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Vectors and Geometry
MATHematics MAGnification™

Dr. Ralph deLaubenfels

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Teacher-Scholar Institute

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VECTORS and GEOMETRY MAGNIFICATION

This is one of a series of very short books on math, statistics, and physics called "Math Magnifications." The "magnification" refers to focusing on a particular topic that is pivotal in or emblematic of mathematics.

OUTLINE

We will define (two-dimensional) vectors and their operations purely geometrically, that is, with pictures, and give many examples of how they may be used to give quick and easy proofs of results in geometry.

A much more complete treatment of trigonometry and geometry via vectors and complex numbers will appear in a future book.

For this magnification, students should be familiar with first-year high school algebra and the definition of a polygon and its vertices.

Definition 1. A (two-dimensional) **vector** is a directed line segment, or arrow, with a dot at one end called the **initial point** and an arrowhead at the other end called the **terminal point**, with the understanding that two directed line segments represent the same vector if they have the same length and direction, in going from dot to arrowhead.

In the drawing below, each arrow represents the same vector.

DRAWING 2

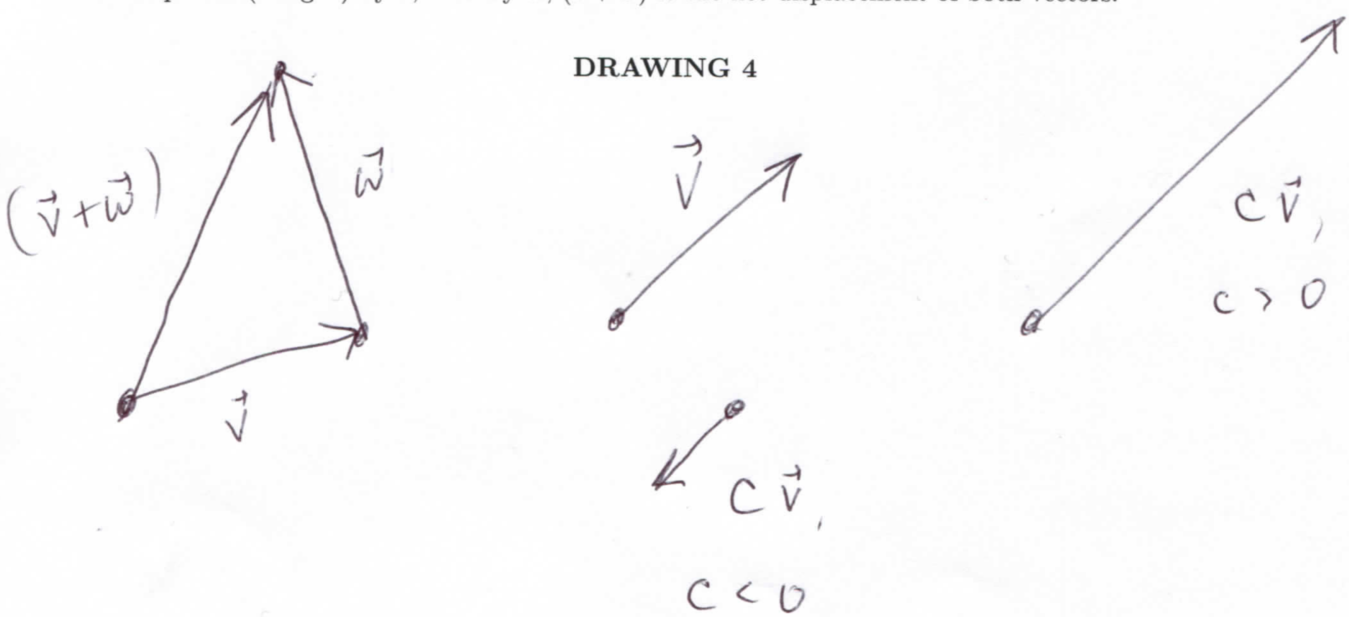
See http://teacherscholarinstitute.com/FreeMathBooksHighschool.html, "Linear Algebra," for rigorous algebraic definitions of vectors and their operations. For this magnification, Definitions 1 and 3 are sufficient definitions.

Definitions 3. We may add two vectors and multiply vectors by real numbers, as drawn below in DRAWING 4, for $\vec{v}$ and $\vec{w}$ vectors, c a real number.

For any vector $\vec{v}$, real number c, the length of $c\vec{v}$ is $|c|$ times the length of $\vec{v}$. If $c > 0$, $c\vec{v}$ points in the same direction as $\vec{v}$; if $c < 0$, then $c\vec{v}$ points in the opposite direction to $\vec{v}$.

For any vectors $\vec{v}, \vec{w}$, think of $(\vec{v} + \vec{w})$ in terms of motion: starting at the initial point of $\vec{v}$, you are displaced (nudged) by $\vec{v}$, then by $\vec{w}$; $(\vec{v} + \vec{w})$ is the net displacement of both vectors.

