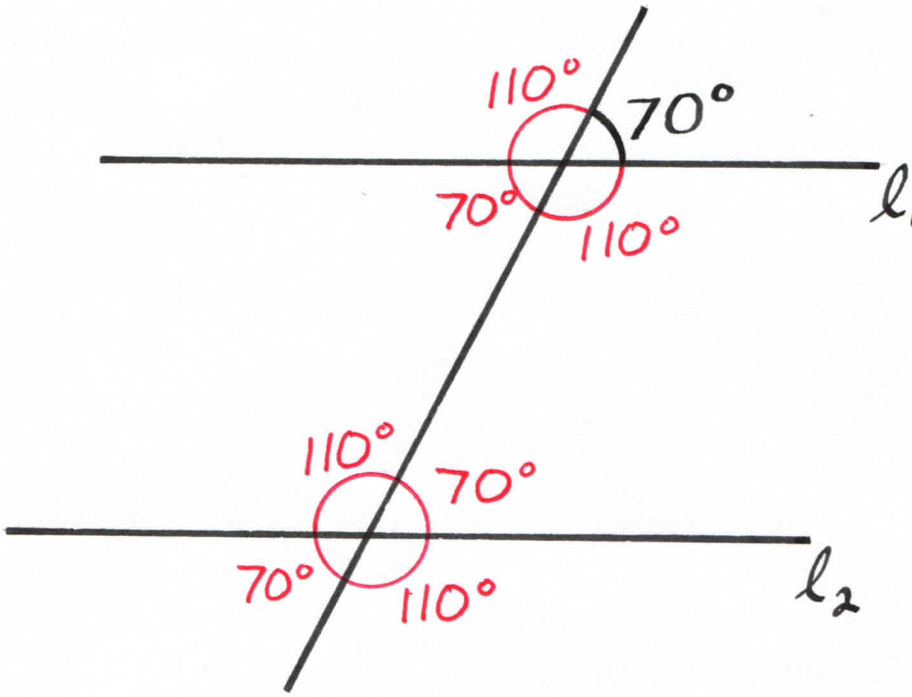
DRAWING 14.35



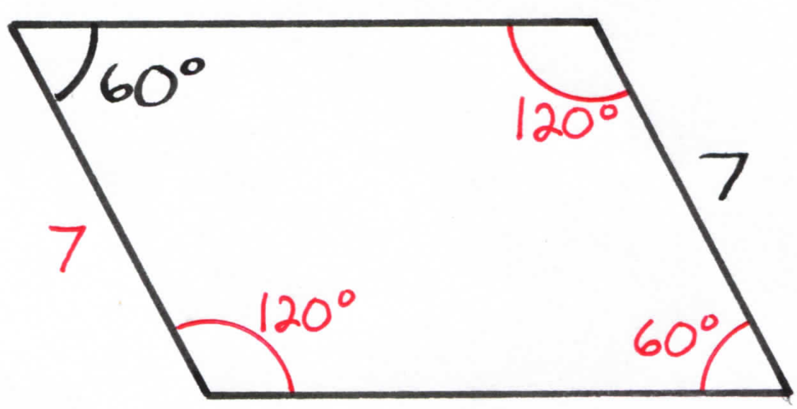
DRAWING 14.36



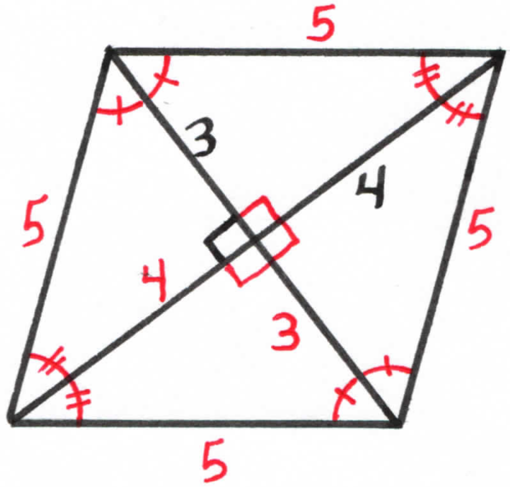
DRAWING 14.37



DRAWING 14.38



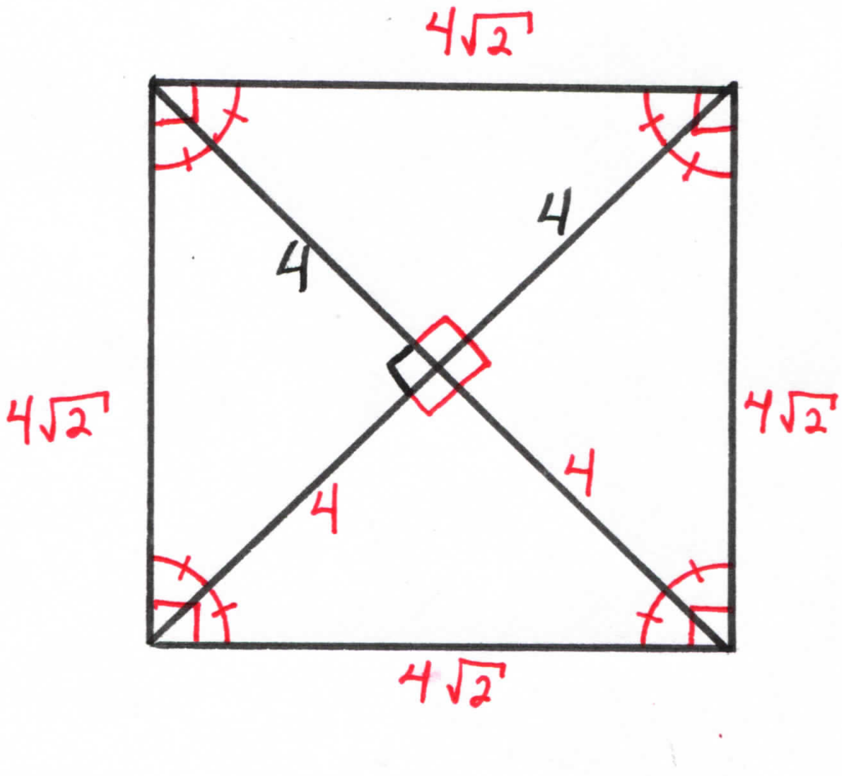
DRAWING 14.39



$\nmid = \cos^{-1}\left(\frac{3}{5}\right)$
$\cong 53.13°$

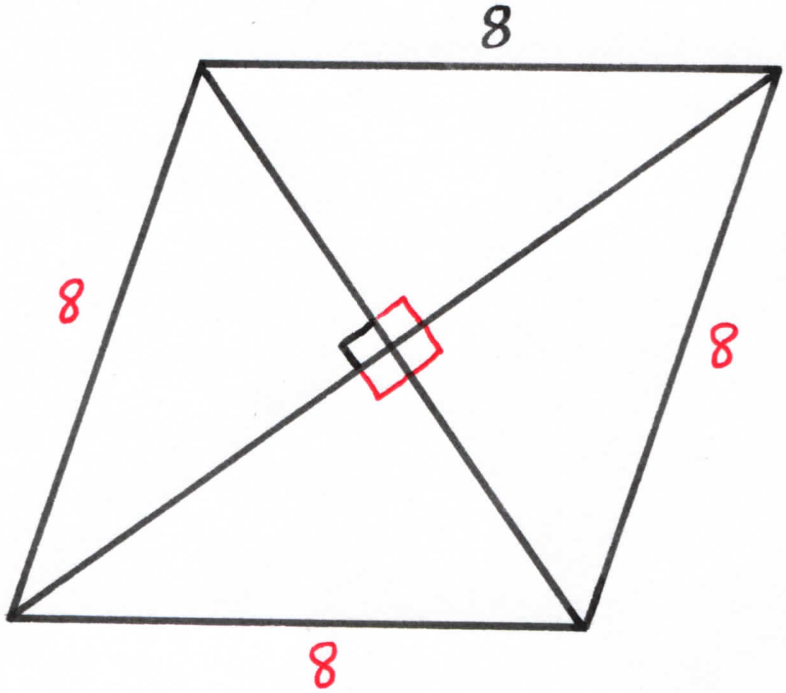
$\not\parallel = \cos^{-1}\left(\frac{4}{5}\right)$
$\cong 36.87°$

DRAWING 14.40

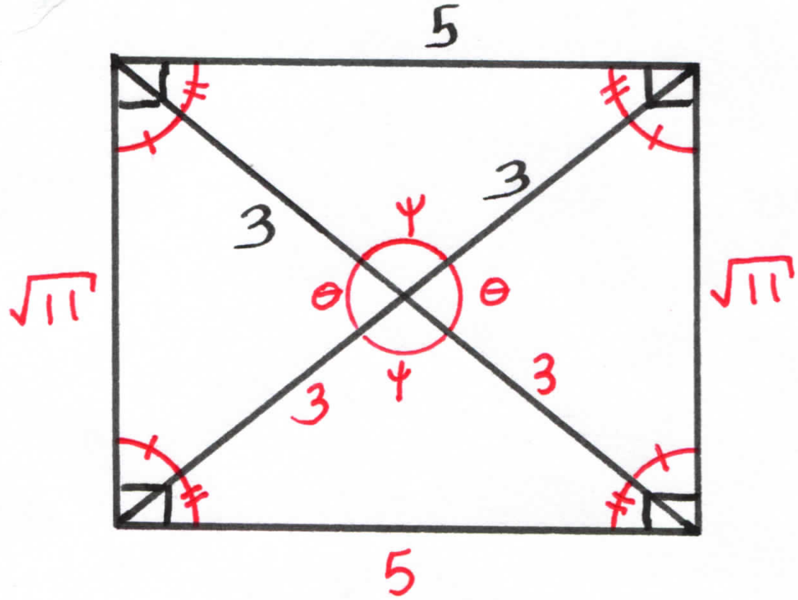


$\nmid = 45°$

DRAWING 14.41



DRAWING 14.42



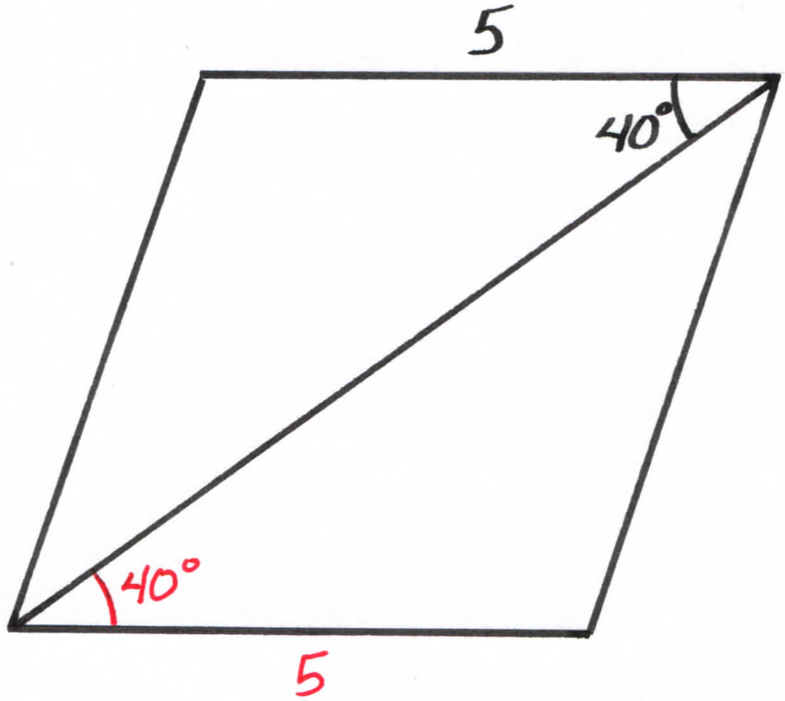
$) = \sin^{-1}\left(\frac{5}{6}\right)$
$\cong 56.44°$

$\theta = \pi - 2)$

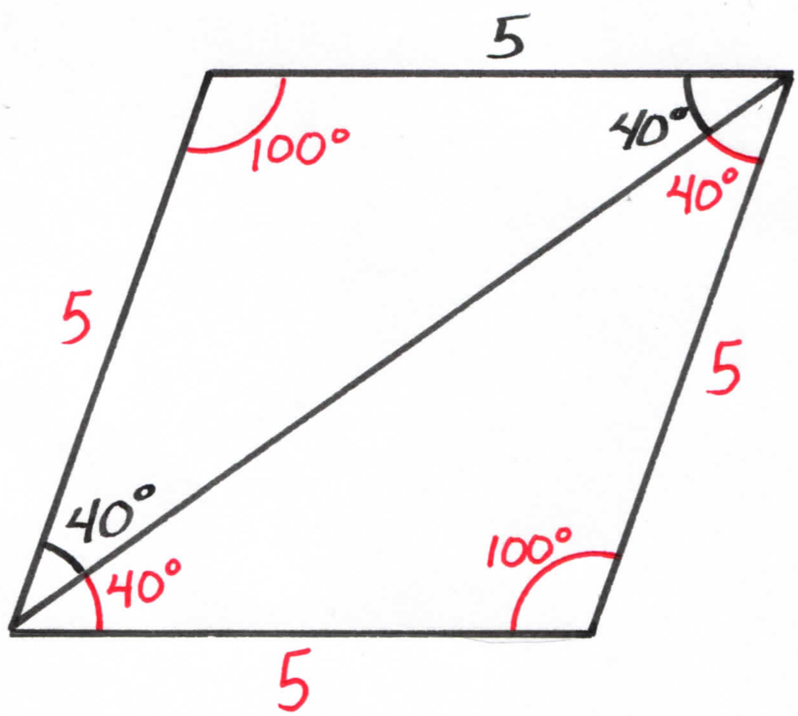
$\psi = \pi - 2\not)$

$\not) = \cos^{-1}\left(\frac{5}{6}\right)$
$\cong 33.56°$

DRAWING 14.43

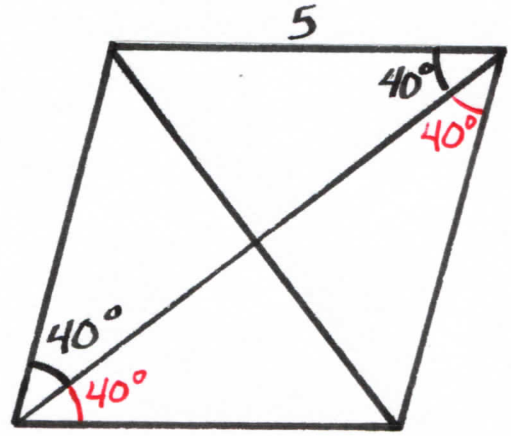


DRAWING 14.44

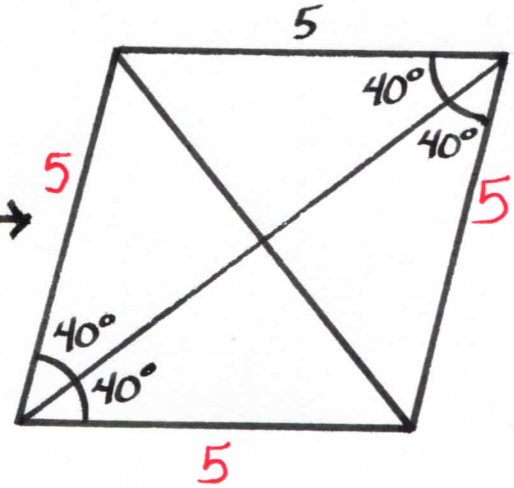


DRAWING 14.45

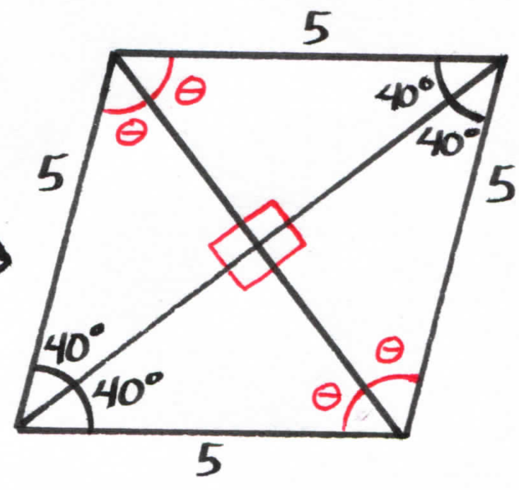
HWIII.2 $\longrightarrow$



Proposition 11.4 and Proposition 3.3 $\longrightarrow$

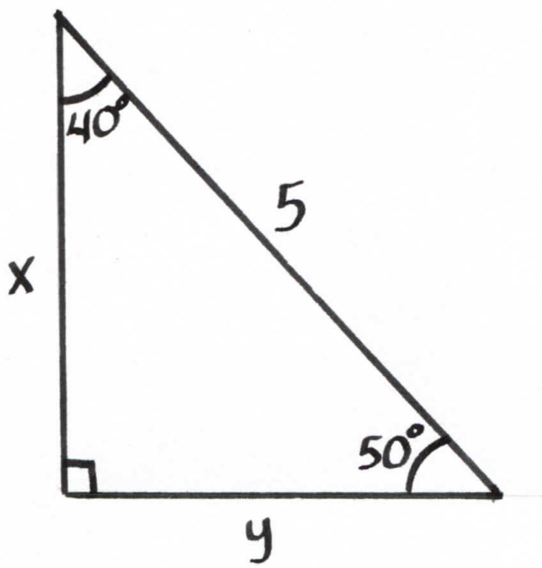


Proposition 11.4, Proposition 3.8, and Proposition 5.5 $\longrightarrow$



$\longrightarrow\ 2\theta + 80 = 180 \ \longrightarrow\ \theta = 50°$

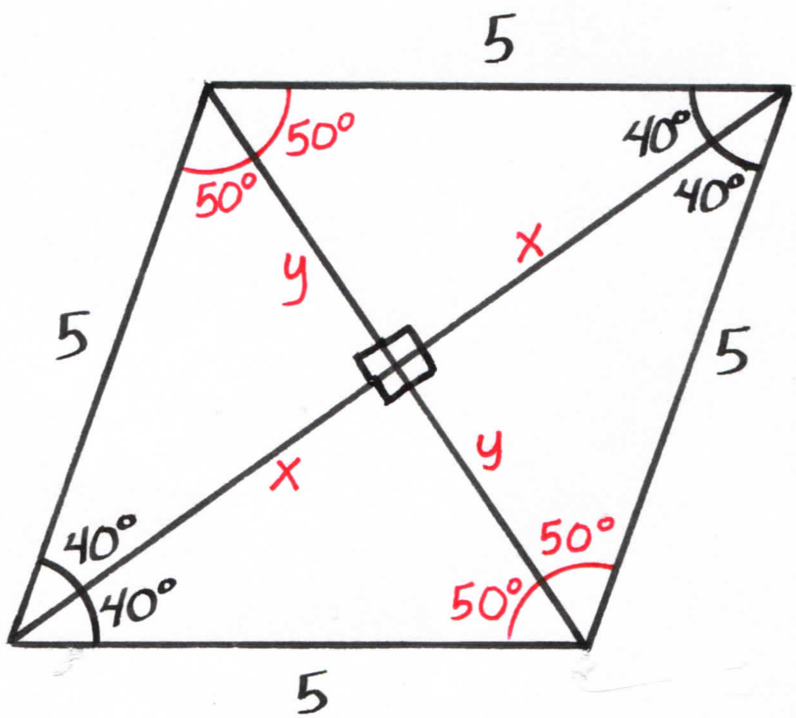
DRAWING 14.45 (continued)



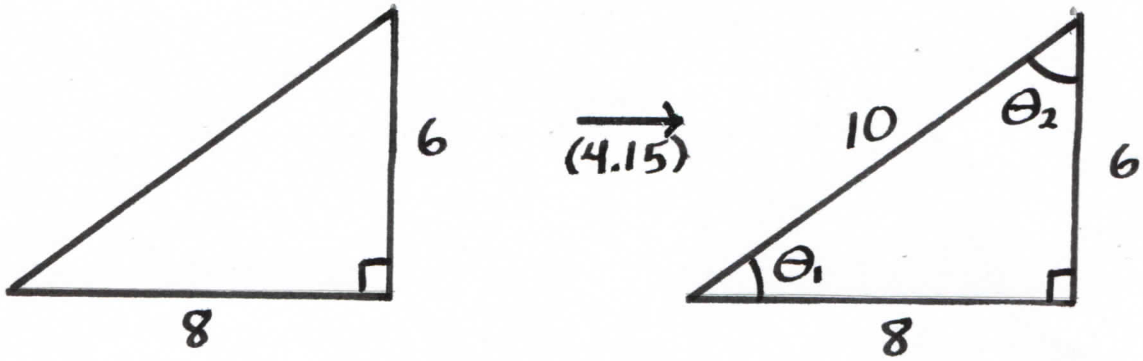
$\longrightarrow$

$x = 5\sin(50°)$
$\cong 3.83$

$y = 5\cos(50°)$
$\cong 3.21$



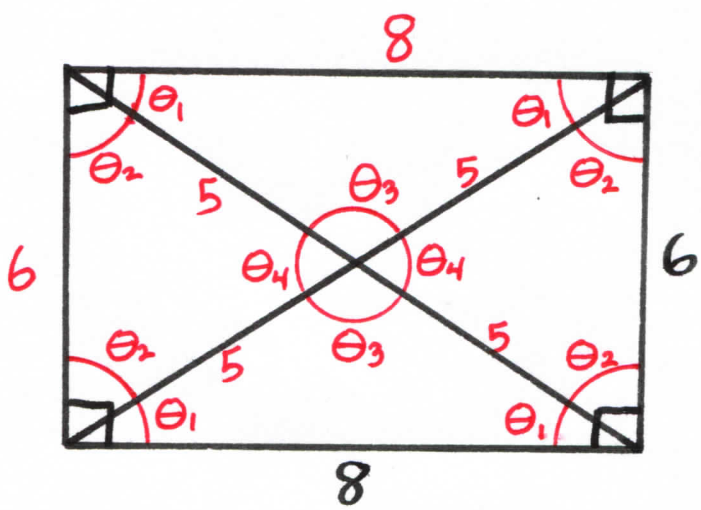
DRAWING 14.46



$$\theta_1 = \cos^{-1}\left(\tfrac{8}{10}\right) \sim 36.87°$$

$$\longrightarrow$$

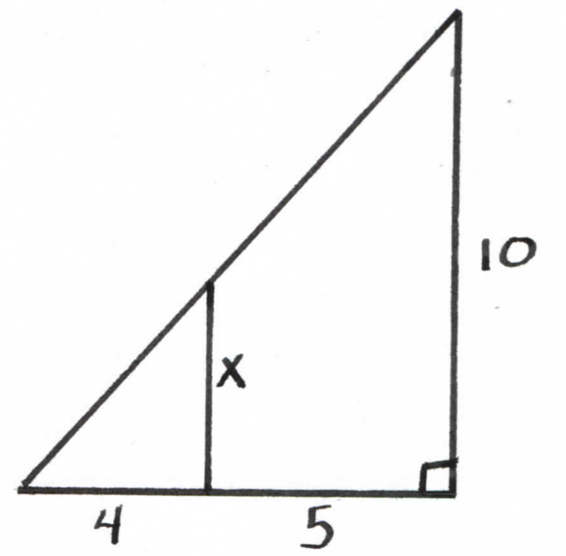
$$\theta_2 = \cos^{-1}\left(\tfrac{6}{10}\right) \sim 53.13°$$