DRAWING 4



Definitions 5. Motivated by DRAWING 4, we define two vectors to be **parallel** if one is a real multiple of the other.

The **zero vector**, denoted $\vec{0}$, is the directed line segment with zero length, a single dot; note that $0\vec{v} = \vec{0}$ and $\vec{0} + \vec{v} = \vec{v}$,for any vector $\vec{v}$.

The **midpoint** of a vector $\vec{a}$ is (the terminal point of) $\frac{1}{2}\vec{a}$; said point **bisects** $\vec{a}$.

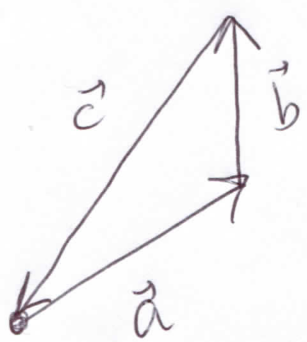
Vector definitions and Drawings 6. Recall that a **triangle** is a polygon with three sides and a **quadrilateral** is a polygon with four sides.

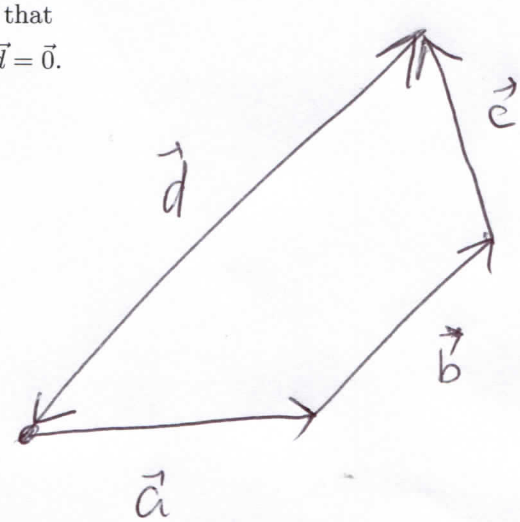
DRAWING 4 tells us that we may describe any triangle with three vectors $\vec{a}, \vec{b}, \vec{c}$ such that

$$\vec{a} + \vec{b} + \vec{c} = \vec{0}$$

and any quadrilateral with four vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ such that

$$\vec{a} + \vec{b} + \vec{c} + \vec{d} = \vec{0}.$$

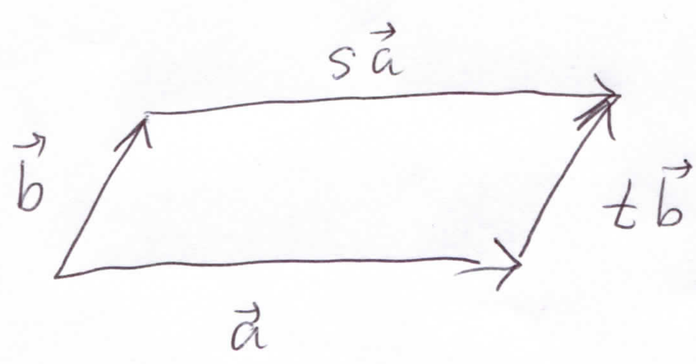




Definition 7. A **parallelogram** is a quadrilateral with nonconsecutive sides parallel.

This may be described with two vectors $\vec{a}, \vec{b}$ and two real numbers s, t, as drawn below in DRAWING 8. We will soon improve this picture (DRAWING 11) by finding out what c and d must equal (Geometry Theorem 9).

DRAWING 8



Geometry Theorem 9. In a parallelogram, nonconsecutive sides have equal length.

Proof: Let $\vec{a}, \vec{b}, c$, and d be as in Definition 7 and DRAWING 8. Then

$$\vec{0} = \vec{a} + t\vec{b} - s\vec{a} - \vec{b},$$

thus

$$(1-t)\vec{b} = (1-s)\vec{a};$$

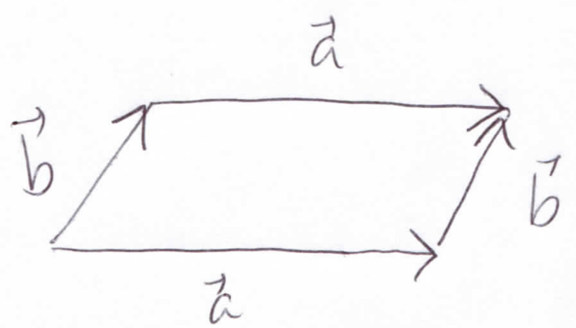
since $\vec{a}$ and $\vec{b}$ are not parallel,

$$(1-t) = 0 = (1-s),$$

so that $s = 1 = t$.

Vector definitions and Drawing 10. For any parallelogram, by Geometry Theorem 9 and DRAWING 8 there are vectors $\vec{a}, \vec{b}$ that **determine** the parallelogram, as in DRAWING 11 below.

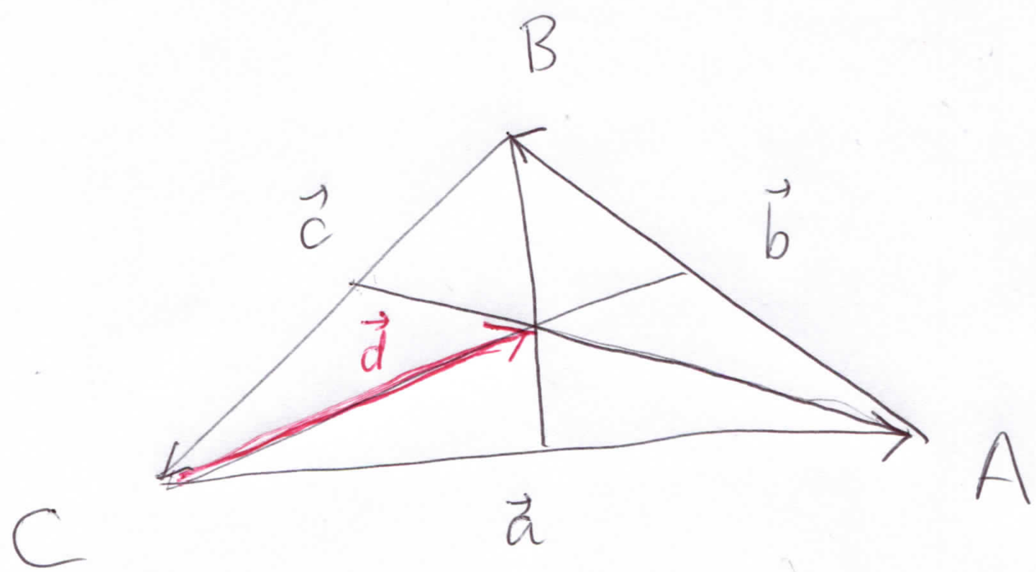
DRAWING 11



Geometry Theorem 12. In any triangle, the lines from vertices to midpoints of opposite sides all intersect at the same point.

Proof: Let $\vec{a}, \vec{b}, \vec{c}$ be as in Vector definitions and Drawing 6 for a triangle with vertices A, B, C. Let $\vec{d}$ be the vector drawn below in DRAWING 13 with the same initial point as $\vec{a}$, and terminal point the intersection of the line from C to the midpoint of $\vec{b}$ with the line from A to the midpoint of $\vec{c}$.

DRAWING 13



There are real numbers s and t so that, using $\vec{a}+\vec{b}+\vec{c}=\vec{0}$,

$$s(\vec{a}+\frac{1}{2}\vec{b})=\vec{d}=\vec{a}+t\left(-\vec{a}+\frac{1}{2}(-\vec{c})\right)=\vec{a}+t\left(-\vec{a}+\frac{1}{2}(\vec{a}+\vec{b})\right)=\vec{a}+\frac{t}{2}(\vec{b}-\vec{a}),$$

thus

$$\frac{1}{2}(s-t)\vec{b}=(1-\frac{t}{2}-s)\vec{a},$$

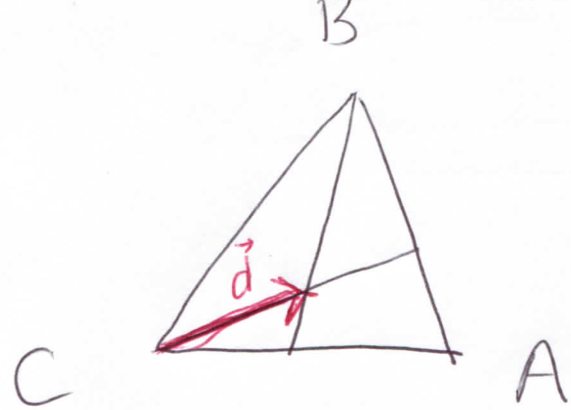
so that, since $\vec{a}$ and $\vec{b}$ are not parallel,

$$\frac{1}{2}(s-t)=0=(1-\frac{t}{2}-s),$$

implying that $s=t=\frac{2}{3}$.

In other words, the line from C to its opposite midpoint intersects the line from A to its opposite midpoint at $\vec{d}=\frac{2}{3}(\vec{a}+\frac{1}{2}\vec{b})$, two thirds of the way from C. An almost identical argument shows that the line from C to its opposite midpoint intersects the line from B to its opposite midpoint at the same place (DRAWING 14). Thus all three lines, from vertex to midpoint of opposite side, intersect at the same place, namely the terminal point of $\vec{d}$.