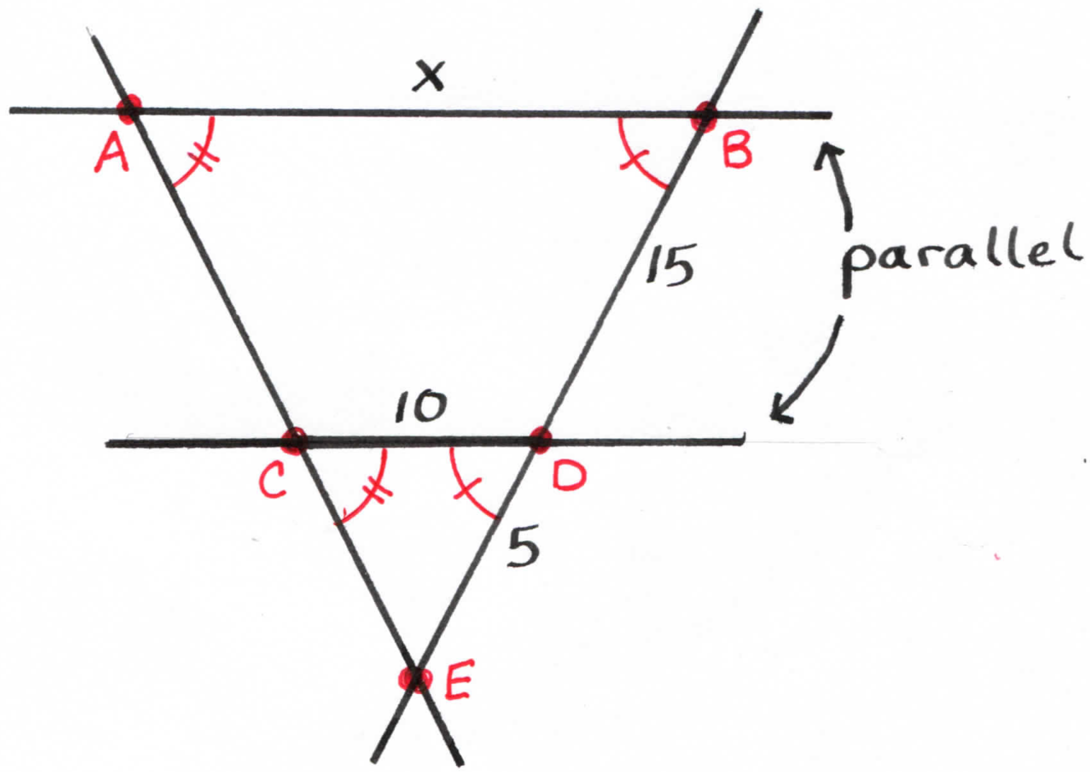
$$\theta_3 = \pi - 2\theta_1$$

$$\theta_4 = \pi - 2\theta_2$$

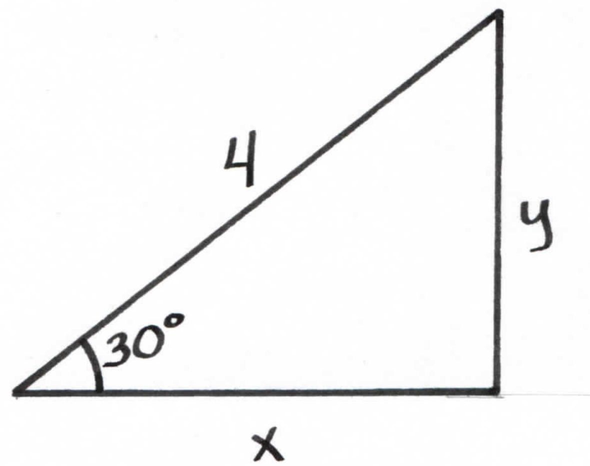
DRAWING 14.47



DRAWING 14.48



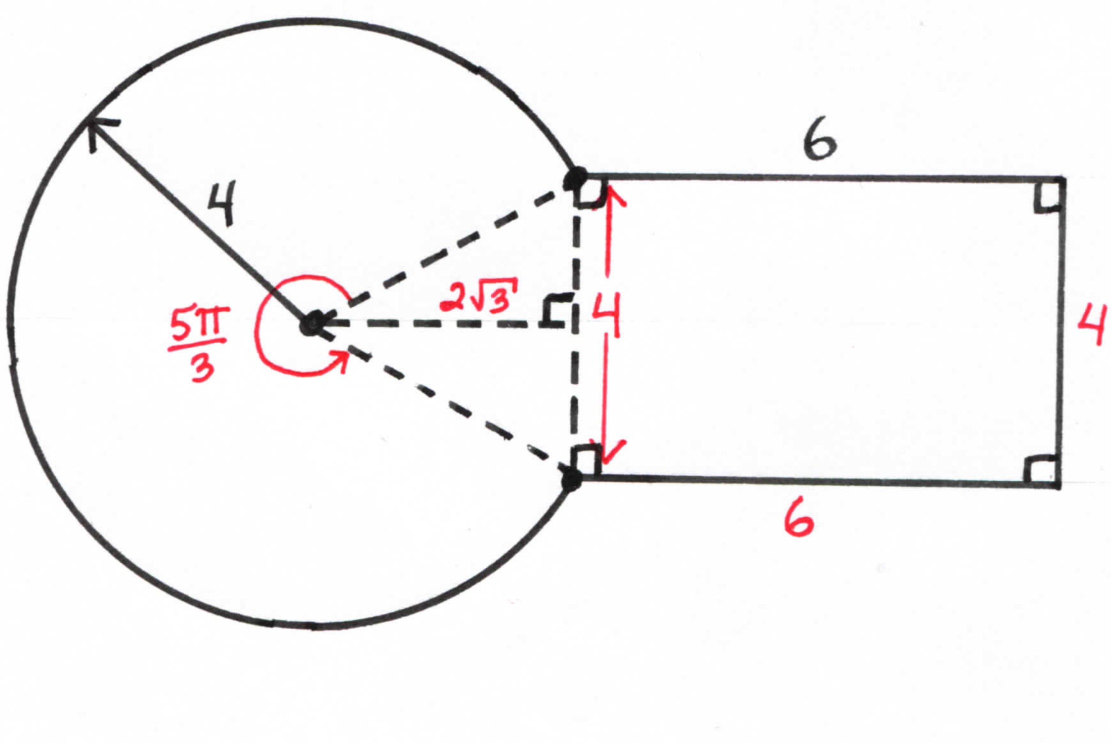
DRAWING 14.49



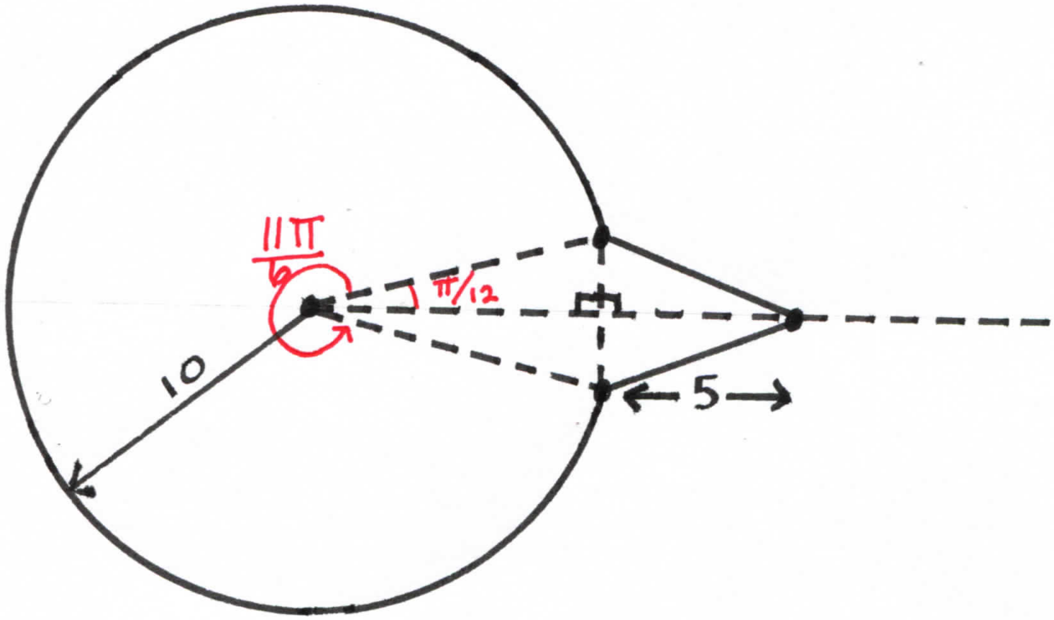
$$y = 4\sin(30°) = 2$$

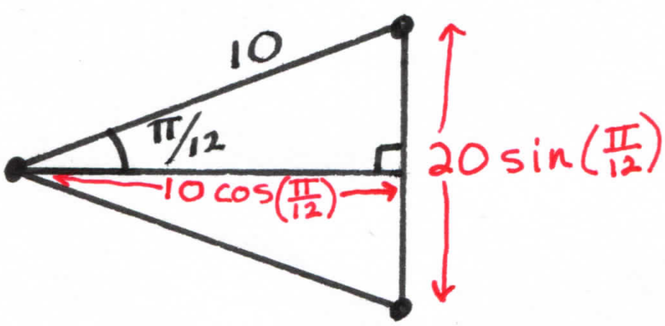
$$x = 4\cos(30°) = 2\sqrt{3}$$

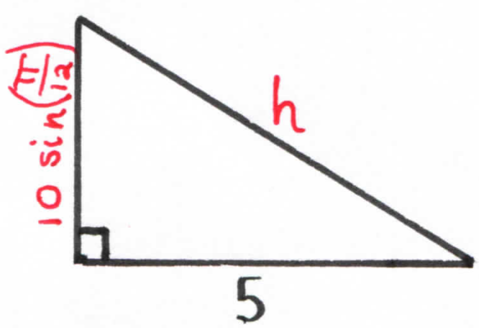
DRAWING 14.50



DRAWING 14.51

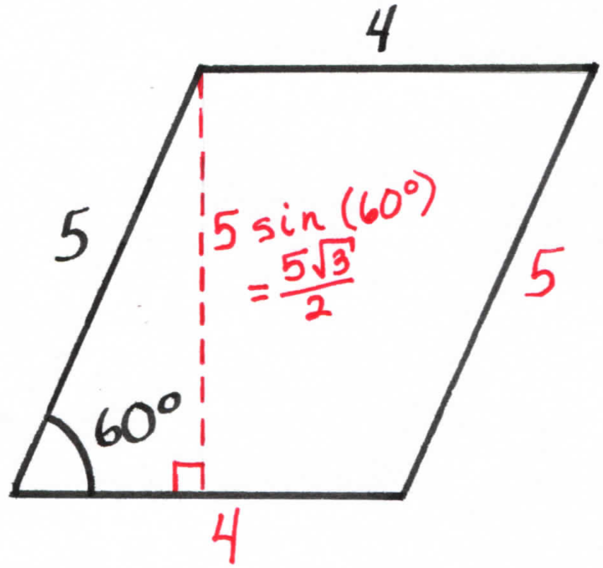




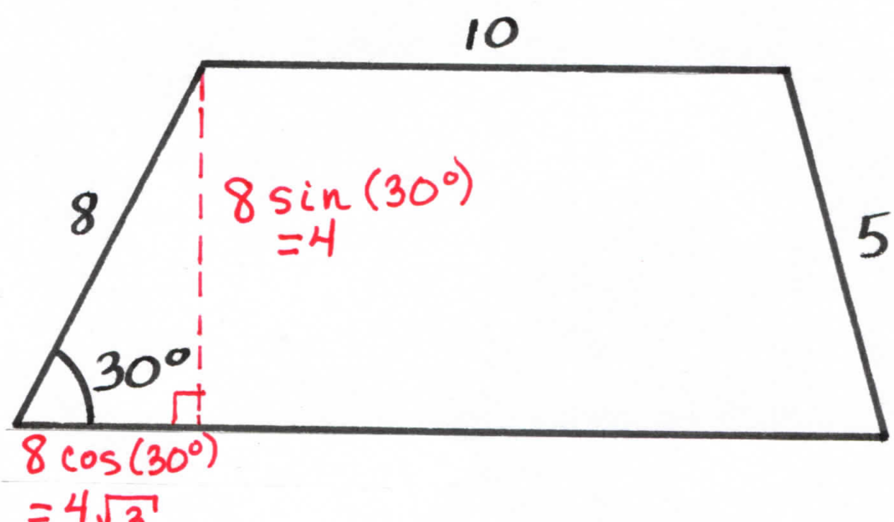


$$\rightarrow h=\sqrt{25+100\sin^2(\tfrac{\pi}{12})}$$

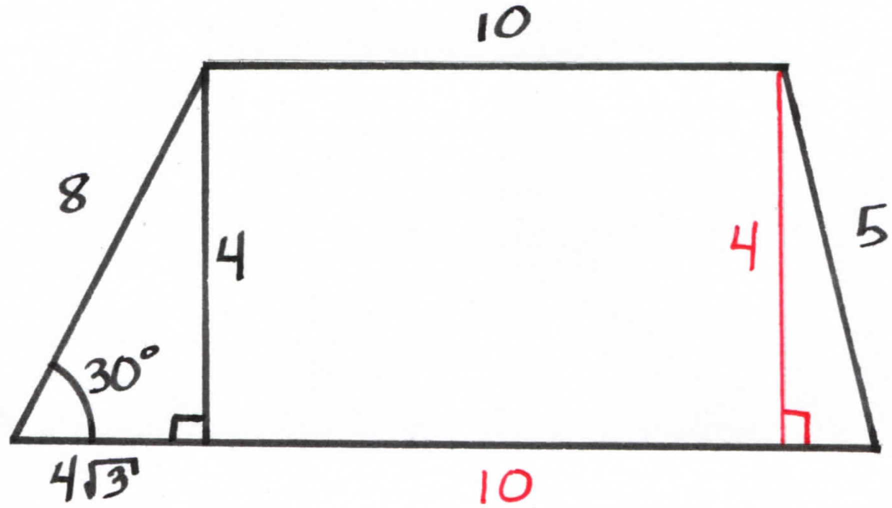
DRAWING 14.52



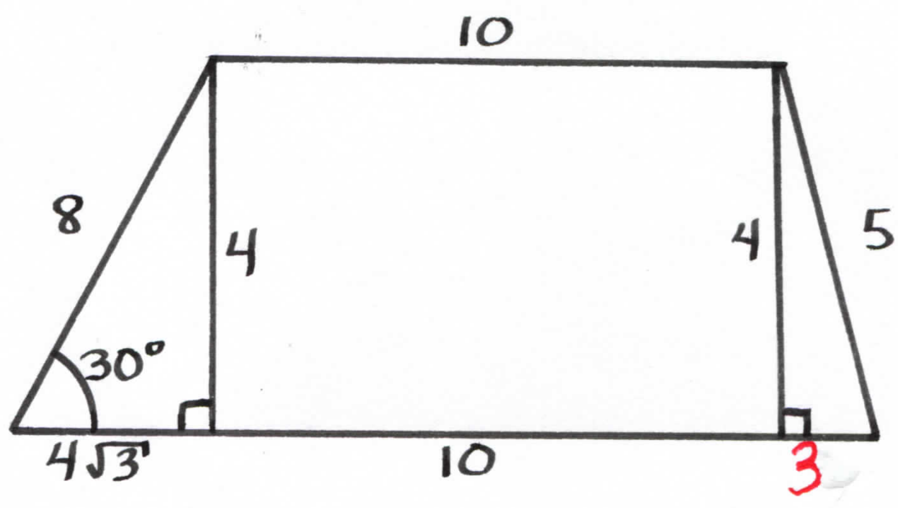
DRAWING 14.53



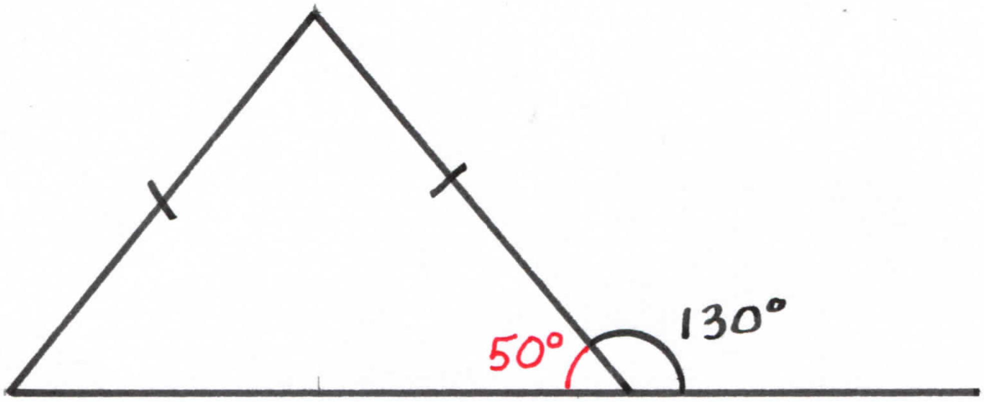
DRAWING 14.54

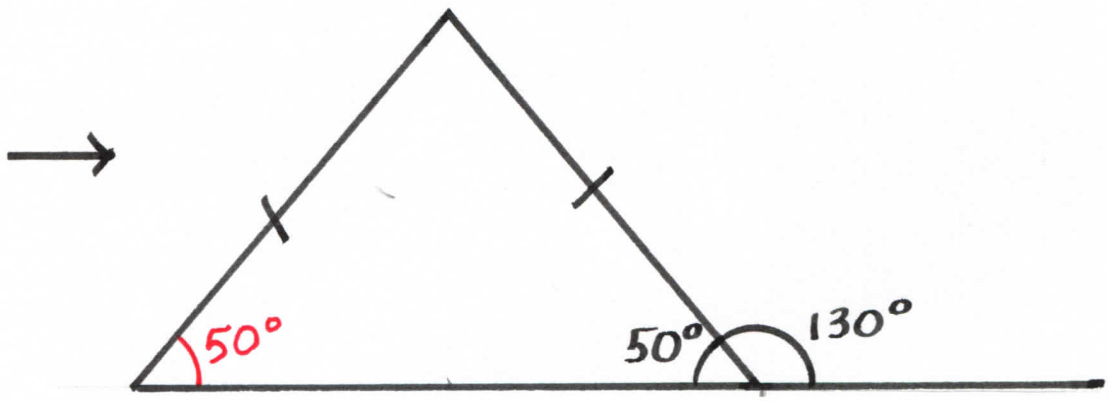


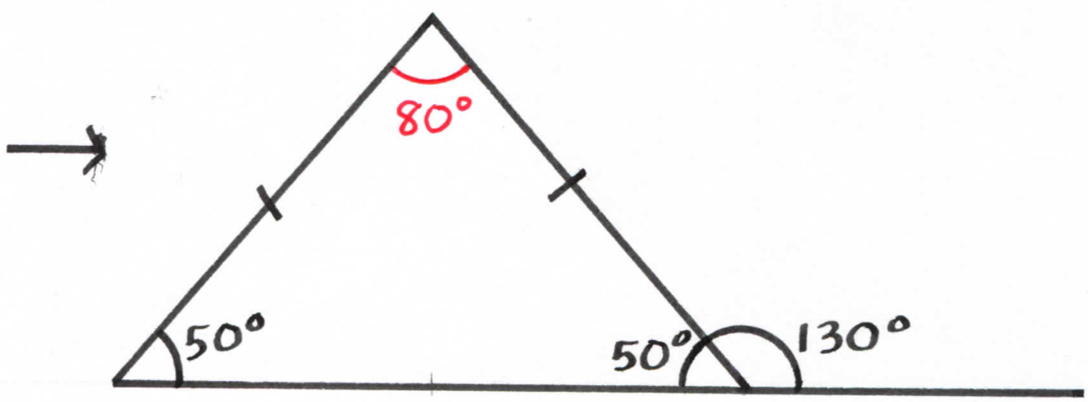
DRAWING 14.55



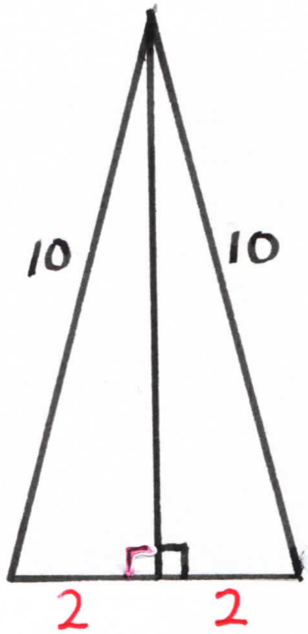
DRAWINGS 14.56



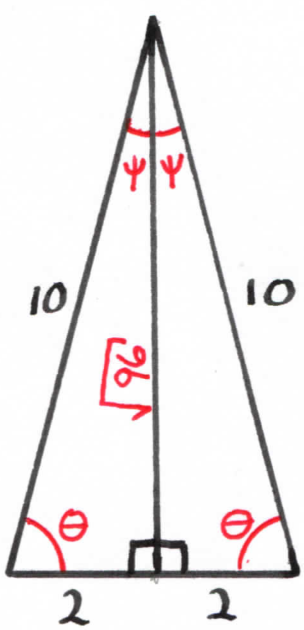




DRAWING 14.57



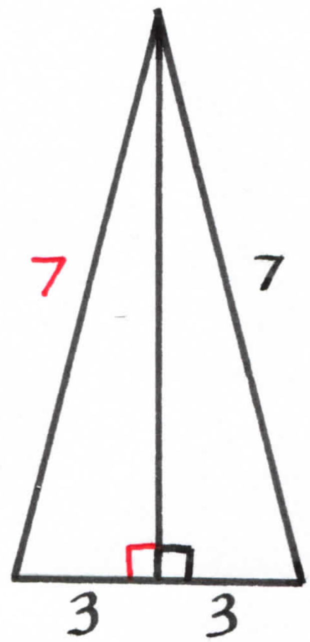
$\longrightarrow \quad 10^2 - 2^2 = 96$



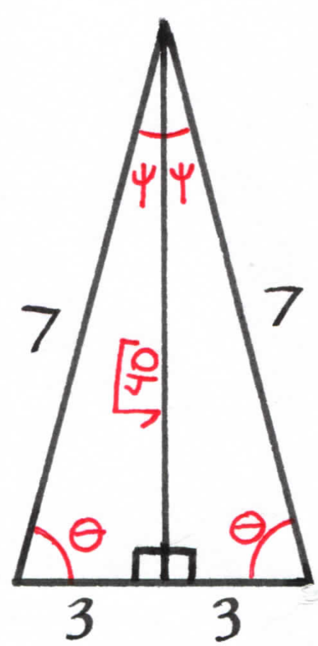
$\cos\theta = \frac{2}{10} \longrightarrow \theta \sim 78°$