DRAWING 14

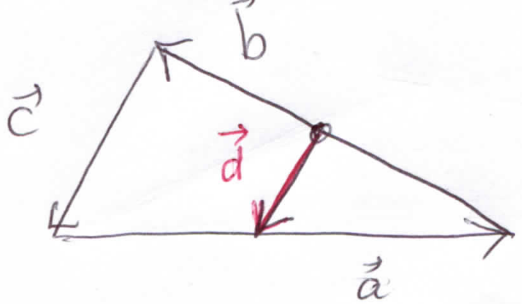


Remark 15. Notice that we inadvertently proved more than we intended in Geometry Theorem 12, namely that, for each vertex, the point of common intersection is two-thirds of the way from said vertex to the midpoint of the opposite side.

Geometry Theorem 16. In any triangle, the line from the midpoint of one side to the midpoint of another side is parallel to the third side and half as long.

Proof: Drawn below (DRAWING 17) is the picture of a triangle as in Vector definitions and Drawings 6, with the additional vector, denoted $\vec{d}$, from midpoint of $\vec{b}$ to midpoint of $\vec{a}$, drawn in.

DRAWING 17



We have

$$\frac{1}{2}\vec{a} = \vec{a} + \frac{1}{2}\vec{b} + \vec{d},$$

thus

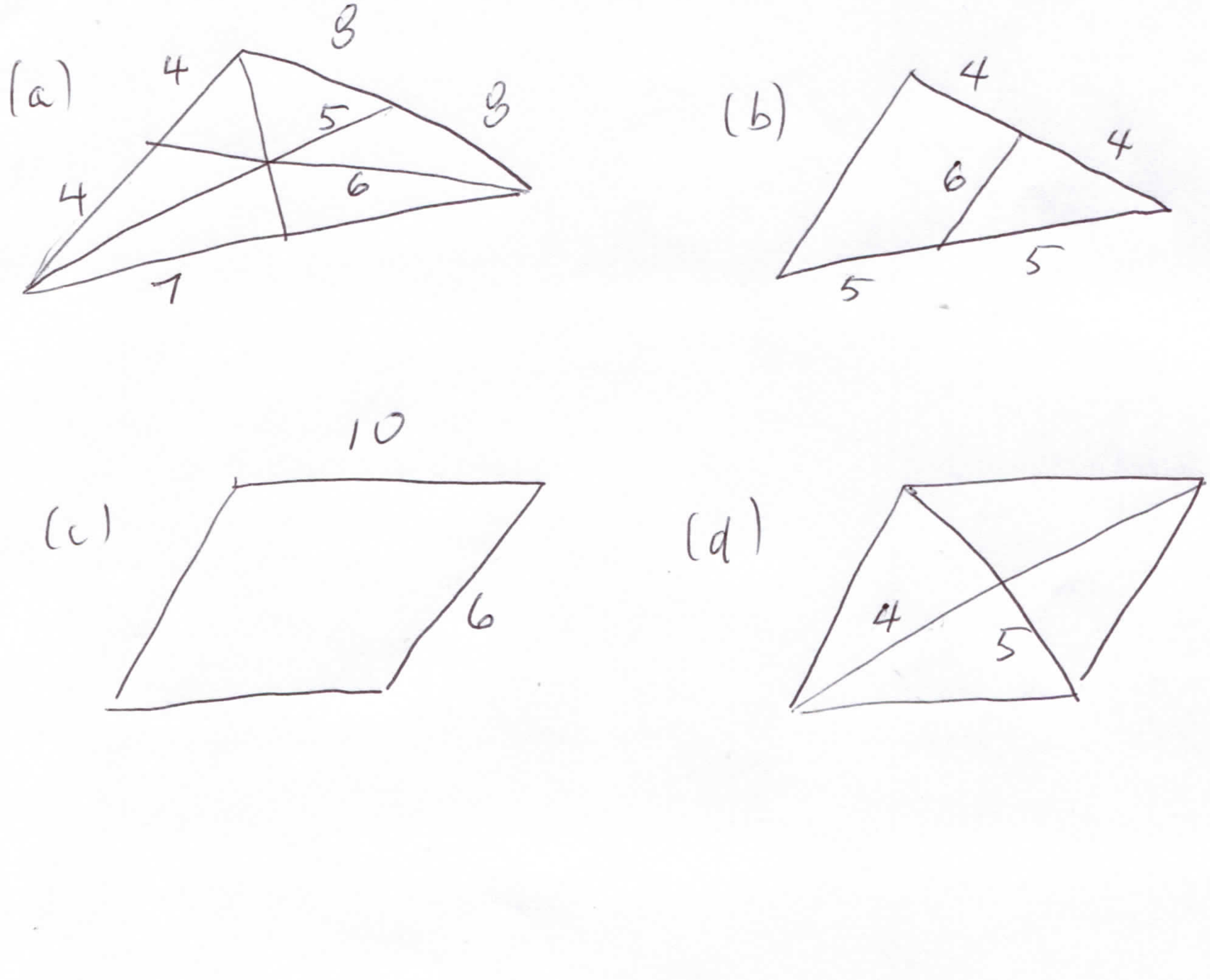
$$\vec{d} = \frac{1}{2}(-\vec{a} - \vec{b}) = \frac{1}{2}\vec{c},$$

since $\vec{a} + \vec{b} + \vec{c} = \vec{0}$.

Geometry Theorem 18. The diagonals in a parallelogram bisect each other.

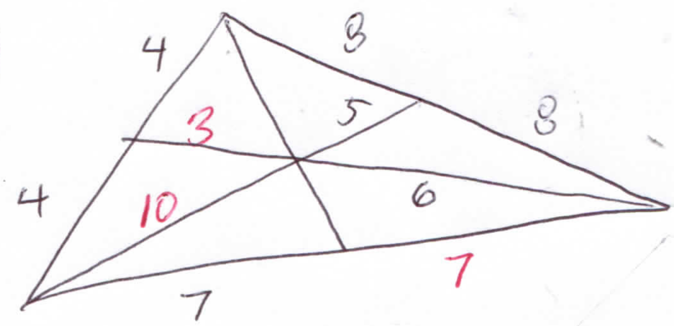
Proof: is in the homework. We advise beginning with expressing the diagonals in DRAWING 11 in terms of the vectors $\vec{a}$ and $\vec{b}$ that determine a parallelogram (see DRAWING 11).

Examples 19. Fill in missing lengths, wherever possible, with the results of this magnification. Assume all quadrilaterals are parallelograms.

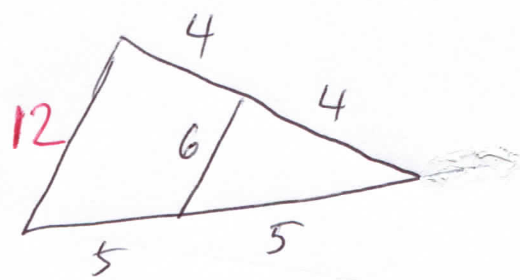


Solutions.

(a)



(b)



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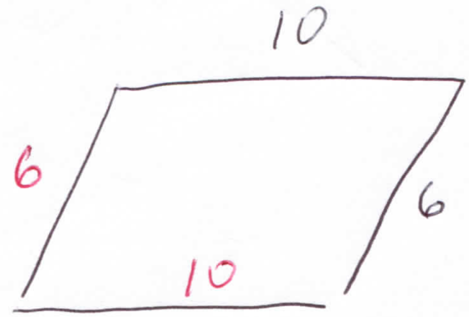

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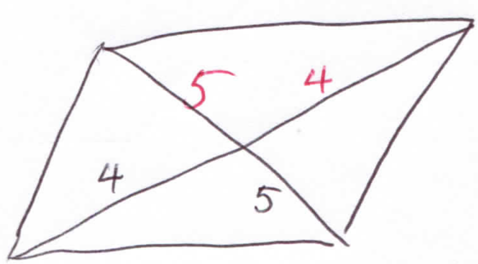


HOMEWORK

1. Use vectors to prove the following:

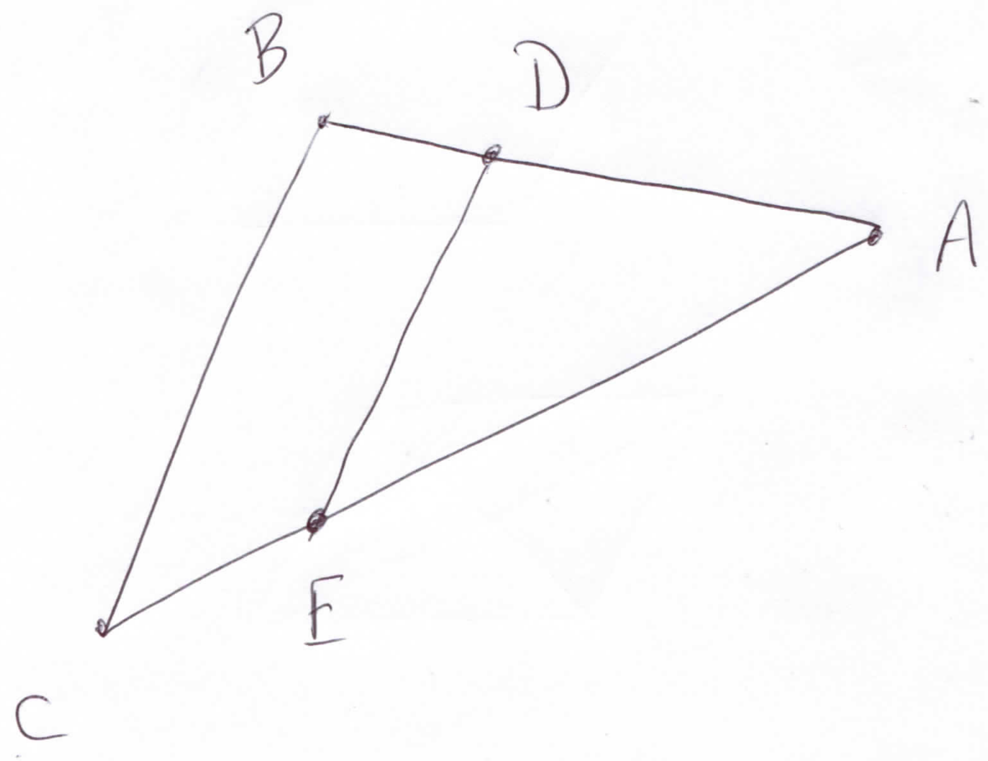
(a) If a quadrilateral has two sides that are parallel and of equal length, then the quadrilaterial is a parallelogram.