$\sin\psi = \frac{2}{10} \longrightarrow \psi \sim 12°$

DRAWINGS 14.58



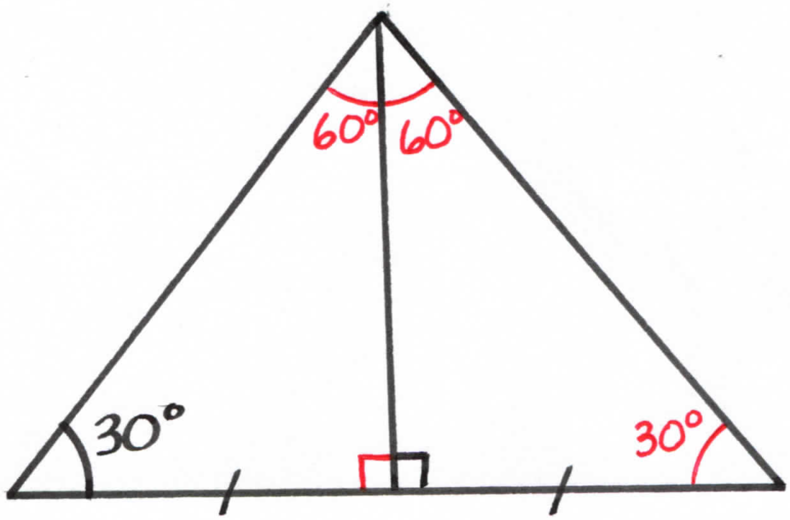
$\longrightarrow \quad 7^2 - 3^2 = 40$



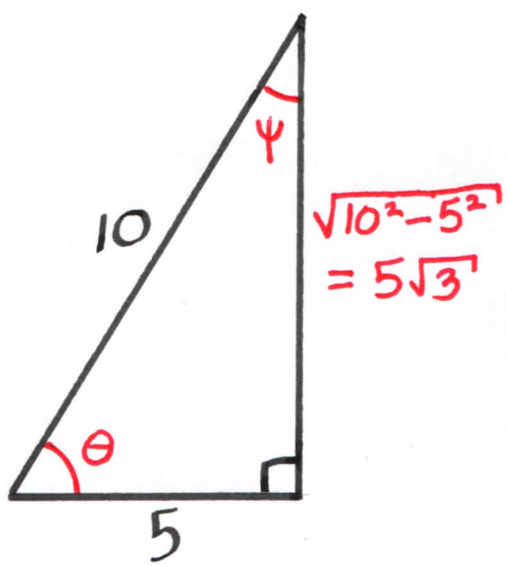
$\cos\theta = \frac{3}{7} \longrightarrow \theta \sim 65°$

$\sin\psi = \frac{3}{7} \longrightarrow \psi \sim 25°$

DRAWING 14.59



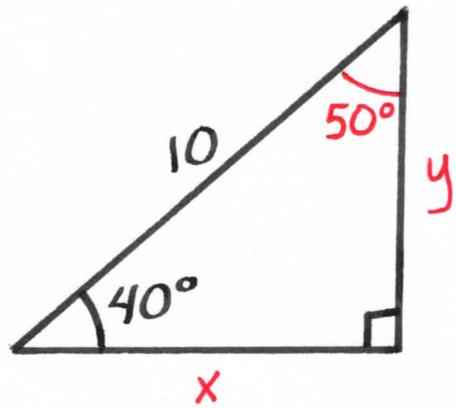
DRAWING 14.60



$\cos\theta = \frac{5}{10}$
$\longrightarrow \theta = 60°$

$\sin\psi = \frac{5}{10}$
$\longrightarrow \psi = 30°$

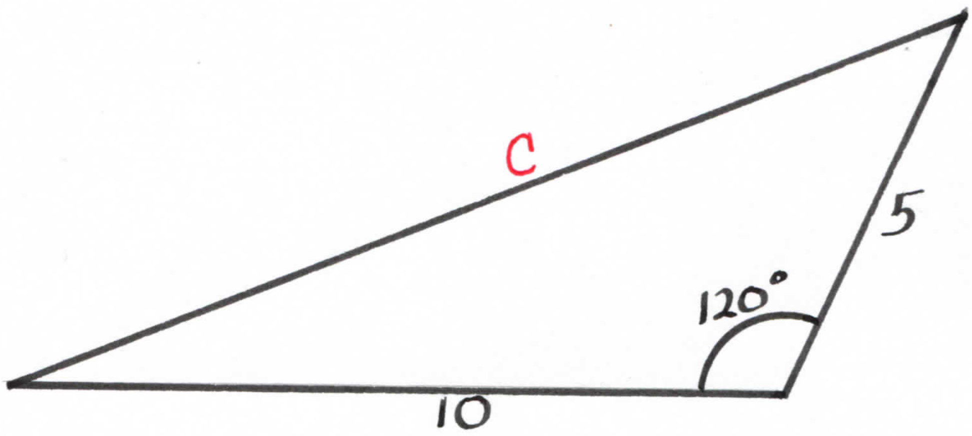
DRAWING 14.61



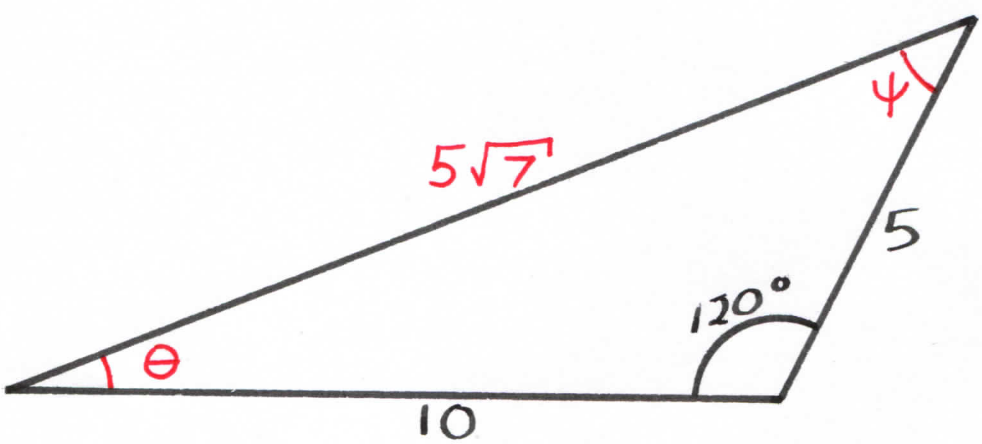
$$x = 10\cos(40^\circ) \sim 7.66$$

$$y = 10\sin(40^\circ) \sim 6.43$$

DRAWINGS 14.62



$$\left.\begin{aligned} c^2 &= 10^2+5^2-2\cdot 10\cdot 5\cos(120°) \\ &= 125+100\cos(60°) \\ &= 125+50 \\ &= 175 \end{aligned}\right\} \longrightarrow \begin{aligned} c &= \sqrt{175} \\ &= 5\sqrt{7} \end{aligned}$$

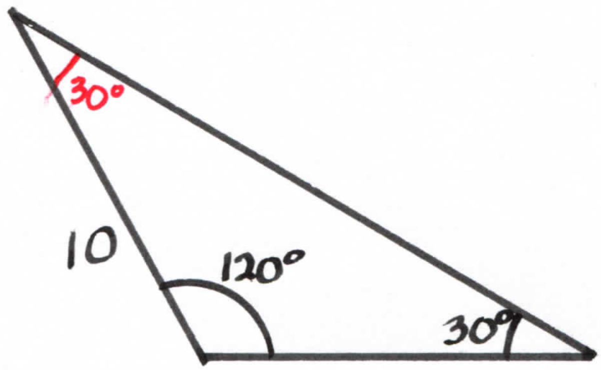


$$\frac{\sin\psi}{10}=\frac{\sin\theta}{5}=\frac{\sin(120°)}{5\sqrt{7}}=\frac{\sin(60°)}{5\sqrt{7}}=\frac{\sqrt{3}/2}{5\sqrt{7}}=\frac{\sqrt{3}}{10\sqrt{7}}$$

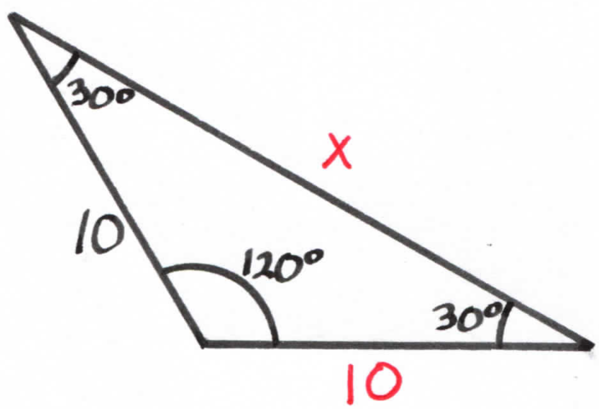
$$\longrightarrow \quad \sin\psi=\frac{\sqrt{3}}{\sqrt{7}} \; ; \quad \sin\theta=\frac{\sqrt{3}}{2\sqrt{7}}$$

$$\longrightarrow \quad \psi\sim 41° \; ; \quad \theta\sim 19°$$

DRAWINGS 14.63



$\longrightarrow$



$\sin(120°) = \sin(60°) = \frac{\sqrt{3}}{2}$

$\sin(30°) = \frac{1}{2}$

$\cos(120°) = -\cos(60°) = -\frac{1}{2}$

Law of Cosines:

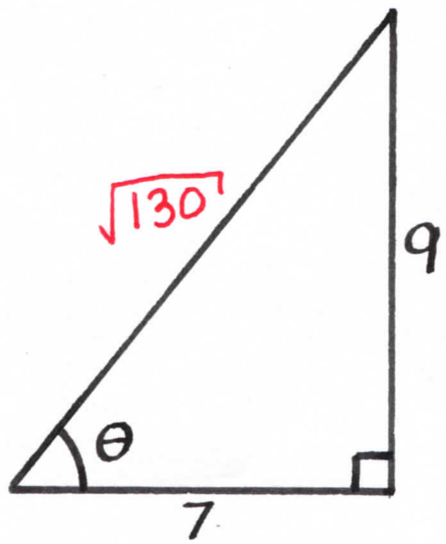
$$x^2 = 10^2 + 10^2 - 2 \cdot 10 \cdot 10 \cos(120°)$$

OR

Law of Sines:

$$\frac{x}{\sin(120°)} = \frac{10}{\sin(30°)}$$

DRAWING 14.64



$$7^2+9^2=130$$

$\longrightarrow$ $\cos\theta=\frac{7}{\sqrt{130}}$; $\sin\theta=\frac{9}{\sqrt{130}}$; $\tan\theta=\frac{9}{7}$

$$\cos(2\theta)=\cos^2\theta-\sin^2\theta=\frac{49-81}{130}=\frac{-32}{130}$$

$$\sin(2\theta)=2\sin\theta\cos\theta=\frac{126}{130}$$

$$\cos\left(\tfrac{1}{2}\theta\right)=\sqrt{\tfrac{1}{2}(1+\cos\theta)}=\sqrt{\tfrac{1}{2}\left(1+\tfrac{7}{\sqrt{130}}\right)}$$

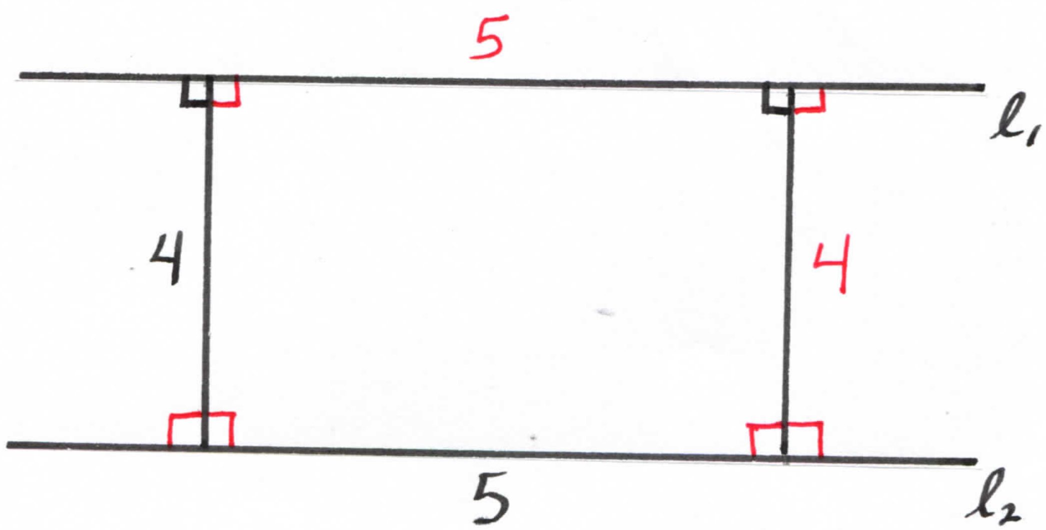
$$\sin\left(\tfrac{1}{2}\theta\right)=\sqrt{\tfrac{1}{2}(1-\cos\theta)}=\sqrt{\tfrac{1}{2}\left(1-\tfrac{7}{130}\right)}$$

$$\theta=\cos^{-1}\left(\frac{7}{\sqrt{130}}\right)\sim 52^\circ$$

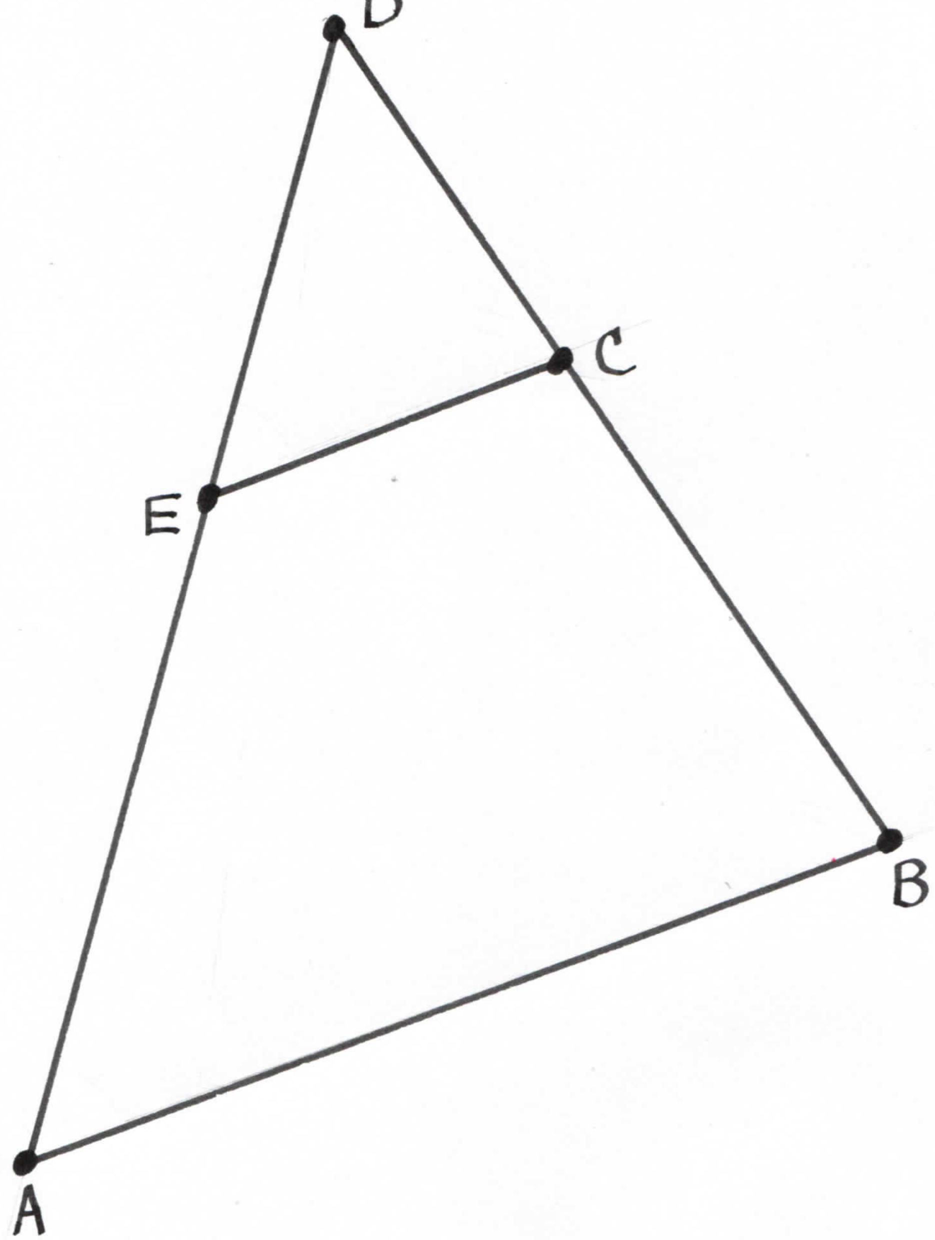
$$\text{area}=\tfrac{1}{2}\cdot 7\cdot 9=\frac{63}{2}$$

$$\text{perimeter}=7+9+\sqrt{130}=16+\sqrt{130}$$

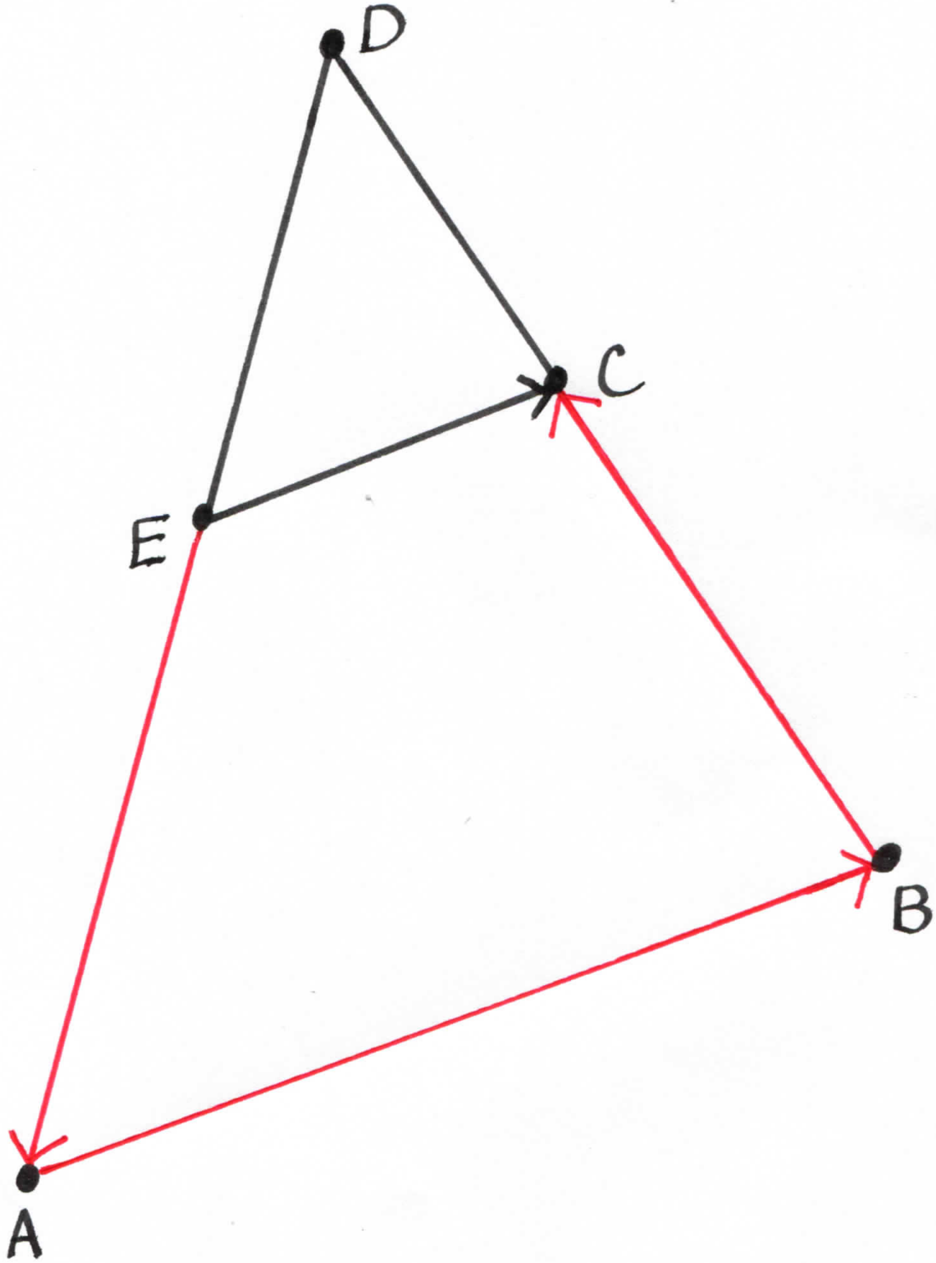
DRAWING 14.65



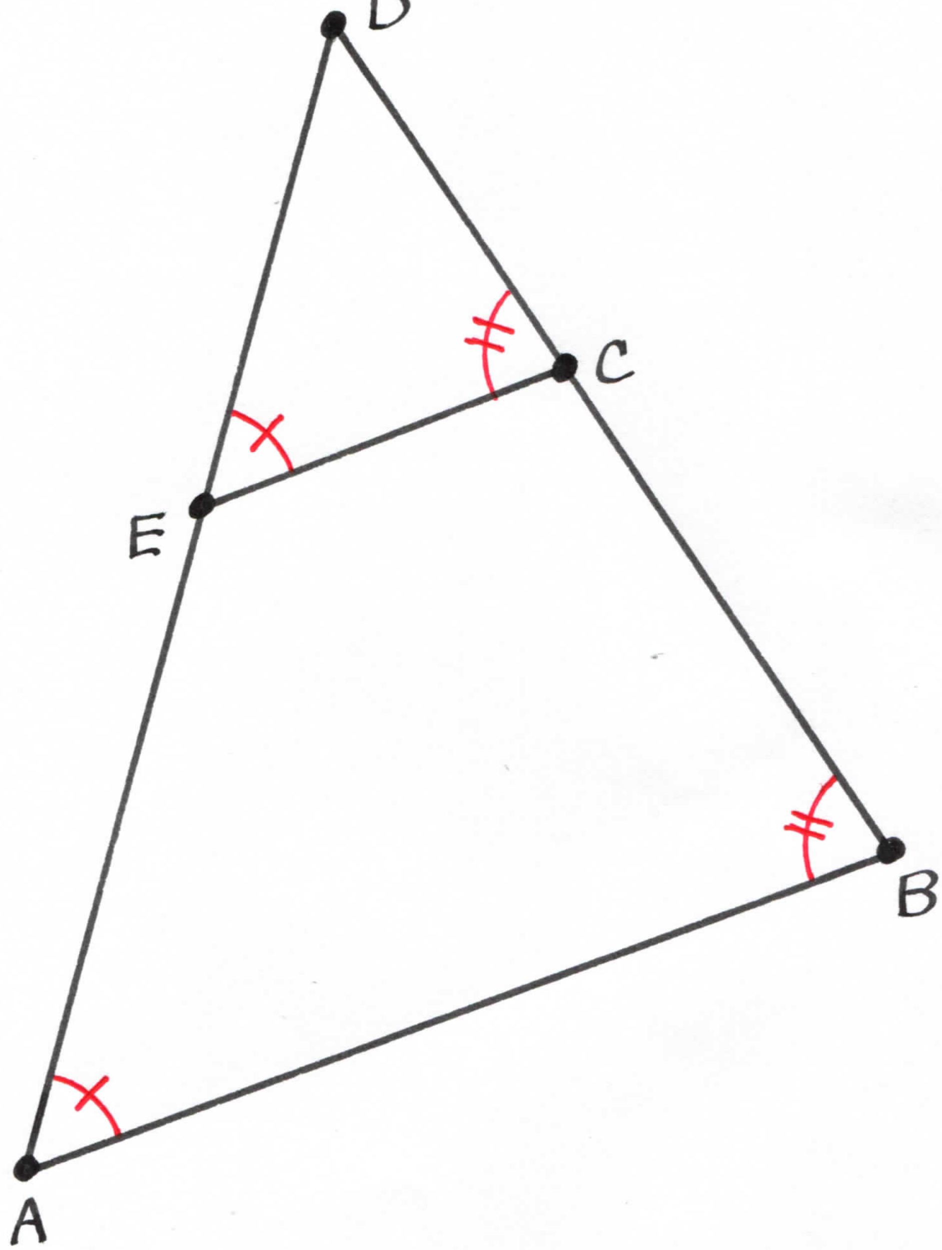
DRAWING 14.66



DRAWING 14.67



DRAWING 14.68



APPENDIX ZERO: The Language of Sets, Logic, and Functions.