(b) Geometry Theorem 18.

(c) Without using the proof of Geometry Theorem 12, prove that the lines from vertices to midpoints of opposite sides in a triangle intersect.

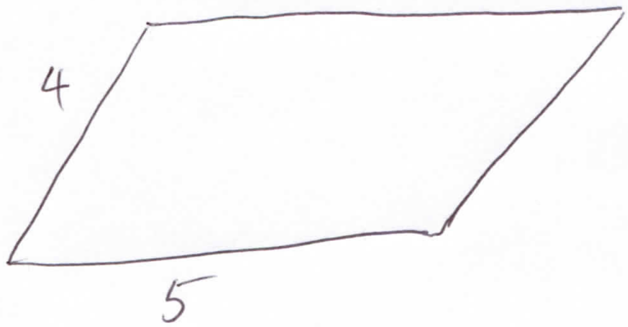
(d) Let A, B, and C be the vertices of a triangle. Suppose D is a point on the line from B to A such that the distance from D to A is twice the distance from B to D, and E is a point on the line from C to A such that the distance from E to A is twice the distance from C to E. See drawing below.

Prove that the line segment from D to E is parallel to the line segment from B to C. What is the relationship between the lengths of the line segments in the previous sentence? Prove whatever assertion you make about said lengths.



2. In each of the following, fill in lengths of sides, where possible with the results of this magnification. Assume all quadrilaterals are parallelograms.

(a)



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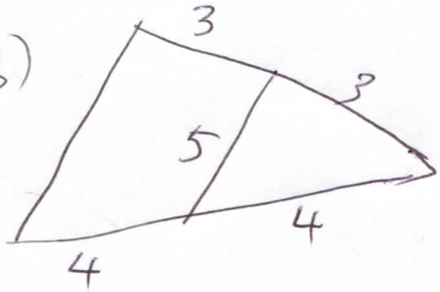

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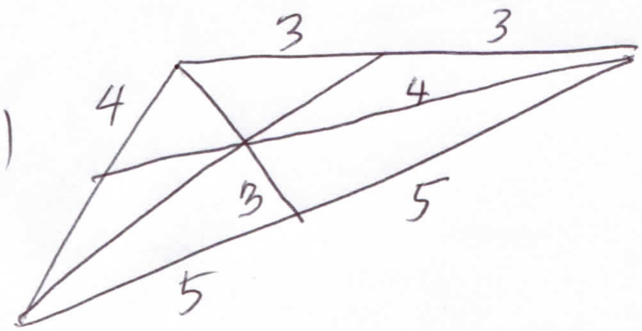

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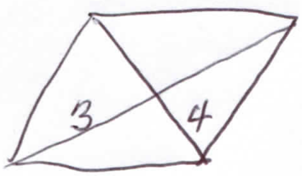
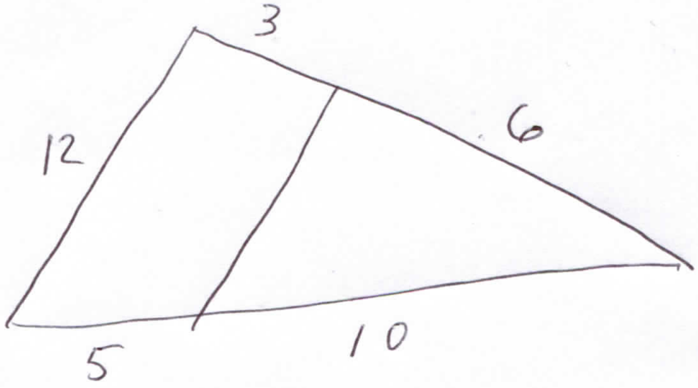
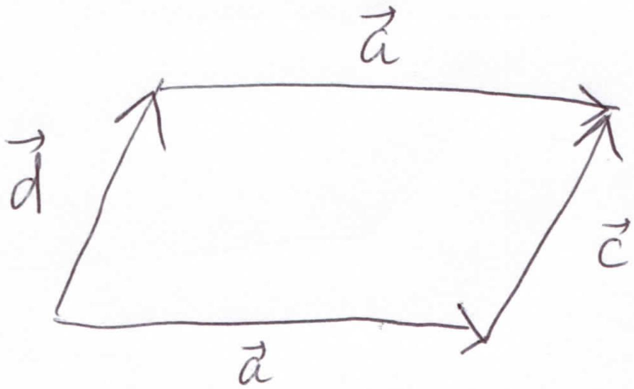


(e)



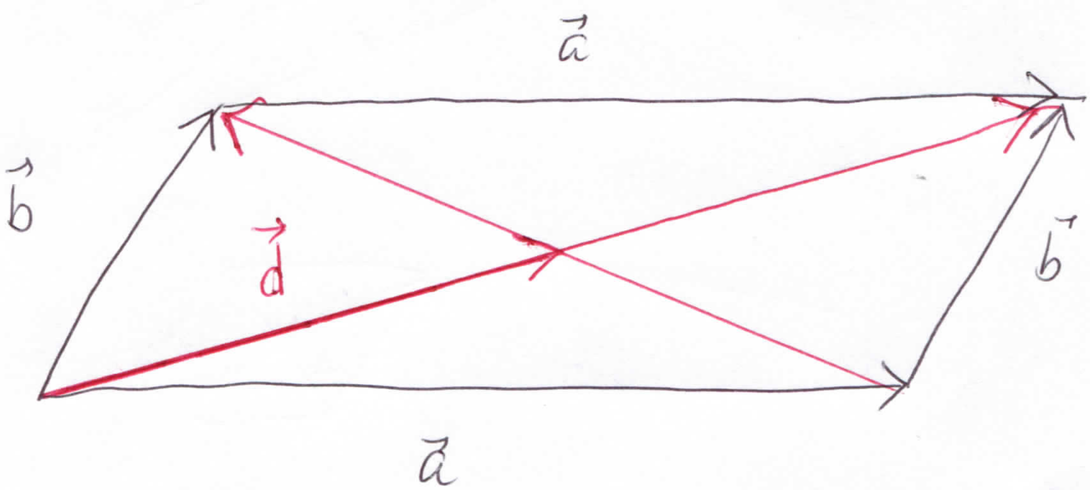
HOMEWORK HINTS

1(a) The quadrilateral may be drawn as follows.



(b) Here is an arbitrary parallelogram, as in DRAWING 11, with the diagonals $(\vec{a} + \vec{b})$ and $(\vec{b} - \vec{a})$ and a vector $\vec{d}$, with terminal point at the intersection of the diagonals, added, in red.

Argue analogously to Geometry Theorem 12.



(c) Show that the lines from vertices to opposite sides cannot be parallel (see Definitions 5).

(d) See the proof of Geometry Theorem 16. Make a drawing like DRAWING 17, with $\vec{d}$ going from D to E.

2. See Examples 19.

HOMEWORK ANSWERS

1(a) See the drawing under **1**(a) Hints. Then

$$\vec{a} + \vec{c} - \vec{a} - \vec{d} = \vec{0},$$

which implies that $\vec{c} = \vec{d}$.

1(b) See the drawing under **1**(b) Hints. There are real numbers s and t so that

$$\vec{a} + s(\vec{b} - \vec{a}) = \vec{d} = t(\vec{a} + \vec{b}),$$

thus

$$(1 - t - s)\vec{a} = (t - s)\vec{b},$$

which implies, since $\vec{a}$ and $\vec{b}$ are not parallel, that

$$(1 - t - s) = 0 = (t - s);$$

solving for t and s gives $t = \frac{1}{2} = s$.

1(c) Referring to DRAWING 13, we will show that the line from C to the midpoint of the opposite side cannot be parallel to the line from B to the midpoint of the opposite side, hence must intersect. The same argument will apply to the other lines from vertices to midpoints. See the drawing below, where A, B, C and $\vec{a}, \vec{b}, \vec{c}$ are as in DRAWING 13.

The lines drawn in red below are

$$(\vec{a} + \frac{1}{2}\vec{b}) \text{ and } (\vec{c} + \frac{1}{2}\vec{a}) = (-(\vec{a} + \vec{b}) + \frac{1}{2}\vec{a}) = -(\frac{1}{2}\vec{a} + \vec{b}).$$

If those lines were parallel, there would be a real number c so that

$$(\vec{a} + \frac{1}{2}\vec{b}) = c\left(-(\frac{1}{2}\vec{a} + \vec{b})\right),$$