Aside from the social aspect of making harmonious noise, the purpose of speaking or writing is the communication of ideas. The language of mathematics is compact, (almost) unambiguous, and precise. Here we present a brief dictionary.

Definitions APP0.1: Sets. A **set** is, informally, a collection of objects, or a bunch of things in a paper bag; in practice, a paper bag is replaced by a pair of set brackets $\{\dots\}$, with something between them (unless it is the *empty set* ϕ, to be thought of as a pair of set brackets with nothing between them $\{\ \}$).

For example, $\{2, \sqrt{5}, \text{universal truth}, \text{my pet wolverine}, \{\pi\}\}$ is the set containing $2, \sqrt{5}$, universal truth, my pet wolverine, and the set $\{\pi\}$.

A set is not defined rigorously, because any system of definitions must begin with something that is undefined (try using a dictionary to look up a word, then look up the words defining that word, then look up the words you last read, ...; eventually, you will come full circle to the original word you first looked up). Sets are the one and only undefined idea in mathematics, from which all other ideas are defined. For example,

$$0 \equiv \phi \text{ (the empty set)}, 1 \equiv \{\phi, \{\phi\}\} = \{0, \{0\}\}, 2 \equiv \{\{0, \{0\}\}, \{\{0, \{0\}\}\}\} = \{1, \{1\}\},$$

etc.; all the *counting numbers* $0, 1, 2, \dots$ are generated this way, with

$$(n + 1) \equiv \{n, \{n\}\},$$

for $n = 0, 1, 2, 3, \dots$.

Here are the primary operations on sets. See DRAWINGS APP0.1 at the end of this appendix for the standard pictures called *Venn diagrams*, with a large rectangle for the *universe* of all possibilities of interest at the moment.

The **complement** of a set A, denoted A^c, is everything not in A.

The **union** of two sets A and B, denoted $(A \bigcup B)$, is everything in A or B or both.

The **intersection** of two sets A and B, denoted $(A \bigcap B)$, is everything in both A and B.

Definitions APP0.2: Logic. All reasoning begins with an unproven assumption, called a **postulate** or **postulates**, analogous to all definitions beginning with something that is not defined, such as a set.

Logic connects facts that are true (or assumed to be true) to other facts that will inevitably also be true. Those initial, possibly assumed, facts are called *premises*, and the inevitably true facts that follow are called *conclusions*. A sequence of logically connected assertions beginning with the premises and ending with the conclusions is called an *argument*. Perhaps we should say *logical argument* to distinguish from emotional or threatening tirades, or ad hominem or non sequitur diversions, that persuade for all the wrong reasons; a **mathematical proof**, or **proof** for short, consists purely of logical arguments, thus the "logical" is usually left assumed but not stated. Expositions of mathematics should try to make it clear when an assertion is made without presenting its argument.

A natural goal in a personal, or shared, set of beliefs, is to minimize the number of postulates, from which logic will produce all the other beliefs.

The most important connective in logic is *implication*:

"**A implies B**," denoted $\mathbf{A} \rightarrow \mathbf{B}$, means that, when A is true, B is automatically also true.

For example, if A is "I am a primate" and B is "I am a mammal," then $A \rightarrow B$, in English, is "being a primate implies being a mammal."

For another example, let A be “I live in Ohio” and B be “I live in North America,” then A $\rightarrow$ B, in English, is “living in Ohio implies living in North America.”

Notice that both examples of implications above involve set containment: the set of all primates is contained in the set of all mammals, and Ohio is contained in North America.

Here are some synonyms for implication: A implies B is also written

“**If A, then B**,” e.g., “If I am a primate, then I am a mammal,” or “If I live in Ohio, then I live in North America”;

“**B if A**,” e.g., “I am a mammal if I am a primate,” or “I live in North America if I live in Ohio”;

“**B is necessary for A**,” e.g., “If I am a primate, then it is necessary that I be a mammal,” or “If I live in Ohio, then it is necessary that I live in North America”;

“**A is sufficient for B**,” e.g., “In order that I be a mammal, it is sufficient that I be a primate,” or “To live in North America, it is sufficient to live in Ohio”;

“**B follows from A**,” e.g., “being a mammal follows from being a primate,” or “being in North America follows from being in Ohio”;

“**B is a consequence of A**,” e.g., “being a mammal is a consequence of being a primate,” or “being in North America is a consequence of being in Ohio”;

“**A only if B**,” e.g., “I am a primate only if I am a mammal,” or “I live in Ohio only if I live in North America”;

“**A, therefore B**,” e.g., “I am a primate, therefore I am a mammal,” or “I live in Ohio, therefore I live in North America.”

There are probably many more such synonyms.

In practice, we create strings of implications, A $\rightarrow$ B, B $\rightarrow$ C, C $\rightarrow$ D, $\ldots$, from which it follows that A $\rightarrow$ C, A $\rightarrow$ D, etc. (implication is *transitive*).

Here is probably the most famous or infamous example of this type of argument: Socrates is a carbon-based life form, all carbon-based life forms all mortal, therefore Socrates is mortal. We could translate this as A $\equiv$ I am Socrates, B $\equiv$ I am a carbon-based life form, C $\equiv$ I am mortal, then A $\rightarrow$ B and B $\rightarrow$ C, thus A $\rightarrow$ C.

There is no limit to how long a string of implications $A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow A_4 \rightarrow \ldots$ can be. Think of this as a sequence of dominoes, with each domino A_k knocking over the next domino A_{k+1}; provided we start the logical process by knocking over the first domino A_1 (asserting the truth of our premise) eventually all the dominoes are knocked over (asserting the truth of our conclusions); that is, we conclude that $A_1 \rightarrow A_2, A_1 \rightarrow A_3, A_1 \rightarrow A_4, \ldots$. See DRAWING APP0.2 at the end of this appendix.

We say that two statements A and B are **equivalent**, denoted **A** $\iff$ **B** or “**A if and only if B**” or “**A is necessary and sufficient for B**,” if we have both (A $\rightarrow$ B) and the converse (defined below) implication (B $\rightarrow$ A).

For example, “my wolverine is between two and three feet long” is equivalent to “my wolverine is less than three feet long and my wolverine is more than two feet long”; in symbols,

$$[2 < x < 3] \iff [x > 2 \text{ and } x < 3]\,,$$

where x is the length of my wolverine.

More generally, any set of n assertions $A_1, A_2, \ldots, A_n$ is said to be **equivalent** if the truth of one of the assertions implies the truth of all the others; that is, for $1 \leq j \leq n$, $A_j \rightarrow A_i$, for $1 \leq i \leq n$.

Equivalence of $A_1, A_2, \ldots, A_n$ is usually shown with a full circle of implications, such as

$$A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow \ldots A_n \rightarrow A_1.$$

The **converse** of (A $\rightarrow$ B) is (B $\rightarrow$ A). For example, the converse of "If I am a primate, I am a mammal" is "If I am a mammal, I am a primate. Although the first implication is true, its converse is false, since there are many mammals that are not primates. Thus being a primate is not equivalent to being a mammal.

The **contrapositive** of (A $\rightarrow$ B) is [(not B) $\rightarrow$ (not A)]. For example, the contrapositive of "If I am a primate, I am a mammal" is "If I am not a mammal, I am not a primate." Every implication is equivalent to its contrapositive.

Closely related to contrapositive is **proof by contradiction**. To prove a conclusion, call it B, we assume B is false and show that this leads to something we know is not true (this is the "contradiction"). In particular, if our desired result is A $\rightarrow$ B, proof by contradiction could consist of assuming B is false and showing that this leads to the contradiction of the assumed A being false; that would be exactly showing the contrapositive (not B) $\rightarrow$ (not A).

Here is an example. A **prime** number is a positive integer that is divisible only by itself and 1. For example, 6 is *not* prime, because it equals 2×3; 6 is divisible by 2.

We will prove by contradiction that there are infinitely many prime numbers:

Suppose there are *not* infinitely many primes; that is, there are only a finite number of prime numbers, call them $p_1, p_2, \ldots, p_N$, where we have listed the prime numbers in increasing order. In particular, our supposition implies that there is a largest prime number, that we have denoted p_N.

Let $M \equiv (p_1 \times p_2 \times p_3 \times \cdots \times p_n)$, the product of all the prime numbers.

If $p_k(k = 1, 2, 3, \ldots N)$ is any prime number, since p_k divides M, it cannot divide $(M + 1)$. This implies (since every positive integer is a product of prime numbers) that no number besides 1 and $(M + 1)$ divides $(M + 1)$. Thus $(M + 1)$ is a prime number larger than p_N. Since p_N was the largest prime number, we have reached a contradiction. This implies that our original assumption that there are finitely many prime numbers must be false; that is, there are infinitely many prime numbers.

An assumption is made **without loss of generality** if it does not limit the scope of an assertion. For example, if we are proving a result about areas of polygons, we may assume, without loss of generality, that one side of the polygon is on the positive x axis, because area is unchanged by rotations and translations (see Theorem 8.3).

Formal logic, with all assertions either true or false at a given moment, is missing the ingredient of *probability.* Aristotle, the founder of formal logic, acknowledged this limitation, as in "there will be a sea battle tomorrow."

For those familiar with conditional probability, the statement "the probability of B, given A, is equal to 100 percent," can be recognized as the implication A $\rightarrow$ B. More interesting and realistic scenarios appear when the "100 percent" is shrunk.

The remaining pair of definitions may not be necessary for this text, but are very useful, fundamental, and simplifying ideas.

Definitions APP0.3: Functions. If X and Y are two sets, a **function** f from X to Y, written

$$f : X \to Y,$$

is a rule that associates, to each x in X, a unique $f(x)$ in Y. See DRAWING APP0.3 at the end of this appendix.

$f(x)$ (reads "f of x") is the **image** of x under f; we also say that f **maps** x **to** $f(x)$.

The word "function" is English usage. For example, the area of a square of side x is $f(x) \equiv x^2$; we would say that "area is a function of the length of a side" to mean that area is determined by the length of a side. Or, closer to home, "your grade in a math class is a function of time spent studying."

Note the practical value of this functional relationship. Length is easy to measure (e.g., pace it off with one-yard steps), while area is difficult to measure directly: you'd need to try to place concrete squares of the same size into the region whose area you want. If you want to know how many seeds to buy to plant in this region, it's very convenient to measure only length.

Note that the function f is neither the set X nor the set Y; it is the *relationship* between points x in X and $f(x)$ in Y. More dynamically, it should be thought of as the action of moving from x to $f(x)$. In the example just given, f is neither length nor area; it is the relationship between length and area, the act of squaring. In general, a function should be thought of as a verb, *doing* something to points in the domain.

The **inverse function**, denoted f^{-1}, of the function f from Definitions APP0.3, is a function from Y to X such that

$$f^{-1}(f(x)) = x \quad \text{for all } x \text{ in } X \quad \text{and} \quad f(f^{-1}(y)) = y \quad \text{for all } y \text{ in } Y.$$

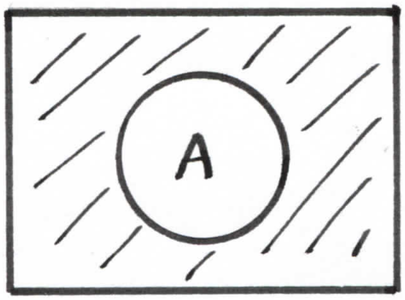
See DRAWING APP0.4 at the end of this appendix.

Note that f^{-1} cannot exist if either of the pictures in DRAWINGS APP0.5 at the end of this appendix occur.

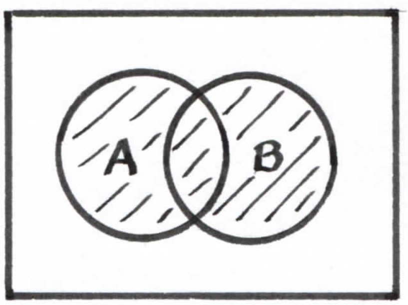
For example, say X is the set of all pets, Y is the set of all pet owners, and, for any pet x, $f(x)$ is that pet's owner. The function f has an inverse only if no pet owner has multiple pets; the inverse function f^{-1} then maps a pet owner to said owner's only pet.

The inverse function f^{-1} undoes whatever f did, leaving you back where you started before f got applied. Examples of this are mythical weight-loss medicines, returning you immediately to your pre-overeating state, and apologetic phone calls, undoing whatever damage your careless words did. We leave it to relationship experts to determine if the careless-words function really has an inverse function.

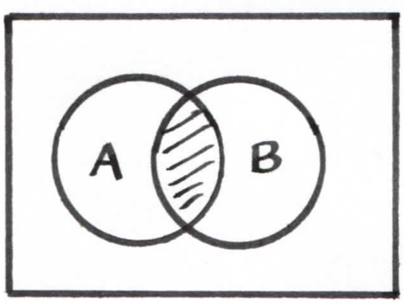
DRAWINGS APPO.1



A^c ("not A")

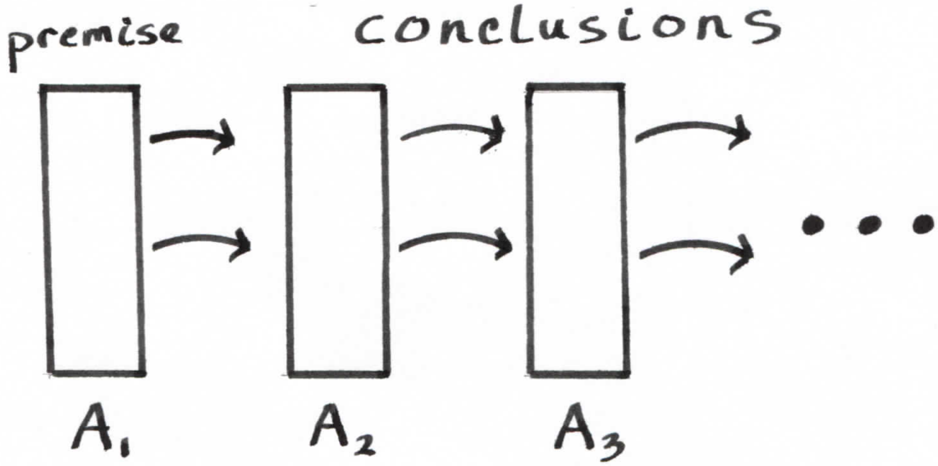


$A \cup B$
("A or B or both")

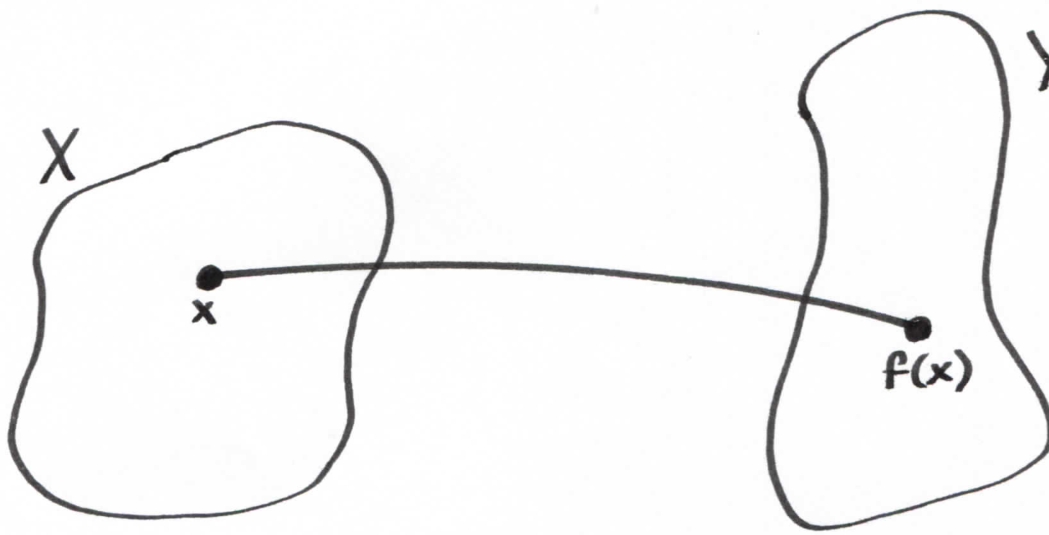


$A \cap B$ ("A and B")

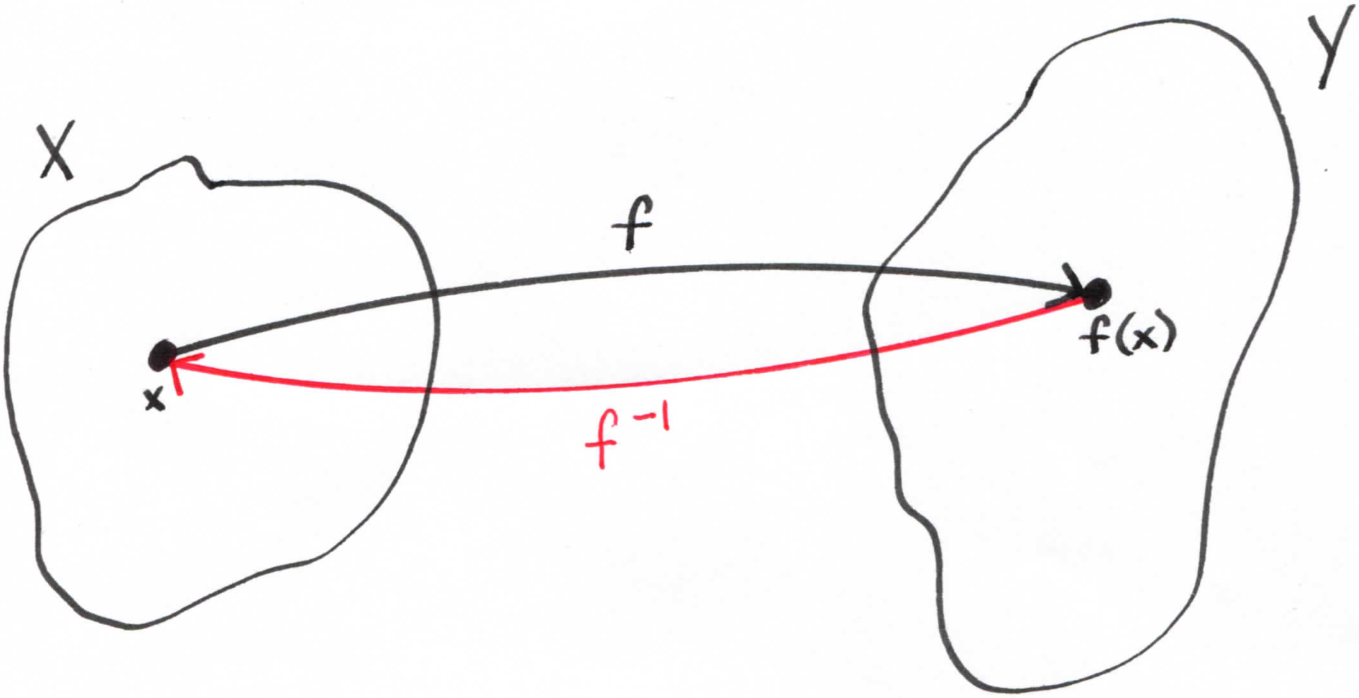
DRAWING APPO.2



DRAWING APPO.3

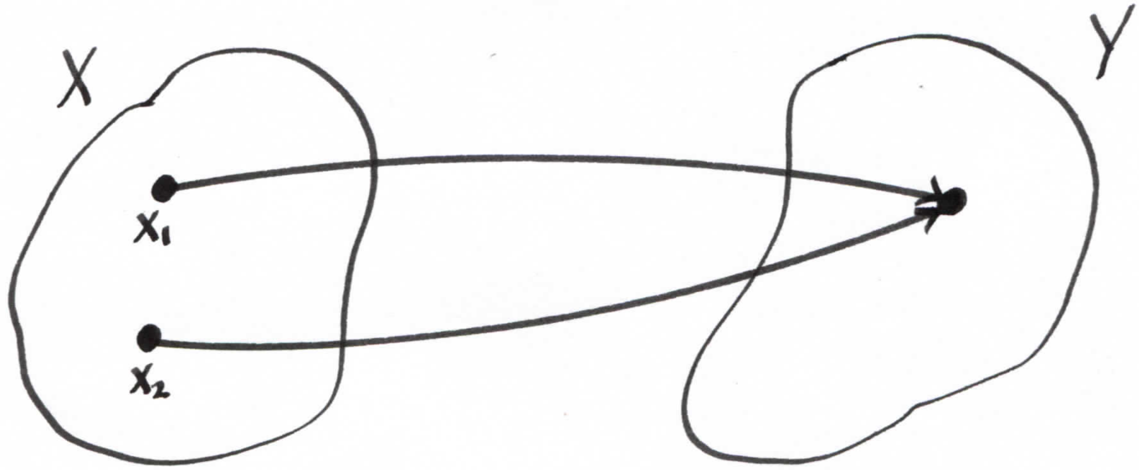


DRAWING APPO.4

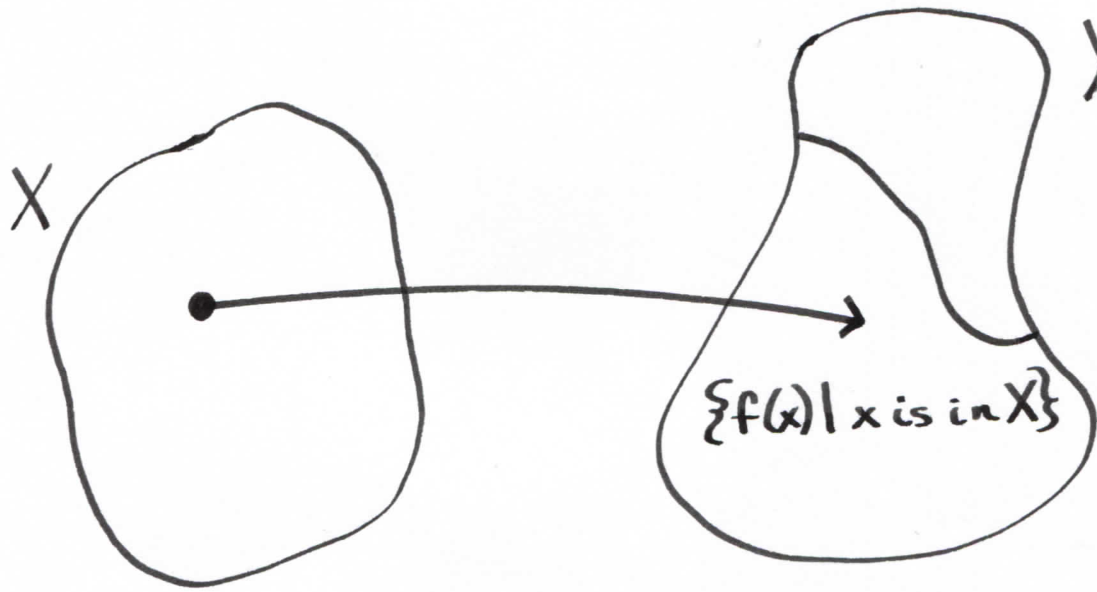


$$y = f(x) \longleftrightarrow x = f^{-1}(y)$$

DRAWINGS APPO.5



OR



APPENDIX ONE: Integration, Area, and Volume.

In this section, APP1.1 through APP1.7 is needed only for the proofs in Chapter IX and the additivity of area mentioned in Remarks 2.5. Proposition APP1.9 is needed for Theorem 12.4. APP1.10 and APP1.11 are needed for Theorem 12.8.

This section will consider two-dimensional regions of the form

$$R \equiv \{(x, y) \,|\, a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\},$$

for $a < b, g_1, g_2$ piecewise-continuous functions on $[a, b]$ with $g_1(x) \leq g_2(x)$ for $a \leq x \leq b$. See Remarks 2.5 and DRAWING APP1.1 at the end of this appendix.

Integration extends the definition of area in the Postulates in the Introduction to the area of R with Riemann sums:

$$\text{area of } R = \int_a^b (g_2(x) - g_1(x))\, dx \equiv \lim_{n\to\infty} \sum_{k=1}^{n} \left[(g_2 - g_1)(a + k\frac{(b-a)}{n})\right] \left[\frac{(b-a)}{n}\right]$$

See DRAWING APP1.2 at the end of this appendix.

The first thing worth noting is that area is additive because integration is additive.

Proposition APP1.1. If R_1 and R_2 are regions that overlap at most on a curve, then the area of the union of R_1 and R_2 equals the sum of the areas of R_1 and R_2.

Here are some special cases of and terminology for *rigid motions* of Definition 8.2 and Definitions 8.1.

Definitions APP1.2. For x_0, y_0 arbitrary real numbers, the **translation** of R by $< x_0, y_0 >$ is

$$(R+ < x_0, y_0 >) \equiv \{(x, y) \,|\, a + x_0 \leq x \leq b + x_0, g_1(x - x_0) + y_0 \leq y \leq g_2(x - x_0) + y_0\}.$$

The **reflection** through the y axis is the map $f(x, y) \equiv (-x, y)$; the **reflection** through the x axis is the map $g(x, y) \equiv (x, -y)$.

The reflection of a set of points S through an axis is

$$\{(-x, y) \,|\, (x, y) \text{ is in } S\} \quad (\text{through the } y \text{ axis}) \quad \text{or} \quad \{(x, -y) \,|\, (x, y) \text{ is in } S\} \quad (\text{through the } x \text{ axis}).$$

See DRAWING APP1.3 at the end of this appendix.

Proposition APP1.3. The area of $(R+ < x_0, y_0 >)$ equals the area of R.

Proof: The area of $(R+ < x_0, y_0 >)$ is

$$\int_{a+x_0}^{b+x_0} ((g_2(x - x_0) + y_0) - (g_1(x - x_0) + y_0))\, dx,$$

which, after the change of variables $s \equiv (x - x_0)$, becomes the integral for the area of R. $\square$

Corollary APP1.4. The area of a triangle with vertices $I, (I + \vec{a})$ and $(I + \vec{b})$ is a function only of $\vec{a}$ and $\vec{b}$.

We may also give quick calculus arguments for invariance of area under reflections through axes.

Proposition APP1.5. The area of R reflected through either axis equals the area of R.

Proof: R reflected through the y axis is

$$\{(x,y)\,|\,-b \leq x \leq -a, g_1(-x) \leq y \leq g_2(-x)\},$$

thus its area is

$$\int_{-b}^{-a} (g_2(-x) - g_1(-x))\; dx = \int_{b}^{a} (g_2(s) - g_1(s))\; d(-s) = \int_{a}^{b} (g_2(s) - g_1(s))\; ds.$$

R reflected through the x axis is

$$\{(x,y)\,|\,a \leq x \leq b, -g_2(x) \leq y \leq -g_1(x)\},$$

whose area is

$$\int_{a}^{b} (-g_1(x) - (-g_2(x)))\; dx = \int_{a}^{b} (g_2(x) - g_1(x))\; dx.$$

$\square$

Corollary APP1.6. Let $\vec{a}, \vec{b}$ be vectors. The area of a parallelogram formed by $\vec{a}, \vec{b}$ is twice the area of a triangle formed by $\vec{a}, \vec{b}$.

Proof: See DRAWING APP1.4 at the end of this appendix. By Proposition APP1.5, the triangle formed by $(-\vec{a}), (-\vec{b})$ has the same area as the triangle formed by $\vec{a}, \vec{b}$, since the former triangle (with $\vec{a}, \vec{b}$ in standard position) is the reflection, through both axes, of the latter triangle. Since a parallelogram formed by $\vec{a}, \vec{b}$ is the union of those two triangles, the result follows. $\square$

Corollary APP1.7. The area of a right triangle with a horizontal leg is one half the product of the lengths of the two perpendicular sides.

Proof: This follows from Corollary APP1.6, with the parallelogram a rectangle as in Postulate (2) in the Introduction. $\square$

Fundamental Theorem of Calculus APP1.8. If the derivative f' is continuous on an interval $[a, b]$, then

$$\int_{a}^{b} f'(x)\, dx = f(b) - f(a).$$

See APP3.6 for derivatives of sines and cosines.

Proposition APP1.9. Suppose $0 \leq \theta \leq \frac{\pi}{2}$. Then the closed sector of the unit disc $x^2 + y^2 \leq 1$ determined by the arc $\{e^{i\psi}\,|\,0 \leq \psi \leq \theta\}$ (see DRAWINGS APP1.5 at the end of this appendix) has area $\frac{\theta}{2}$.

Proof: This area is (see DRAWINGS APP1.5 at the end of this appendix)

$$\frac{1}{2}\sin\theta\cos\theta + \int_{\cos\theta}^{1} \sqrt{1-x^2}\, dx = \frac{1}{4}\sin(2\theta) + \int_{\theta}^{0} \sqrt{1-(\cos\phi)^2}\, d(\cos\phi) = \frac{1}{4}\sin(2\theta) + \int_{\theta}^{0} \sqrt{(\sin\phi)^2}\,(-\sin\phi)\, d\phi$$

$$= \frac{1}{4}\sin(2\theta) + \int_{0}^{\theta} (\sin\phi)^2\, d\phi = \frac{1}{4}\sin(2\theta) + \int_{0}^{\theta} \frac{1}{2}(1-\cos(2\phi))\, d\phi = \frac{1}{4}\sin(2\theta) + \frac{1}{2}\left(\phi - \frac{1}{2}\sin(2\phi)\right)|_0^\theta = \frac{\theta}{2},$$

by APP1.7, 6.3, and 6.4. $\square$

Definitions APP1.10. As with area at the beginning of this section, we could extend the definition of the volume of a box, in Definitions 12.6, via two-dimensional analogues of Riemann sums. We prefer the following approach, which will allow us to use only one-dimensional integrals.

Let R be a subset of $\mathbf{R}^3$, a and b real numbers such that

$$R \text{ is contained in } \{(x,y,z)\,|\,a \leq z \leq b\}.$$

For any fixed z_0 with $a \le z_0 \le b$, we want to intersect R with the plane $z = z_0$ parallel to the Cartesian plane: let

$$R_{z_0} \equiv R \cap \{(x, y, z_0) \,|\, (x, y) \text{ is in } \mathbf{R}^2\},$$

and let $A(z_0) \equiv$ the area of R_{z_0}. See DRAWING APP1.6 at the end of this appendix.

Then the **volume** of R is

$$V(R) = \int_a^b A(z)\,dz.$$

Think of R as a loaf of bread, with the variable z running through the length of R. If we slice the bread at a particular value of z, we are exposing a cross-section of R, like the side of a slice of the loaf; $A(z)$ is the corresponding area of the side of that slice. We could cut up the entire loaf into thin slices, of varying areas $A(z)$. Intuitively, $A(z)\,dz$ is the volume of one of those thin slices of bread; integrating those areas dz is like putting the slices back together to get the entire loaf and its volume.

Implicit in this volume formula is the assumption that, for each z, we may calculate the area $A(z)$ of R_z, and that the function $z \mapsto A(z)$ is continuous.

Now we may address the particular volume formulas from Theorem 12.8.

APP1.11: Proof of Theorem 12.8. We use Definitions APP1.10.

(b) For R a generalized cylinder of height H and base Ω, for any z, $0 \le z \le H$,

$$A(z) = (\text{area of } \Omega),$$

thus

$$V(R) = \int_0^H A(z)\,dz = \int_0^H (\text{area of } \Omega)\,dz = (\text{area of } \Omega)H.$$

(a) follows from (b) and Theorem 12.4.

(d) For R a generalized cone of height H and base Ω, by Theorem 8.3,

$$A(z) = \left(\frac{H - z}{H}\right)^2 (\text{area of } \Omega) \quad (0 \le z \le H),$$

thus

$$V(R) = \int_0^H A(z)\,dz = \int_0^H \left(\frac{H - z}{H}\right)^2 (\text{area of } \Omega)\,dz = (\text{area of } \Omega)\left(\frac{-1}{3H^2}\right)(H-z)^3|_0^H = \frac{1}{3}(\text{area of } \Omega)H.$$

(c) follows from (d) and Theorem 12.4.

(e) As with length and area, volume is not affected by translation, thus we may assume $(a, b, c) = (0, 0, 0)$; that is, R is $\{(x, y, z) \,|\, x^2 + y^2 + z^2 \le r^2\}$. See DRAWING APP1.7 at the end of this appendix.

For $-r \le z \le r$ (see DRAWING APP1.7 at the end of this appendix),

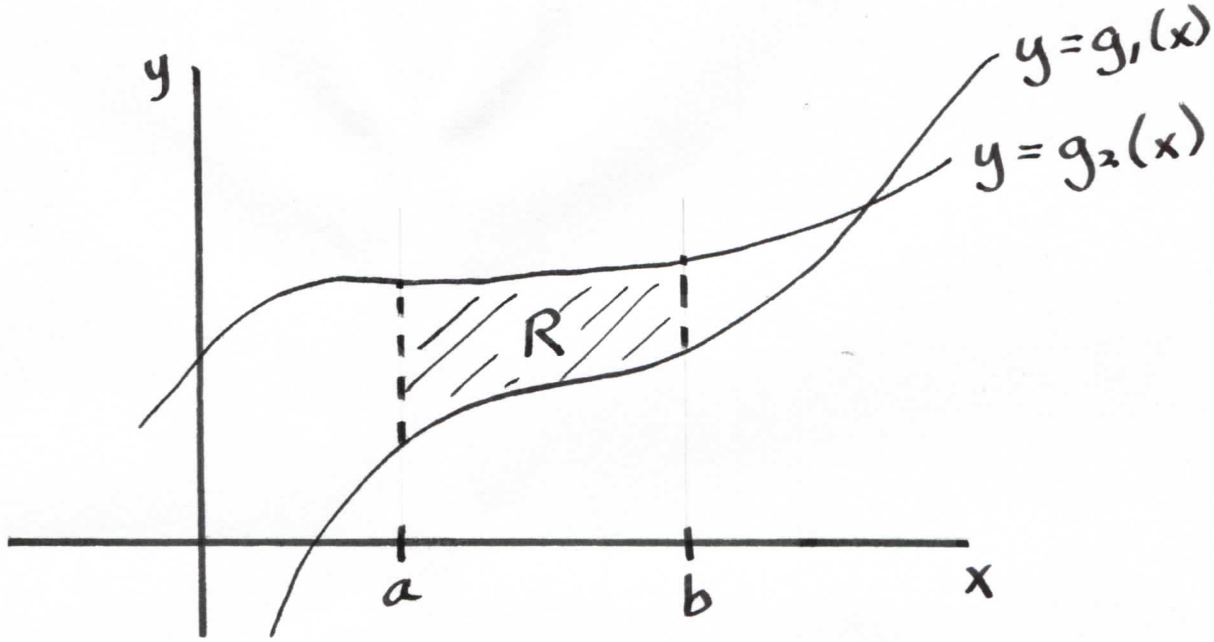
$$A(z) = \text{area of } \{(x, y) \,|\, x^2 + y^2 \le (r^2 - z^2)\} = \pi(r^2 - z^2),$$

by Theorem 12.4, thus

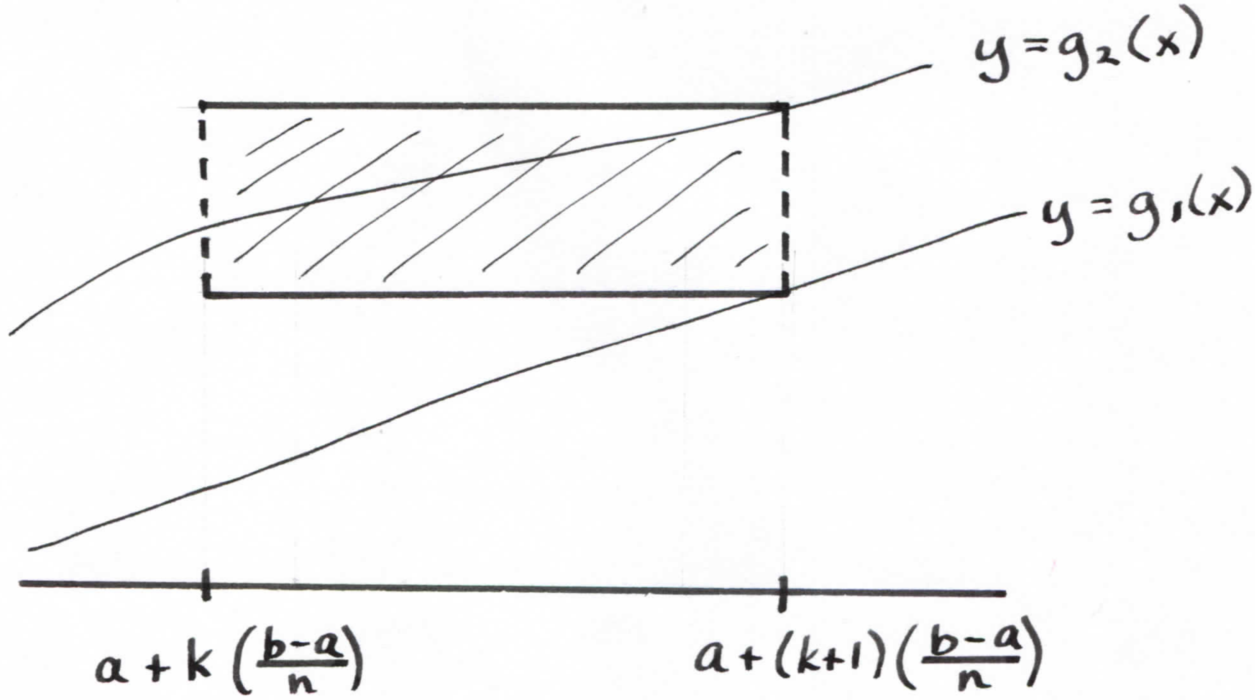
$$V(R) = \int_{-r}^r \pi(r^2 - z^2)\,dz = 2\int_0^r \pi(r^2 - z^2)\,dz = 2\pi(r^2 z - \frac{1}{3}z^3)|_0^r = 2\pi(r^3 - \frac{1}{3}r^3) = \frac{4}{3}\pi r^3.$$

$\square$

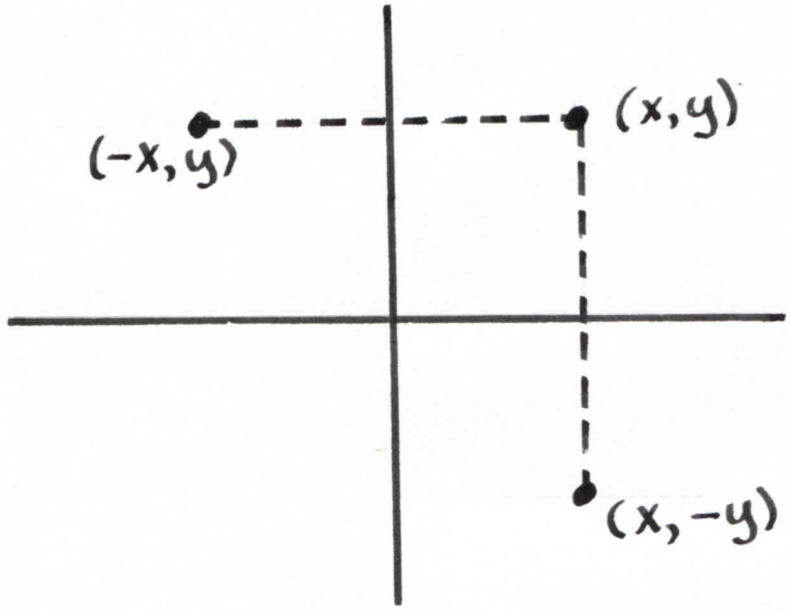
DRAWING APP1.1



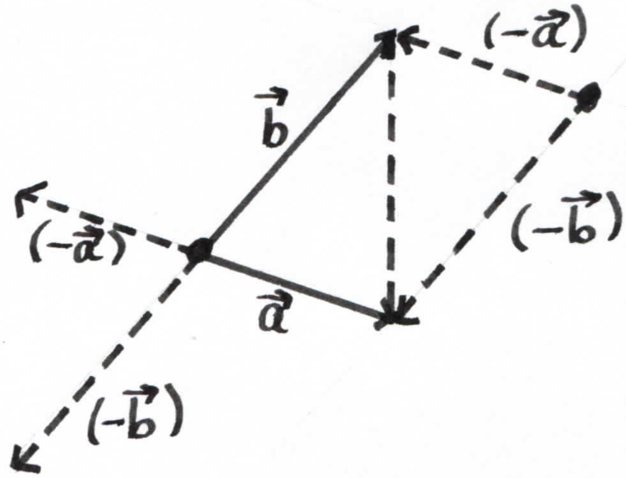
DRAWING APP1.2



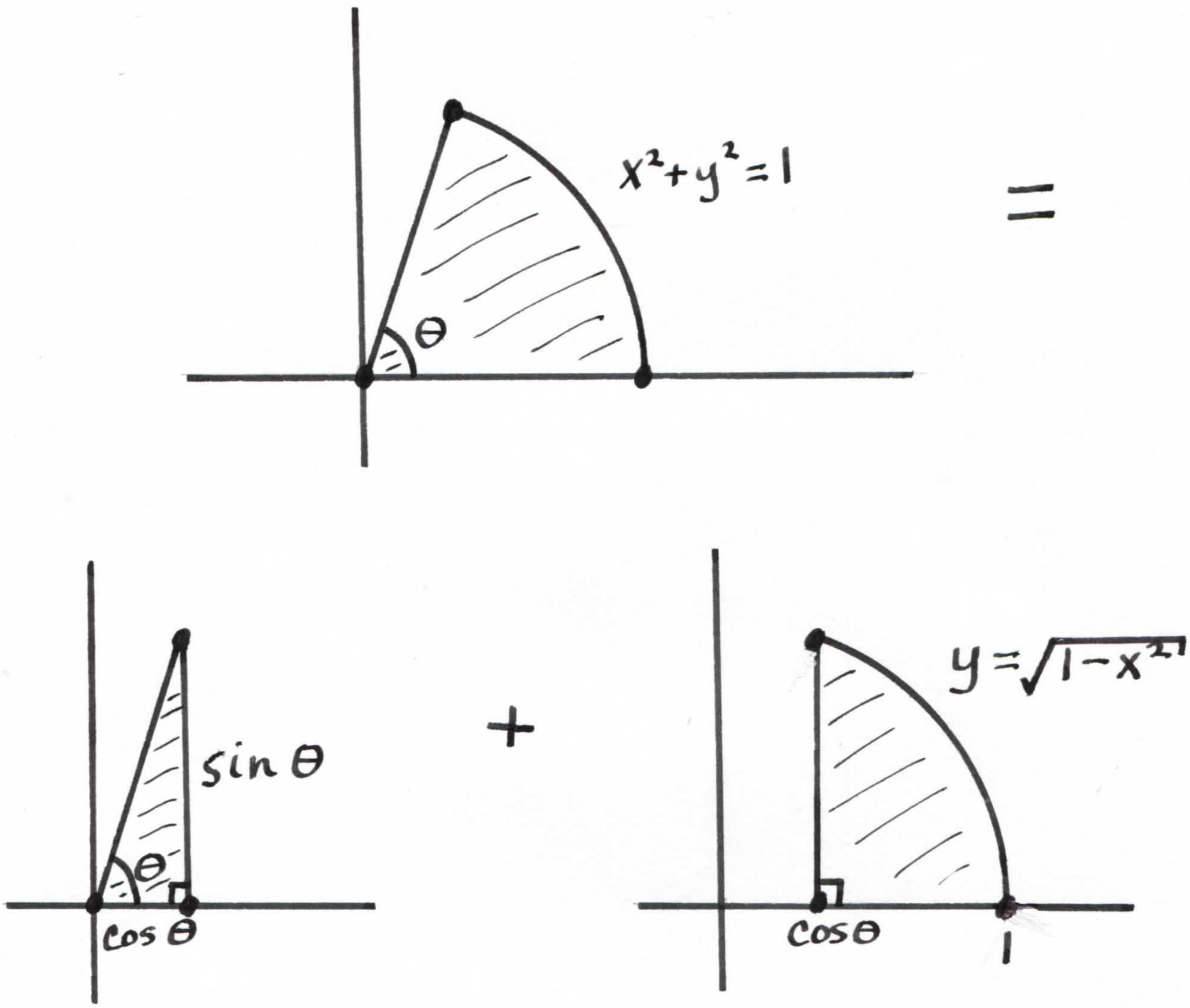
DRAWING APP1.3



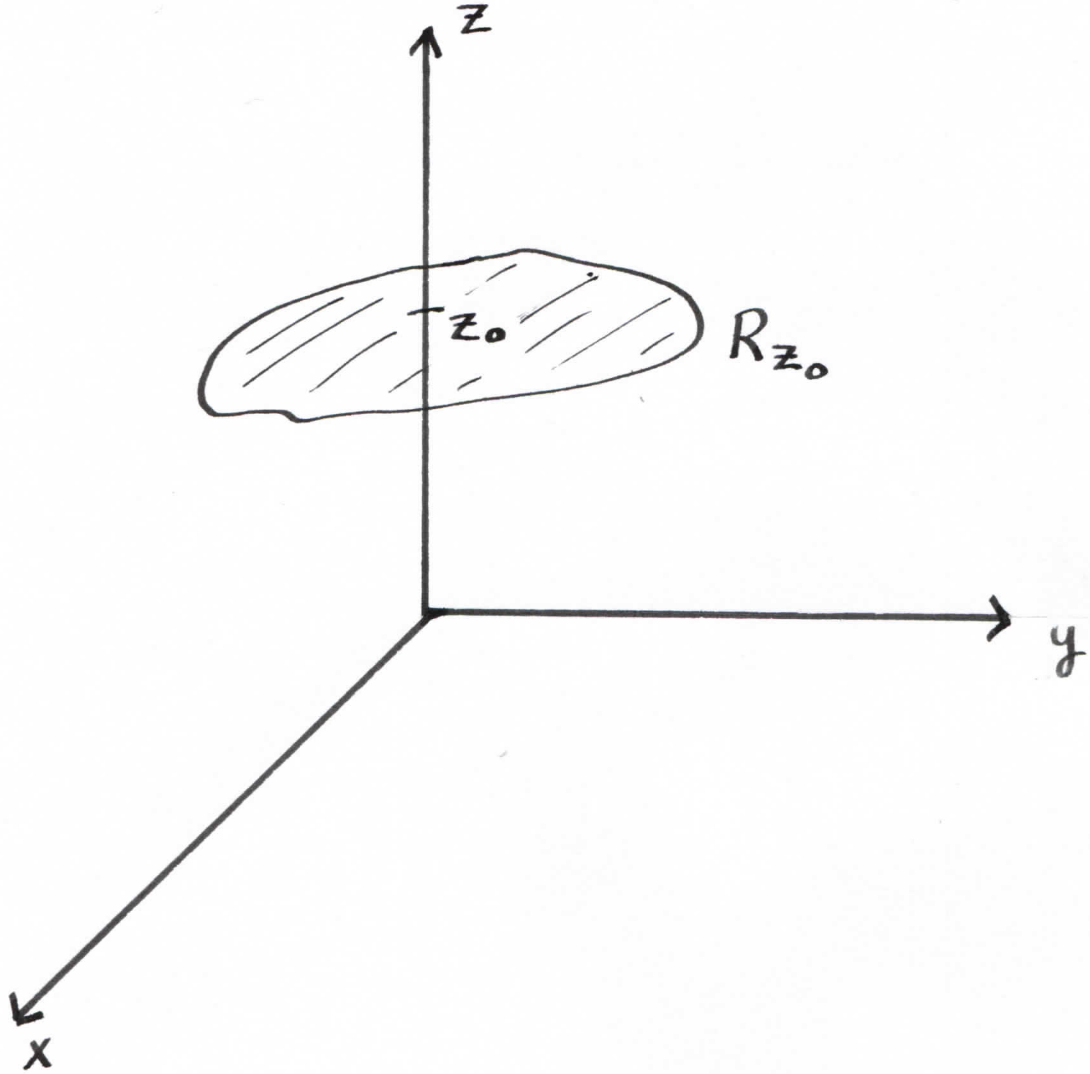
DRAWING APP1.4



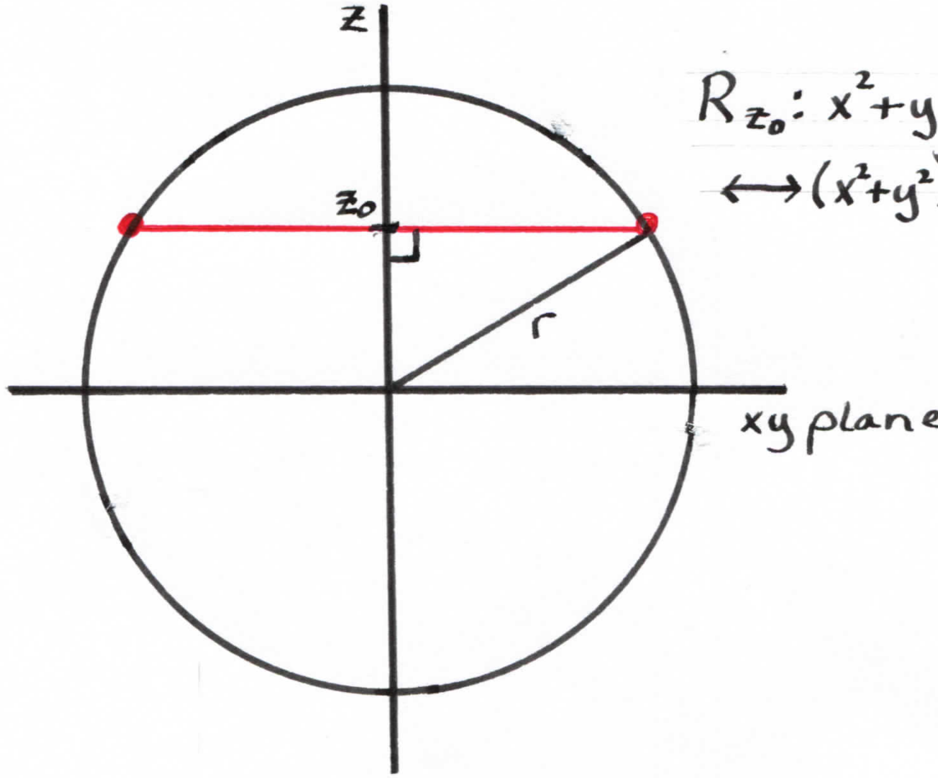
DRAWINGS APP1.5



DRAWING APP1.6

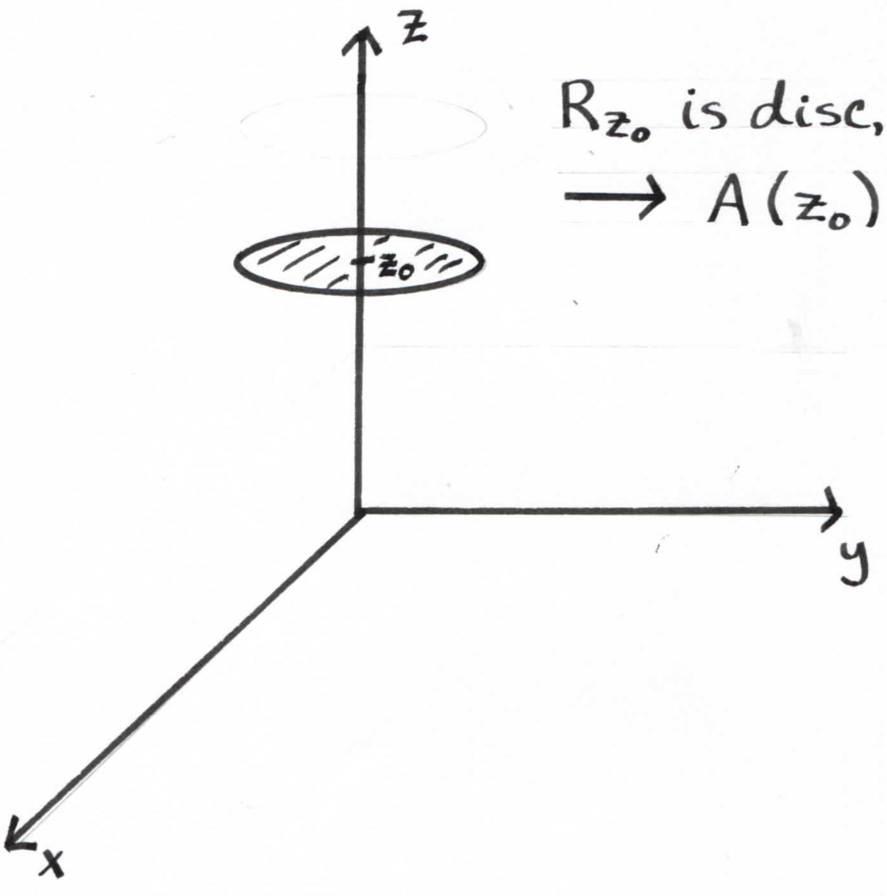


DRAWING APP1.7



$R_{z_0}: x^2+y^2+z_0^2 \leq r^2$

$\longleftrightarrow (x^2+y^2) \leq (r^2-z_0^2)$



R_{z_0} is disc, of radius $\sqrt{(r^2-z_0^2)}$

$\longrightarrow A(z_0) = \pi(r^2-z_0^2)$

APPENDIX TWO: Complex Integration and Arclength.

This section will consider curves in the complex plane. We will produce the additivity of length mentioned in Remarks 2.5. This section will also be used for the beginning of the proof of Theorem 8.3, regarding translation. We will also use Definitions APP2.1 in APP3.3 and APP3.5, in Appendix Three.

Definitions APP2.1. A **curve** in $\mathbf{R}^2$ or $\mathbf{C}$ is the image

$$L \equiv \{f(t) \,|\, a \leq t \leq b\}$$

of a continuous function $f : [a, b] \to \mathbf{C}, a < b$.

As with a line, there is an implied sense of motion, if we think of t as being time: a slug begins at $f(a)$ and travels, as t increases, along L until it arrives at $f(b)$. The curve L consists only of the trail left by the slug, without any arrows or other indications of a trip occurring.

See DRAWING APP2.1 at the end of this appendix.

Integration, at least when f' is piecewise continuous and $f(t_1) = f(t_2)$ for only finitely many $t_1 \neq t_2$, extends the definition of length in the Postulates in the Introduction to the **length** of L with what turn out to be almost Riemann sums

$$\text{length of } L = \int_a^b |f'(t)|\, dt = \lim_{n\to\infty} \sum_{k=1}^{n} |f(t_k) - f(t_{k-1})| \quad (t_j \equiv (a + \frac{j}{n}(b-a)), j = 0, 1, 2 \ldots, n).$$

Note that, thinking of $f(t)$ as a point in $\mathbf{R}^2$, for $a \leq t \leq b$, as in 1.16,

$$\overrightarrow{f(t_0)f(t_1)}, \overrightarrow{f(t_1)f(t_2)}, \overrightarrow{f(t_2)f(t_3)}, \ldots, \overrightarrow{f(t_{n-1})f(t_n)}$$

is a sequence of line segments, called a *polygonal path*, approximating L, so that

$$\sum_{k=1}^{n} |f(t_k) - f(t_{k-1})| = \sum_{k=1}^{n} \|\overrightarrow{f(t_{k-1})f(t_k)}\|,$$

the length of said polygonal path, approximates the length of L.

See DRAWING APP2.2 at the end of this appendix, where we have drawn the polygonal path in red.

As with area (see Proposition APP1.1), length is additive because integration is additive.

Proposition APP2.2. If L_1 and L_2 are curves that share an endpoint and otherwise do not overlap, then the length of the union of L_1 and L_2 equals the sum of the lengths of L_1 and L_2.

Again as with area, let x_0, y_0 be arbitrary real numbers and define the **translation** of L by $< x_0, y_0 >$ to be

$$(L+ < x_0, y_0 >) \equiv \{f(t)+ < x_0, y_0 > \,|\, a \leq t \leq b\}.$$

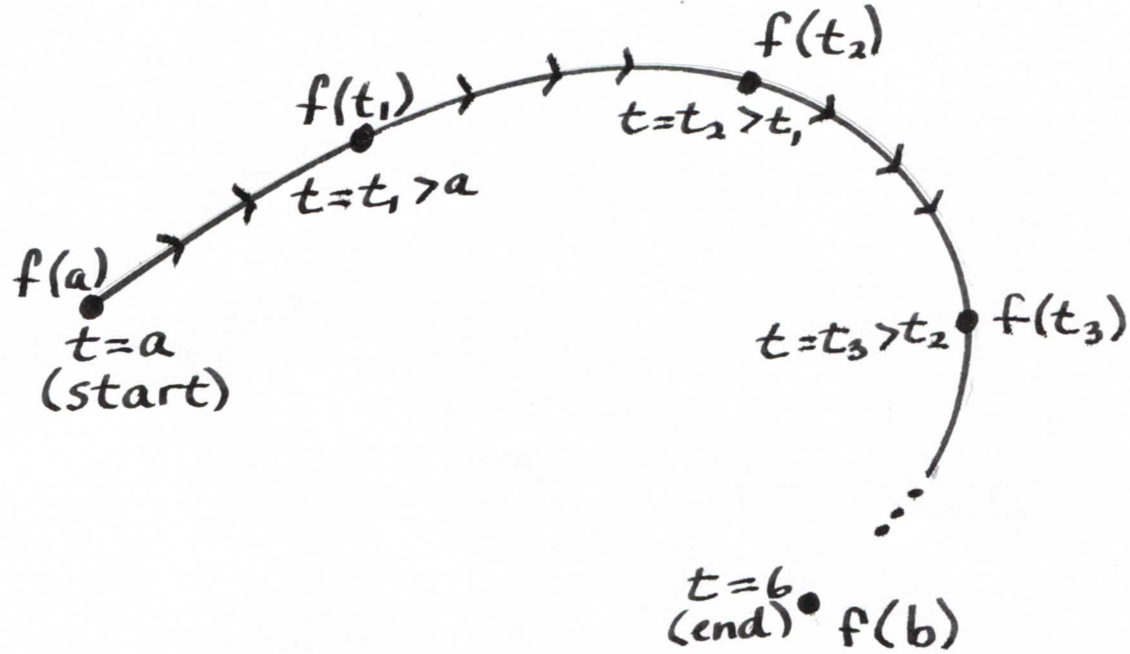
With reflection as in Definition APP1.2, the same arguments as with area give invariance of length under translation and reflection.

Proposition APP2.3. (a) The length of $(L+ < x_0, y_0 >)$ equals the length of L.

(b) The length of L reflected thru either axis equals the length of L.

DRAWING APP2.1

Slug motion



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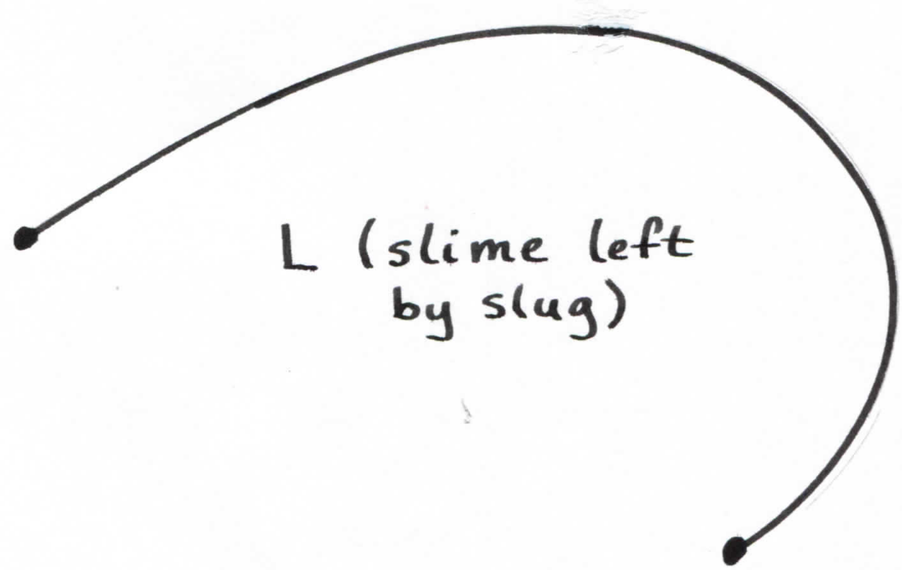
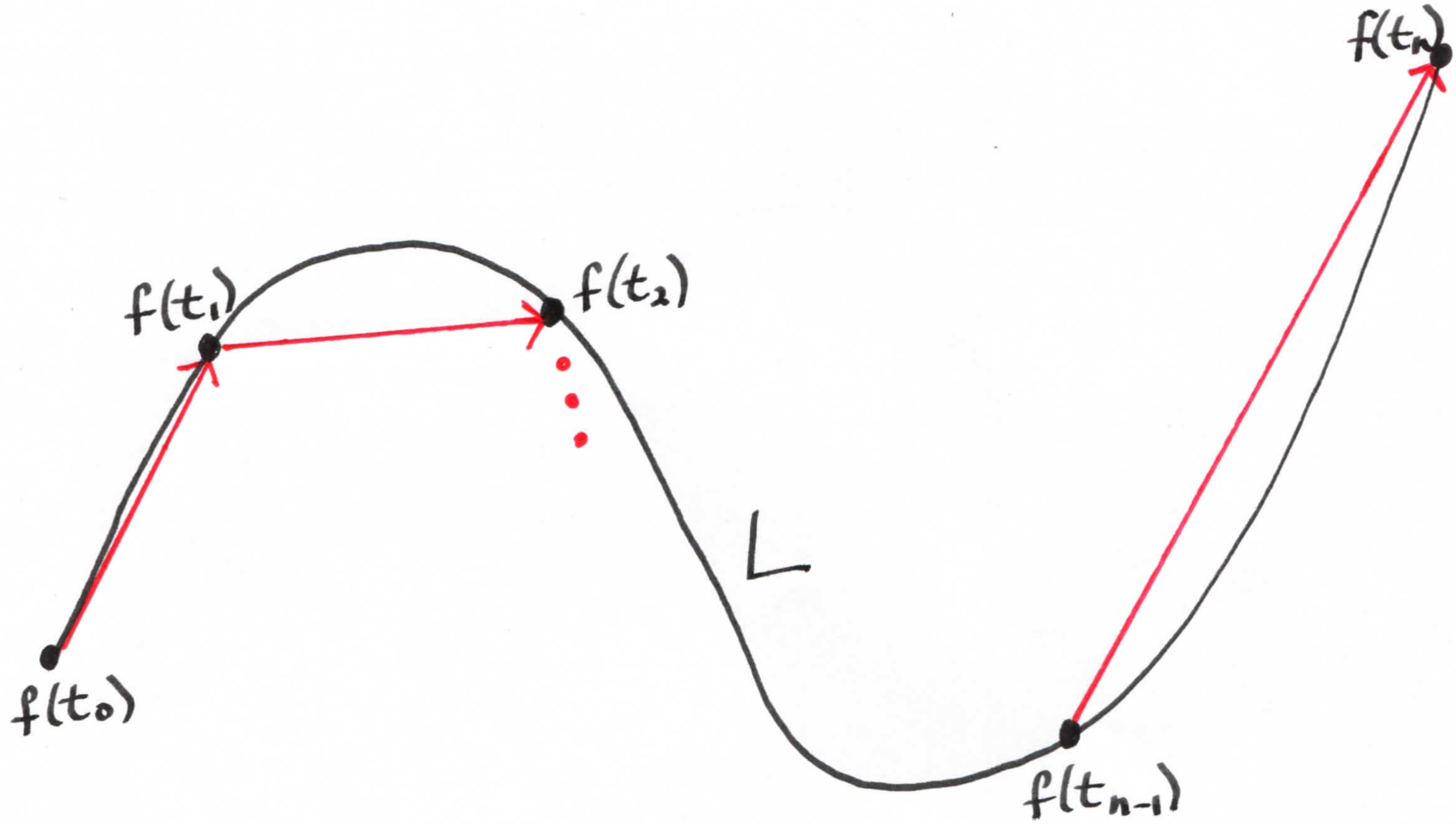
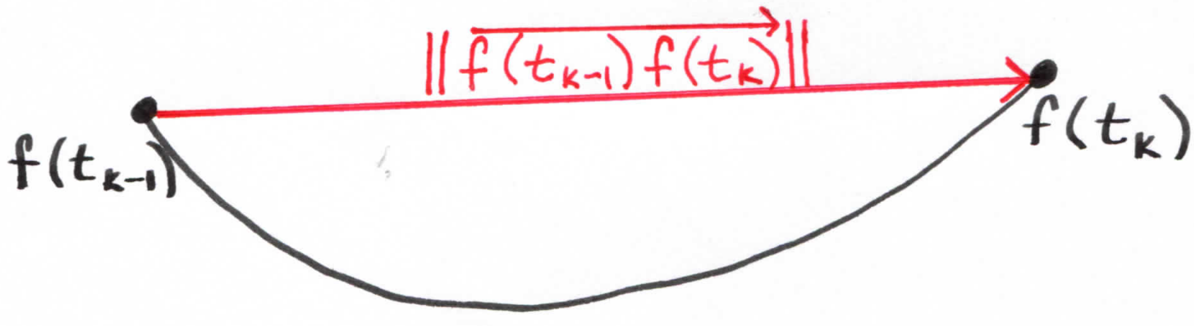


DRAWING APP2.2





APPENDIX THREE: Complex Exponentials, Arclength, and Angles.

Definition APP3.1. Define a function $\exp : \mathbf{C} \to \mathbf{C}$ by $\exp(z) \equiv \sum_{k=0}^{\infty} \frac{z^k}{k!}$.

The following is Theorem 1.12, with an additional factoid (3) added.

Theorem APP3.2. For any complex z, w,

(1)
$$\exp(z+w) = \exp(z)\exp(w),$$

(2)
$$\exp(\overline{z}) = \overline{\exp(z)},$$

(3) $\frac{d}{dz}\exp(z) = \exp(z)$.

Proof: (1)

$$\exp(z+w) = \sum_{k=0}^{\infty} \frac{(z+w)^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} z^j w^{k-j} = \sum_{j=0}^{\infty}\sum_{k=j}^{\infty} \frac{z^j w^{k-j}}{j!(k-j)!}$$
$$= \sum_{j=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{z^j w^{\ell}}{j!(\ell)!} = \left(\sum_{j=0}^{\infty} \frac{z^j}{j!}\right)\left(\sum_{\ell=0}^{\infty} \frac{w^\ell}{\ell!}\right) = (\exp(z))\,(\exp(w))\,.$$

(2) follows from Conjugation Lemma 1.10(2) and (3).

(3) $\frac{d}{dz}\exp(z) = \sum_{k=1}^{\infty} \frac{1}{k!}\left(kz^{k-1}\right) = \sum_{k=1}^{\infty} \frac{z^{k-1}}{(k-1)!} = \sum_{j=0}^{\infty} \frac{z^j}{j!} \equiv \exp(z)$.

Here is an explicit definition or formula for π, from Definition 2.7.

Proposition APP3.3. $\pi = \int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}}$.

Proof: The upper half of the unit circle $\{(x,y)\,|\,y \geq 0 \text{ and } x^2+y^2=1\}$ equals
$$\{f(t)\,|\,-1 \leq t \leq 1\}, \quad \text{where} \quad f(t) \equiv (t, \sqrt{1-t^2}),$$
thus, by Definitions APP2.1,
$$\pi = \int_{-1}^{1} |f'(t)|\,dt = \int_{-1}^{1} |(1, \frac{-t}{\sqrt{1-t^2}})|\,dt = \int_{-1}^{1} \sqrt{1+\left(\frac{-t}{\sqrt{1-t^2}}\right)^2}\,dt = \int_{-1}^{1} \sqrt{1+\frac{t^2}{(1-t^2)}}\,dt$$
$$= \int_{-1}^{1} \sqrt{\frac{1}{(1-t^2)}}\,dt = \int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}}.$$
□

Part (b) of the following is a very special case of very general results about uniqueness of solutions of differential equations, whose proof we include so that we may say first-year calculus is sufficient prerequisite for the appendices.

We use here the terminology, for $f : [0,\pi] \to [-1,1]$,
$$f'(0^+) \equiv \lim_{\theta \to 0^+} f'(\theta).$$

Lemma APP3.4. (a) $\left[\mathrm{Re}(e^{i\theta})\right]' = -\left[\mathrm{Im}(e^{i\theta})\right]$, for all real θ.

(b) Suppose $f : [0,\pi] \to [-1,1]$ is continuous on $[0,\pi]$ and twice differentiable on $(0,\pi)$ with
$$f''(\theta) = -f(\theta) \;\; (0 < \theta < \pi), \;\; f(0)=1, f'(0^+) = 0. \quad (*)$$

Then

$$f(\theta) = \mathrm{Re}(e^{i\theta}),$$

for $0 \le \theta \le \pi$.

Proof: (a) For any real θ,

$$e^{i\theta} \equiv \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} = \sum_{m=0}^{\infty} \frac{(i\theta)^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{(i\theta)^{2m+1}}{(2m+1)!} = \left[\sum_{m=0}^{\infty}(-1)^m \frac{\theta^{2m}}{(2m)!}\right] + i\left[\sum_{m=0}^{\infty}(-1)^m \frac{\theta^{2m+1}}{(2m+1)!}\right],$$

thus

$$\left[\mathrm{Re}(e^{i\theta})\right]' = \left[\sum_{m=1}^{\infty}(-1)^m (2m)\frac{\theta^{2m-1}}{(2m)!}\right] = \left[\sum_{m=1}^{\infty}(-1)^m \frac{\theta^{2m-1}}{(2m-1)!}\right] = \left[\sum_{n=0}^{\infty}(-1)^{n+1} \frac{\theta^{2(n+1)-1}}{(2(n+1)-1)!}\right]$$

$$= -\left[\sum_{n=0}^{\infty}(-1)^n \frac{\theta^{2n+1}}{(2n+1)!}\right] = -\left[\mathrm{Im}(e^{i\theta})\right].$$

(b) A calculation very similar to the proof of (a) shows that $\left[\mathrm{Im}(e^{i\theta})\right]' = \left[\mathrm{Re}(e^{i\theta})\right]$, for all real θ. Combined with (a), this shows that

$$f(\theta) = \mathrm{Re}(e^{i\theta})$$

is a solution of (*).

To show that this is the *only* solution, suppose g is another solution; that is,

$$g''(\theta) = -g(\theta) \ \ (0 < \theta < \pi), \ \ g(0) = 1, g'(0^+) = 0.$$

Define $h \equiv (f - g)$. Then h satisfies

$$h''(\theta) = -h(\theta) \ \ (0 < \theta < \pi), \ \ h(0) = 0, h'(0^+) = 0.$$

Showing that g must equal f (equivalent to the uniqueness of f) is equivalent to showing that h must be the zero function; that is, $h(\theta) = 0$ for $0 \le \theta \le \pi$.

Using the fundamental theorem of calculus twice, and performing a change of variables (see DRAWING APP3.1 at the end of this appendix), for $0 < \theta < \pi$:

$$h'(\theta) = h'(\theta) - h'(0^+) = \int_0^{\theta} h''(t)\,dt \quad \rightarrow$$

$$h(\theta) = h(\theta) - h(0) = \int_0^{\theta} h'(s)\,ds = \int_0^{\theta}\int_0^{s} h''(t)\,dt\,ds = \int_0^{\theta}\int_t^{\theta} h''(t)\,ds\,dt = \int_0^{\theta} (\theta - t)h''(t)\,dt = -\int_0^{\theta} (\theta - t)h(t)\,dt.$$

Let "max" be short for "maxiumum." The integral expression we just derived for $h(\theta)$ implies that

$$\left[\max_{0 \le \theta \le \frac{1}{2}} |h(\theta)|\right] \le \int_0^{\frac{1}{2}} \frac{1}{2}\left[\max_{0 \le \theta \le \frac{1}{2}} |h(\theta)|\right] dt = \frac{1}{4}\left[\max_{0 \le \theta \le \frac{1}{2}} |h(\theta)|\right];$$

this implies that $\left[\max_{0 \le \theta \le \frac{1}{2}} |h(\theta)|\right] = 0$, so that

$$h(\theta) = 0 \ \text{ for } \ 0 \le \theta \le \frac{1}{2}.$$

This also implies that $h'(\theta) = 0$ for $0 \le \theta \le \frac{1}{2}$.

A very similar argument (see DRAWING APP3.2 at the end of this appendix)) shows now that, for $\frac{1}{2} \le \theta \le 1$,

$$h(\theta) = \int_{\frac{1}{2}}^{\theta}\int_0^{s} h''(t)\,dt\,ds = -\int_{\frac{1}{2}}^{\theta} (\theta - t)h(t)\,dt - \int_0^{\frac{1}{2}} (\theta - \frac{1}{2})h(t)\,dt = -\int_{\frac{1}{2}}^{\theta} (\theta - t)h(t)\,dt,$$

and we may very similarly argue with $\left[\max_{\frac{1}{2}\leq\theta\leq 1}|h(\theta)|\right]$ to show that

$$h(\theta) = 0 \text{ for } \frac{1}{2} \leq \theta \leq 1.$$

Continue advancing θ in steps of $\frac{1}{2}$ to eventually reach

$$h(\theta) = 0 \text{ for } 0 \leq \theta \leq \pi,$$

as desired.

$\square$

APP3.5, Proof of Lemma 2.8. First we'll prove the statement about the length of $C_{\theta,R}$ in Lemma 2.8, for $0 \leq \theta \leq 2\pi, R > 0$.

By Definitions APP2.1, Corollary 1.14, and Theorem APP3.2(3), the length of $C_{\theta,R}$ equals

$$\int_0^\theta |\frac{d}{dt}Re^{it}|\,dt = \int_0^\theta |iRe^{it}|\,dt = \int_0^\theta R\,dt = R\theta.$$

For the other statement in Lemma 2.8, that every complex number z has a polar form, we will first make two reductions.

Without loss of generality, we may assume that

(1) $|z| = 1$: if we show Lemma 2.8 for this special case, then, for any nonzero z, since $\frac{z}{|z|}$ has absolute value one, there's real θ such that $\frac{z}{|z|} = e^{i\theta}$; we would then have $z = |z|e^{i\theta}$;

and

(2) $\mathrm{Im}(z) \geq 0$: if we show Lemma 2.8 for this special case, then, for $\mathrm{Im}(z) < 0$, since $\mathrm{Im}(\overline{z}) > 0$, there's real θ so that $\overline{z} = e^{i\theta}$, thus

$$z = \overline{e^{i\theta}} = e^{\overline{i\theta}} = e^{i(-\theta)}$$

by Theorem APP3.2(2).

Thus we are trying to get a polar form for z in

$$\Omega \equiv \{z \,|\, |z| = 1, \mathrm{Im}(z) \geq 0\} = \{(x + i\sqrt{1-x^2} \,|\, -1 \leq x \leq 1\},$$

the upper half of the unit circle, as drawn in red in DRAWING APP3.3 at the end of this appendix.

Of importance is the relationship between points z in Ω and the length of the arc in Ω from z to 1; see the definition of the function A below and DRAWING APP3.3 at the end of this appendix. We will also find it convenient to project Ω onto $[-1, 1]$, so that we may deal with the familiar (in first-year calculus) setting of functions from the real line to itself; see the definition of the function P below and DRAWING APP3.4 at the end of this appendix.

Define $A : \Omega \to [0, \pi]$: for z in Ω, $A(z)$ is the length of the arc of the unit circle clockwise from z to 1, as drawn in DRAWING APP.3.3 at the end of this appendix, with Ω drawn in red.

As in the proof of Proposition APP3.3, if $z = x + i\sqrt{1-x^2}$, for $-1 \leq x \leq 1$, then

$$A(z) = A(x + i\sqrt{1-x^2}) = \int_x^1 \frac{dt}{\sqrt{1-t^2}}.$$

Define also $P : \Omega \to [-1, 1]$ by

$$P(x + i\sqrt{1-x^2}) \equiv x \quad (-1 \leq x \leq 1).$$

See DRAWING APP3.4 at the end of this appendix, with Ω drawn in red.

Finally, define $R : [-1, 1] \to [0, \pi]$ by

$$R \equiv A \circ P^{-1};$$

that is,

$$R(x) \equiv A\left(P^{-1}(x)\right) = A\left(x + i\sqrt{1-x^2}\right) = \int_x^1 \frac{dt}{\sqrt{1-t^2}} \quad (-1 \le x \le 1).$$

Our goal will be to show that, for $0 \le \theta \le \pi$, $A^{-1}(\theta) = e^{i\theta}$, so that, for z in Ω, $z = e^{i\theta}$, with θ equal to the arclength $A(z)$ in DRAWING APP3.3 at the end of this appendix.

By the inverse function theorem, $R^{-1} : [0, \pi] \to [-1, 1]$ is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$, with, for $0 < \theta < \pi$,

$$\left[R^{-1}\right]'(\theta) = \frac{1}{\left[R'\left(R^{-1}(\theta)\right)\right]} = \frac{1}{\left[-\frac{1}{\sqrt{1-(R^{-1}(\theta))^2}}\right]} = -\sqrt{1-(R^{-1}(\theta))^2}, \quad (*)$$

where the second equality is from the fundamental theorem of calculus.

Thus, still for $0 < \theta < \pi$,

$$\left[R^{-1}\right]''(\theta) = \left[-\sqrt{1-(R^{-1}(\theta))^2}\right]' = \frac{\left[R^{-1}\right]'(\theta)\left[R^{-1}\right](\theta)}{\sqrt{1-(R^{-1}(\theta))^2}} = -\left[R^{-1}\right](\theta). \qquad (**)$$

Notice also that $(R^{-1})(0) = 1$, since $R(1) = 0$, while by (*) $\left[R^{-1}\right]'(0^+) = -\sqrt{1-(R^{-1}(0))^2} = 0$, thus by Lemma APP3.4(b) and (**), for $0 \le \theta \le \pi$,

$$R^{-1}(\theta) = \mathrm{Re}(e^{i\theta}). \quad (***)$$

Since $R = A \circ P^{-1}$, we have $A^{-1} = P^{-1} \circ R^{-1}$; that is, for $0 \le \theta \le \pi$, by (*), (***), Lemma APP3.4(a), and the continuity of $A^{-1}(\theta)$ and $e^{i\theta}$ (to extend the equality of $A^{-1}(\theta)$ and $e^{i\theta}$ from $(0, \pi)$ to $[0, \pi]$),

$$\begin{aligned} A^{-1}(\theta) = P^{-1}\left(R^{-1}(\theta)\right) &= \left(R^{-1}(\theta)\right) + i\sqrt{1-(R^{-1}(\theta))^2} = \left(R^{-1}(\theta)\right) + i\left(-\left[R^{-1}\right]'(\theta)\right) \\ &= \mathrm{Re}(e^{i\theta}) + i\left(-\left[\mathrm{Re}(e^{i\theta})\right]'\right) = \mathrm{Re}(e^{i\theta}) + i\mathrm{Im}(e^{i\theta}) = e^{i\theta}. \end{aligned}$$

Given z in Ω, let $\theta \equiv A(z)$. Then $z = A^{-1}(\theta)$, which we have just shown to be equal to $e^{i\theta}$, the desired polar form; see DRAWING APP3.5 at the end of this appendix, with Ω drawn in red. □

A consequence of the Chapter VI approach to sines and cosines (Definition 6.1) is a quite simple derivation of their derivatives.

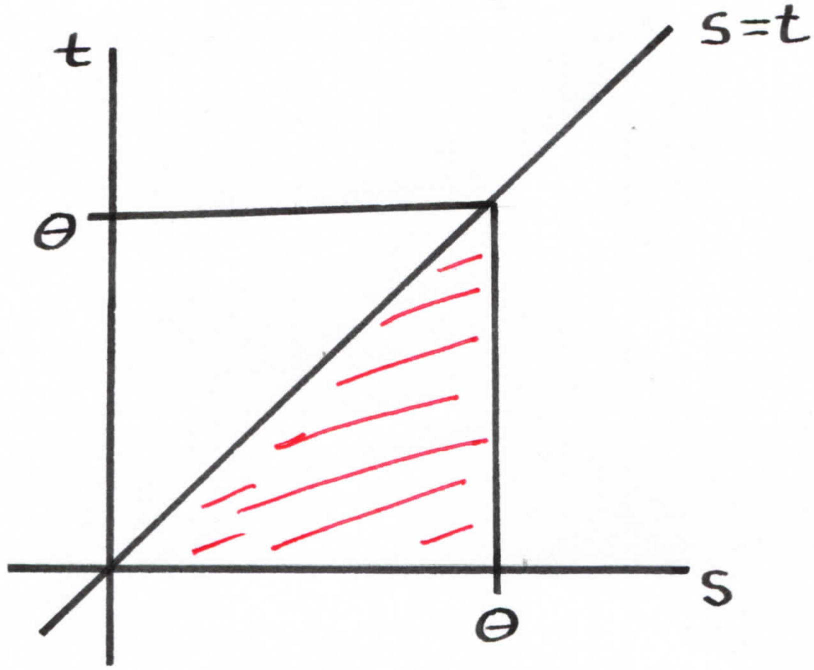
Proposition APP3.6. $\frac{d}{d\theta}\cos\theta = -\sin\theta$, $\frac{d}{d\theta}\sin\theta = \cos\theta$.

Proof: By Theorem APP3.2(3),

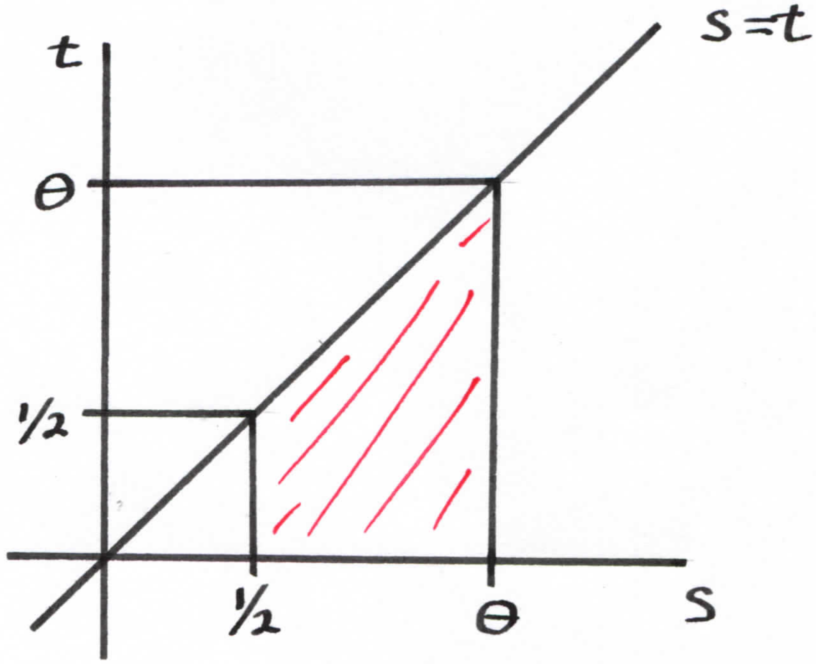
$$\left(\frac{d}{d\theta}\cos\theta\right) + i\left(\frac{d}{d\theta}\sin\theta\right) = \frac{d}{d\theta}(\cos\theta + i\sin\theta) = \frac{d}{d\theta}e^{i\theta} = ie^{i\theta} = i(\cos\theta + i\sin\theta) = (-\sin\theta) + i(\cos\theta).$$

□

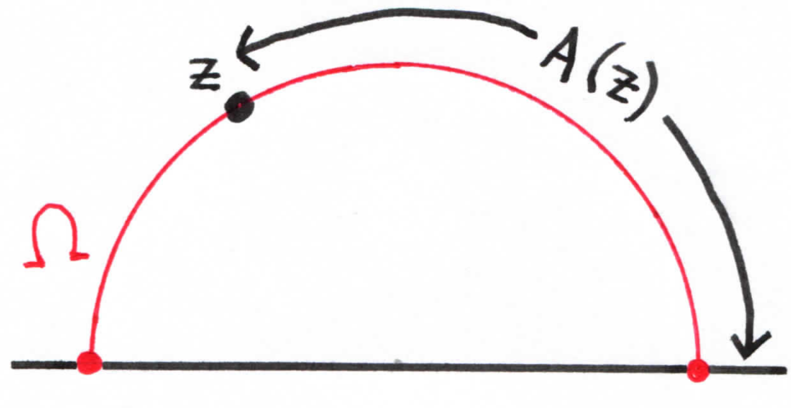
DRAWING APP3.1



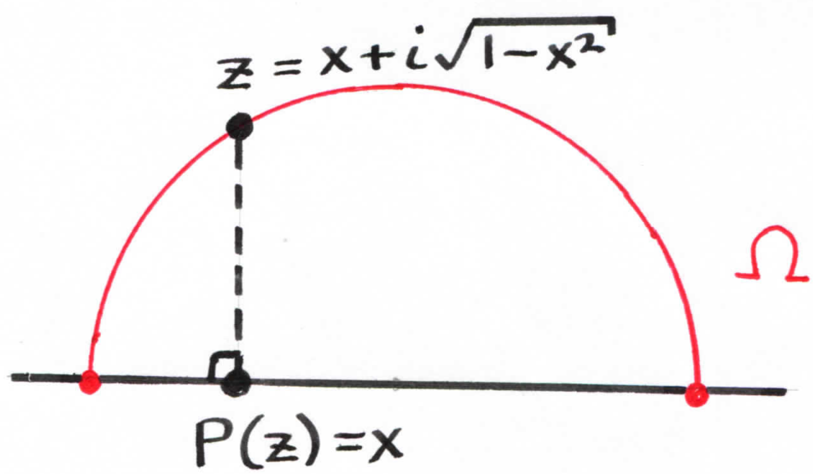
DRAWING APP3.2



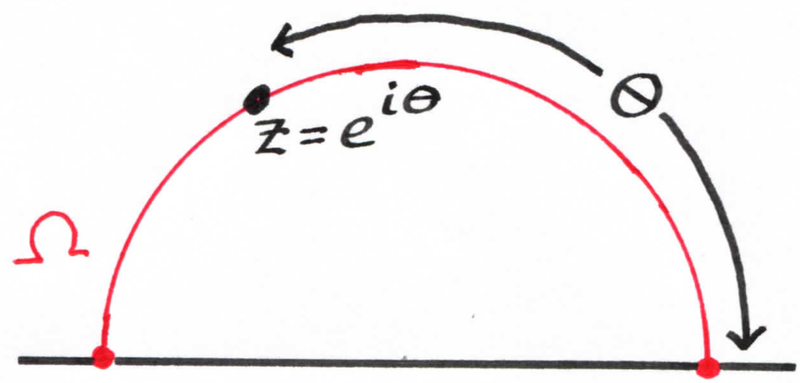
DRAWING APP3.3



DRAWING APP3.4



DRAWING APP3.5



APPENDIX FOUR: Another Approach to Angle, Cosine, and Sine.

This section is independent of the rest of the book; it's not needed for anything. It presents another way to introduce sines and cosines without complex numbers, using integration to begin by defining $\cos^{-1}$.

We don't recommend this approach for starting geometry; note, for example, how much more awkward our definition of angle in Definition APP4.2 is, compared to Definitions 2.10. This section is primarily of interest in its analogy to the calculus definition of exponentials by defining a certain logarithm first.

Since angle will be defined as a certain curve length, we will consider, analogously to Appendices One and Two, one-dimensional curves in $\mathbf{R}^2$

$$L \equiv \{(x, y) \,|\, a \leq x \leq b, y = g(x)\},$$

for $a < b, g$ piecewise continuously differentiable. See DRAWING APP4.1 at the end of this appendix.

Integration extends the definition of length in the Postulates of the Introduction to the length of L with Riemann sums

$$\text{length of } L = \int_a^b \sqrt{1 + (g'(x))^2}\, dx$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{\left(\frac{(b-a)}{n}\right)^2 + \left(g\left(a + k\frac{(b-a)}{n}\right) - g\left(a + (k-1)\frac{(b-a)}{n}\right)\right)^2}.$$

See DRAWING APP4.2 at the end of this appendix.

Note that this definition of length is Definitions APP2.1, with $f(t) \equiv (t, g(t))$, for $a \leq t \leq b$.

We will apply the definition of length to the upper half of the unit circle

$$y = g(x) \equiv \sqrt{1 - x^2};$$

we have

$$\frac{dy}{dx} = g'(x) = \frac{-x}{\sqrt{1-x^2}}, \quad \text{hence} \quad \left((1 + (g'(x))^2\right) = \frac{(1-x^2) + x^2}{(1-x^2)} = \frac{1}{(1-x^2)},$$

thus, for $-1 \leq x_1 \leq x_2 \leq 1$, the length of the arc of the upper half (or the lower half; see Proposition APP2.3) of the unit circle between $x = x_1$ and $x = x_2$ is

$$\int_{x_1}^{x_2} \frac{dt}{\sqrt{1-t^2}}.$$

See DRAWING APP4.3 at the end of this appendix.

Definition APP4.1. $\pi \equiv \int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}}$.

Definition APP4.2. Let $\vec{a}$ and $\vec{b}$ be vectors. The **counterclockwise (clockwise) angle** from $\vec{a}$ to $\vec{b}$ is the same as the **counterclockwise (clockwise) angle** from

$$< x_1, y_1 > \equiv \vec{x}_1 \equiv \frac{\vec{a}}{\|\vec{a}\|} \quad \text{to} \quad < x_2, y_2 > \equiv \vec{x}_2 \equiv \frac{\vec{b}}{\|\vec{b}\|},$$

which we define as follows.

First, for any unit vectors $\vec{x}_1, \vec{x}_2$, the counterclockwise angle from $\vec{x}_1$ to $\vec{x}_2$ equals the clockwise angle from $\vec{x}_2$ to $\vec{x}_1$ equals 2π minus the counterclockwise angle from $\vec{x}_2$ to $\vec{x}_1$.

This means we may restrict ourselves to defining the counterclockwise angle from $\vec{x}_2$ to $\vec{x}_1$, with $x_1 \leq x_2$.

Here are the four cases.

(1) If y_1 and y_2 are nonnegative and $x_1 \le x_2$, then the angle is $\int_{x_1}^{x_2} \frac{dt}{\sqrt{1-t^2}}$.
See DRAWING APP4.3 at the end of this appendix.

(2) If y_1 and y_2 are both nonpositive, then the angle is $\left[2\pi - \int_{x_1}^{x_2} \frac{dt}{\sqrt{1-t^2}}\right]$.
See DRAWING APP4.4 at the end of this appendix.

(3) If y_1 is nonpositive and y_2 is nonnegative, then the angle is $\left[\int_{-1}^{x_2} \frac{dt}{\sqrt{1-t^2}} + \int_{-1}^{x_1} \frac{dt}{\sqrt{1-t^2}}\right]$.
See DRAWING APP4.5 at the end of this appendix.

(4) If y_1 is nonnegative and y_2 is nonpositive, then the angle is $\left[\int_{x_2}^{1} \frac{dt}{\sqrt{1-t^2}} + \int_{x_1}^{1} \frac{dt}{\sqrt{1-t^2}}\right]$.
See DRAWING APP4.6 at the end of this appendix.

Definitions APP4.3. Our approach to cosine and sine will be very analogous to the calculus definition of the natural exponential via its inverse, the natural logarithm $\ln x \equiv \int_1^x \frac{dt}{t}$.

For $-1 \le x \le 1$, define a function I by

$$I(x) \equiv \int_x^1 \frac{dt}{\sqrt{1-t^2}}.$$

Note that $I(x)$ is the arclength of the unit circle $x^2 + y^2 = 1$ counterclockwise from $(1,0)$ to $(x, \sqrt{1-x^2})$. See DRAWING APP4.7 at the end of this appendix.

The **cosine** function, restricted to $[0,\pi]$, is the inverse of I :

$$\cos|_{[0,\pi]} \equiv I^{-1};$$

that is,

$$\int_{\cos\theta}^1 \frac{dt}{\sqrt{1-t^2}} = \theta,$$

for $0 \le \theta \le \pi$.

The **sine** function, restricted to $[0,\pi]$, is defined by

$$\sin|_{[0,\pi]} \equiv \sqrt{1-(I^{-1})^2}.$$

See DRAWING APP4.8 at the end of this appendix.

Extend sine and cosine to the real line as follows.

For $0 \le \theta \le \pi$,

$$(\cos(\theta+\pi), \sin(\theta+\pi)) \equiv -(\cos\theta, \sin\theta).$$

For $0 \le \theta \le 2\pi$, k an integer,

$$(\cos(\theta+2\pi), \sin(\theta+2\pi)) \equiv (\cos\theta, \sin\theta).$$

This is the "unit circle definition" of sine and cosine, representing $(\cos\theta, \sin\theta)$ as the point on the unit circle $x^2 + y^2 = 1$ that has an arclength of θ counterclockwise from $(1,0)$.

See DRAWING APP4.9 at the end of this appendix.

Lemma APP4.4. Suppose f and g are continuous functions on $\mathbf{R}$, D is a discrete subset of $\mathbf{R}$, and, for all real x not in D, $f'(x)$ exists and equals $g(x)$. Then f is differentiable on $\mathbf{R}$, with $f'(x) = g(x)$ for all $x \in \mathbf{R}$.

Proof: Fix $x_0 \in D$. By continuity of g,

$$g(x_0) = \lim_{x\to x_0} g(x) = \lim_{x\to x_0} \left(\frac{\frac{d}{dx}\left(g(x) - g(x_0)\right)}{\frac{d}{dx}(x - x_0)} \right),$$

thus by L'Hôpital's rule,

$$\lim_{x \to x_0} \left(\frac{(g(x) - g(x_0))}{(x - x_0)} \right) \quad \text{exists and equals} \quad g(x_0);$$

that is, f is differentiable at x_0 and $f'(x_0) = g(x_0)$. □

Theorem APP4.5. For all real θ, $\frac{d}{d\theta} \cos\theta = -\sin\theta$ and $\frac{d}{d\theta} \sin\theta = \cos\theta$.

Proof: For $-1 < x < 1$, the fundamental theorem of calculus implies that

$$I'(x) = \frac{-1}{\sqrt{1 - x^2}}.$$

By the definition of cosine and the inverse function theorem, for $0 < \theta < \pi$,

$$\frac{d}{d\theta} \cos\theta = \frac{1}{I'(\cos\theta)} = -\sqrt{1 - (\cos\theta)^2} \equiv -\sin\theta.$$

For $0 < \theta < \pi$,

$$\frac{d}{d\theta} \sin\theta \equiv \frac{d}{d\theta} \left(\sqrt{1 - (\cos\theta)^2} \right) = \frac{(-2\cos\theta)(-\sin\theta)}{2\sqrt{1 - (\cos\theta)^2}} = \cos\theta.$$

Still for $0 < \theta < \pi$,

$$\frac{d}{d\theta} (\cos(\theta + \pi), \sin(\theta + \pi)) \equiv \frac{d}{d\theta} (-\cos\theta, -\sin\theta) = (\sin\theta, -\cos(\theta)) \equiv (-\sin(\theta + \pi), \cos(\theta + \pi)).$$

A similar argument shows that

$$\frac{d}{d\theta} (\cos\theta, \sin\theta) = (-\sin\theta, \cos\theta)$$

for any θ not in $D \equiv \{k\pi \,|\, k \text{ is an integer}\}$. Lemma APP4.4 extends this to all real θ. □

Corollary APP4.6.

$$\cos\theta = \sum_{k=0}^{\infty} (-1)^k \frac{\theta^{2k}}{(2k)!}, \quad \sin\theta = \sum_{k=0}^{\infty} (-1)^k \frac{\theta^{2k+1}}{(2k+1)!},$$

Proof: This follows from Theorem APP4.5, the facts that $\cos(0) = 1$ and $\sin(0) = 0$, and the Taylor series formula

$$g(x) = \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} x^k.$$

Note that the uniform boundedness of all derivatives on the real line guarantees convergence of the series to $g(x)$, for $g(x) \equiv \cos x$ or $\sin x$. □

Definition APP4.7. Define a function **exp**: $\mathbf{C} \to \mathbf{C}$ by $\exp(z) \equiv \sum_{k=0}^{\infty} \frac{z^k}{k!}$.

Two properties of exp are of interest to us, beginning with Chapter VI.

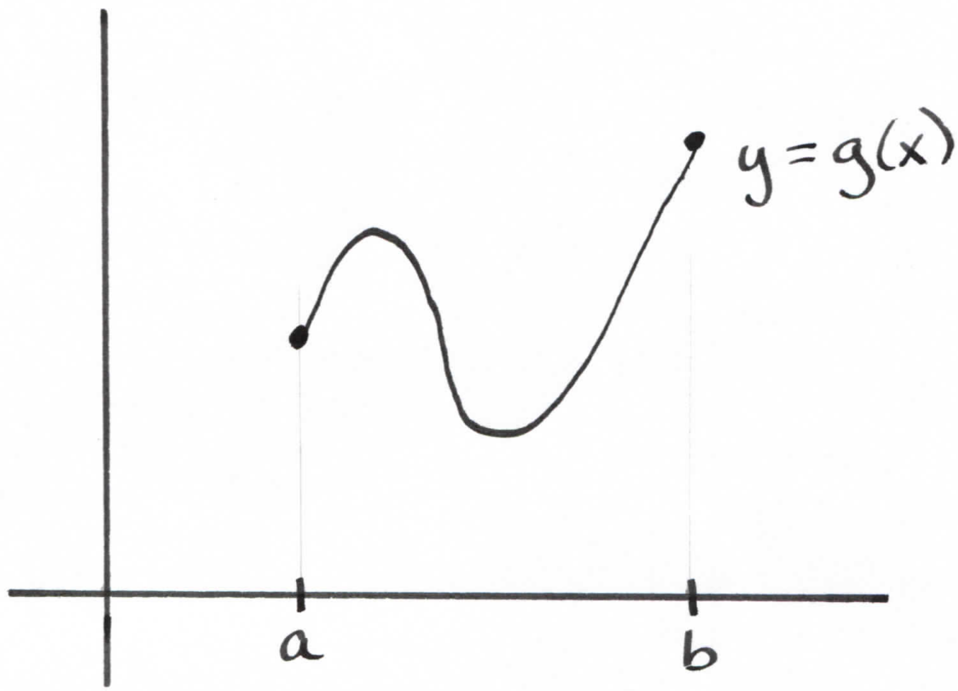
Theorem APP4.8 (a) For any complex z and w, $\exp(z + w) = (\exp(z))\,(\exp(w))$.
(b) (Euler's formula) For any real θ, $\exp(i\theta) = \cos\theta + i\sin\theta$.

Proof: (a) This was done in Theorem APP3.2.

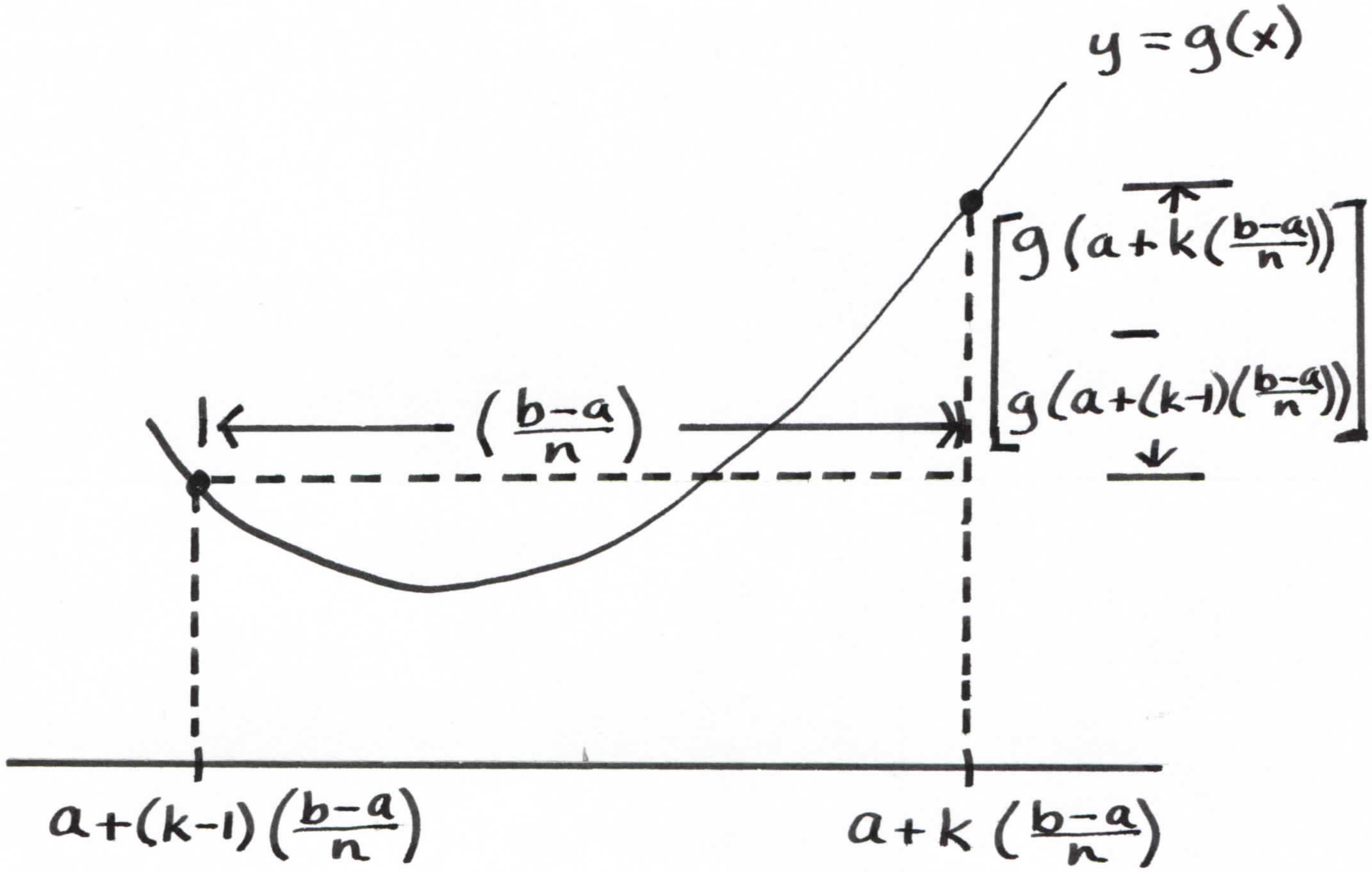
(b) This follows from the definition of exp, Corollary APP4.6, and the behaviour of powers of $(i\theta)$:

$$(i\theta)^{2k} = (-1)^k \theta^{2k}, \quad (i\theta)^{2k+1} = i\left((-1)^k \theta^{2k+1}\right), \quad \text{for any integer } k.$$

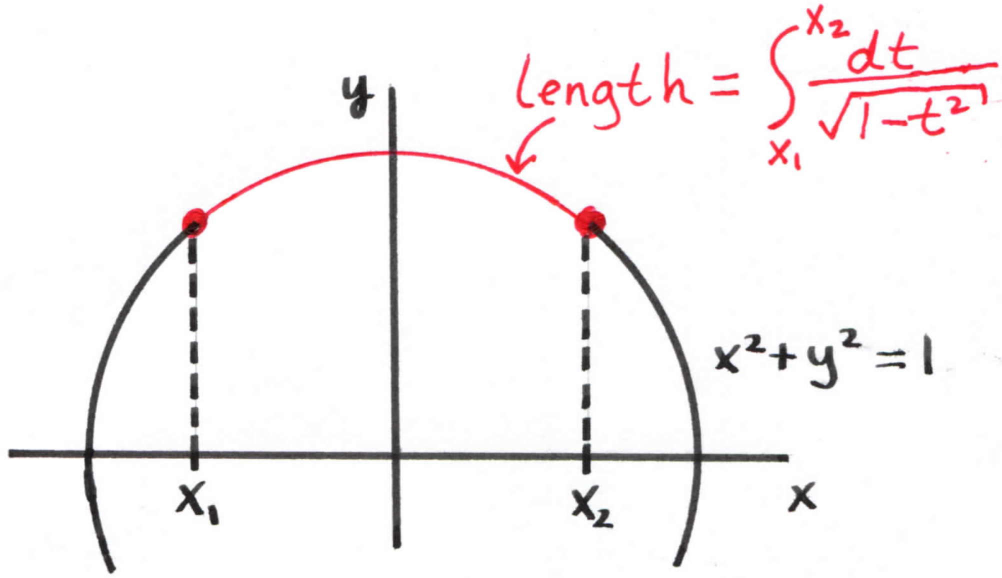
DRAWING APP4.1



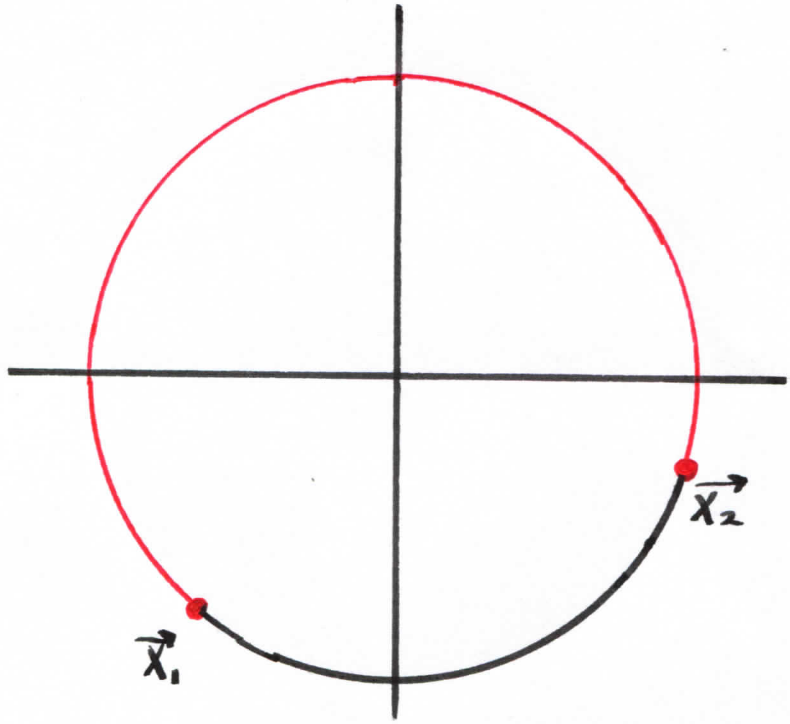
DRAWING APP4.2



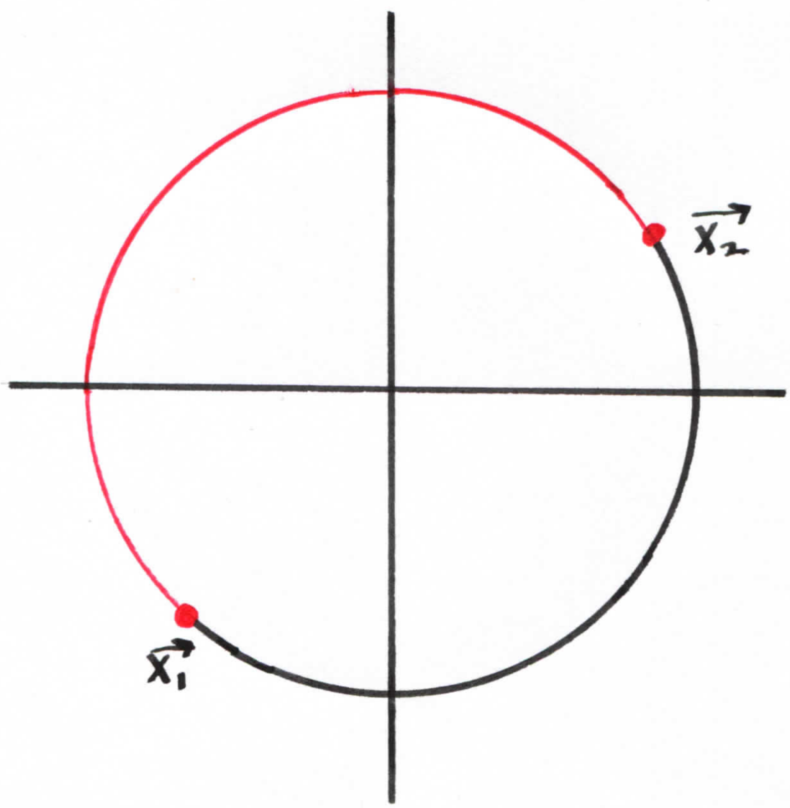
DRAWING APP4.3



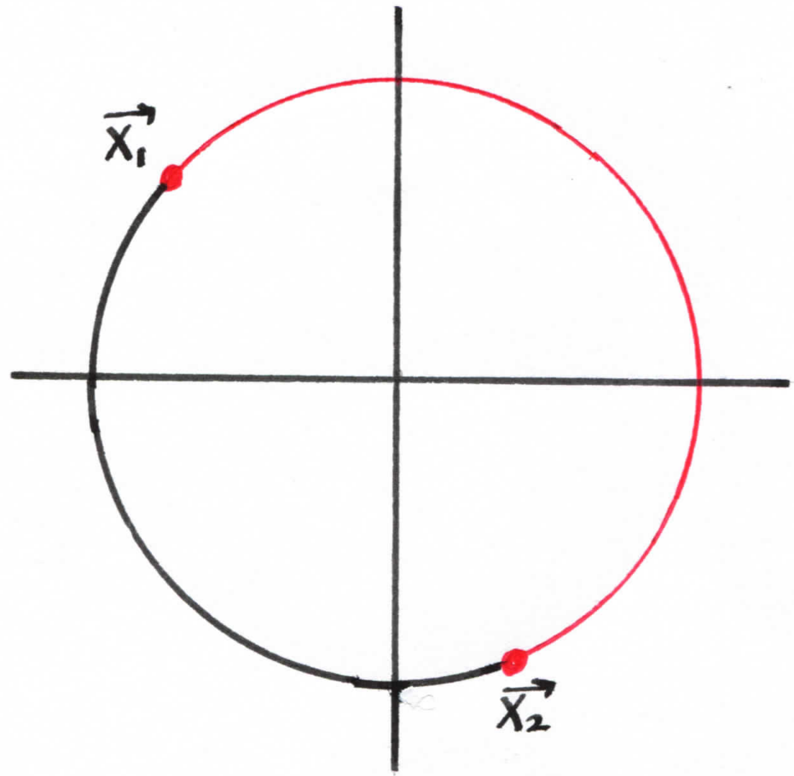
DRAWING APP4.4



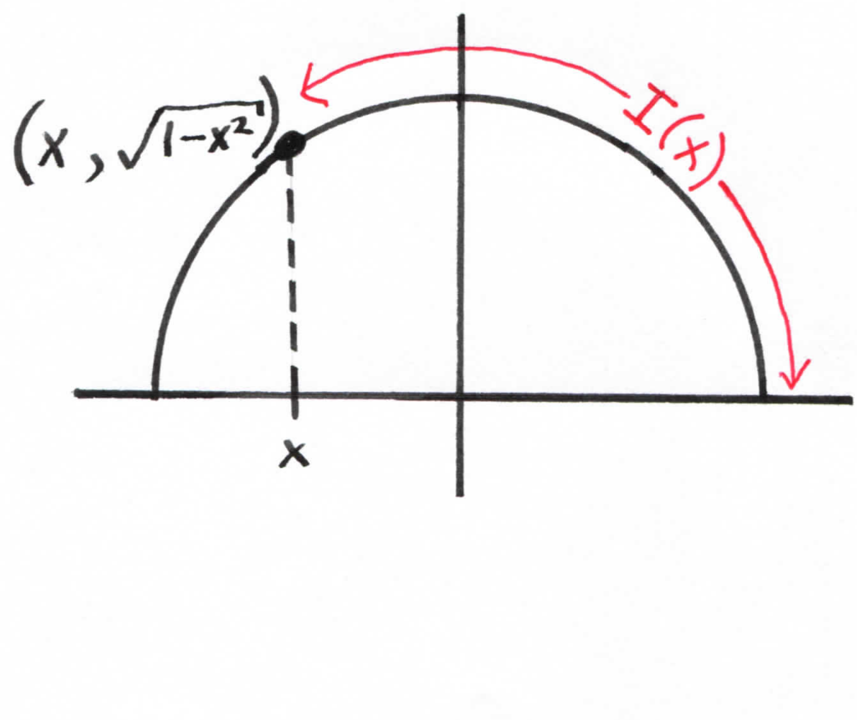
DRAWING APP4.5



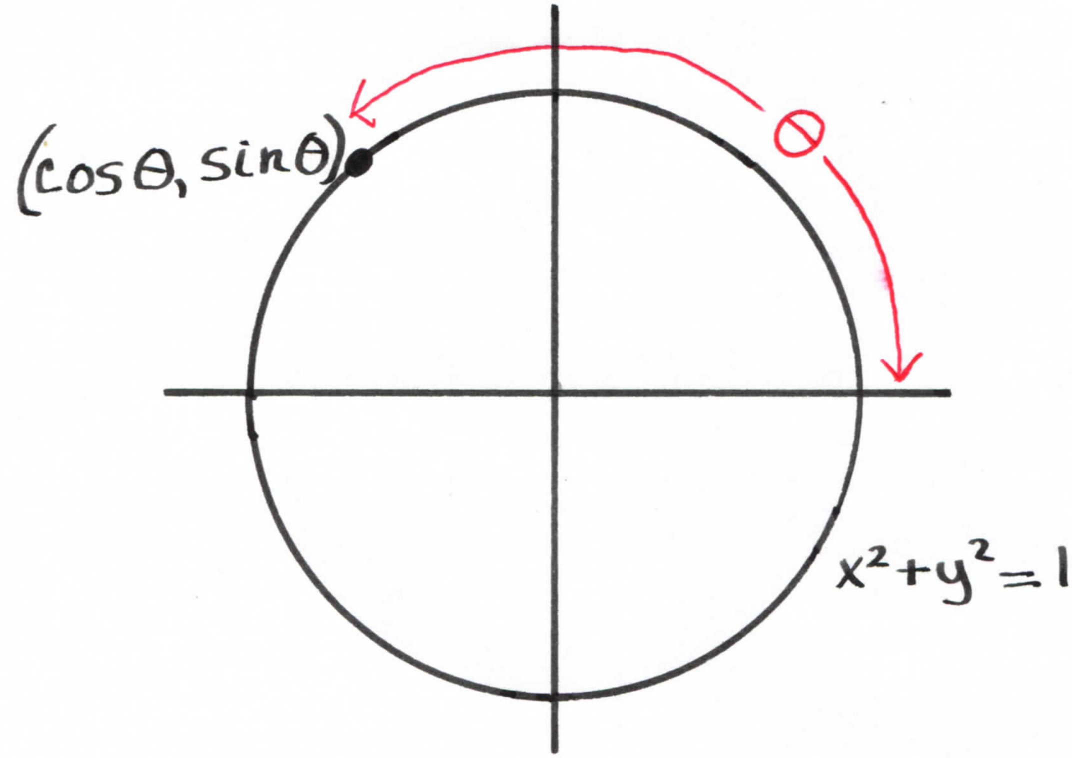
DRAWING APP4.6



DRAWING APP4.7



DRAWING APP4.8



DRAWING APP4.9