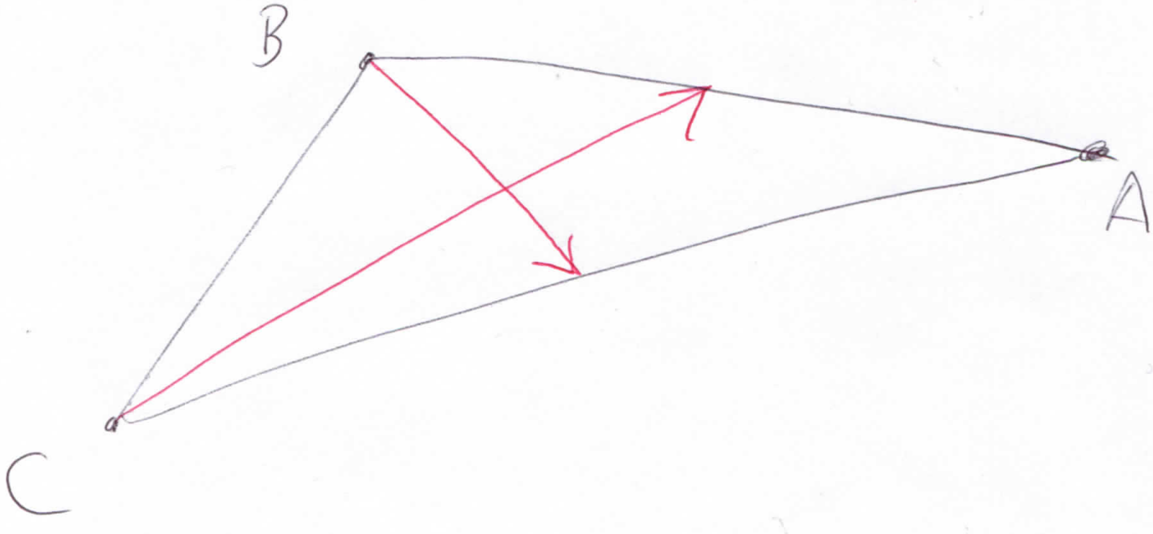
so that

$$(1 + \frac{1}{2}c)\vec{a} = (-c - \frac{1}{2})\vec{b},$$

which implies, since $\vec{a}$ and $\vec{b}$ are not parallel, that

$$(1 + \frac{1}{2}c) = 0 = (-c - \frac{1}{2}).$$

This implies that $c = -2$ and $c = -\frac{1}{2}$, which is impossible, thus the lines drawn in red cannot be parallel.



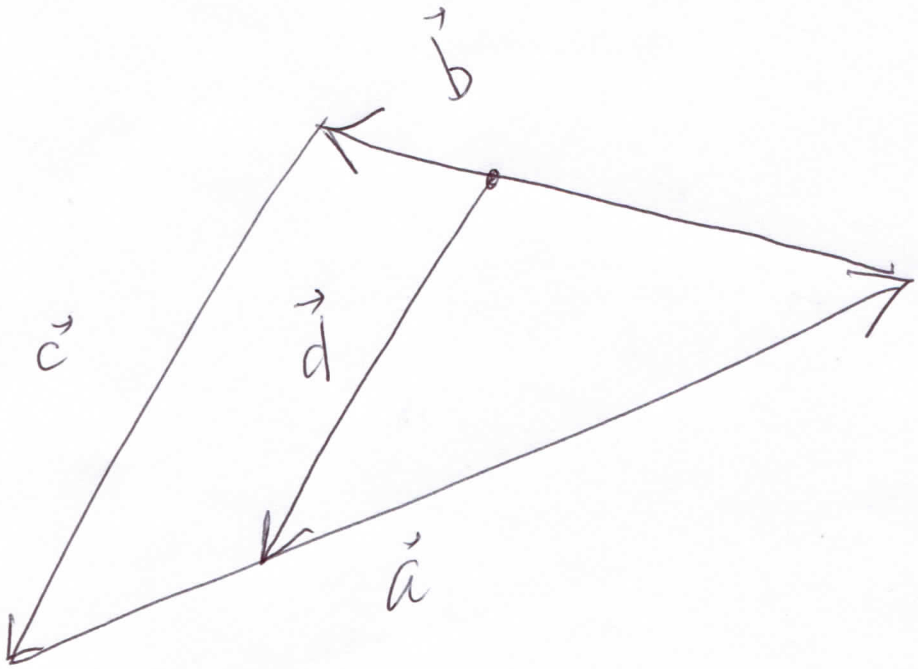
1(d) See the drawing below. We have

$$\frac{1}{3}\vec{a} = \vec{a} + \frac{2}{3}\vec{b} + \vec{d},$$

so that

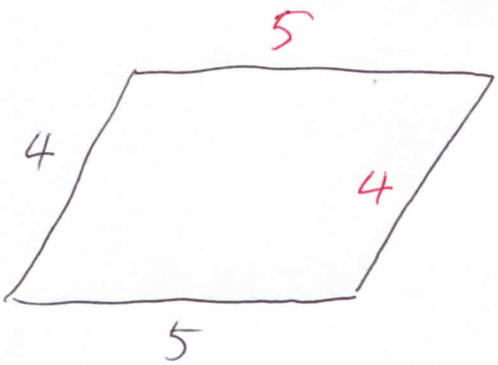
$$\vec{d} = \frac{2}{3}(-\vec{a} - \vec{b}) = \frac{2}{3}\vec{c}.$$

This proves that the line segment from D to E is parallel to the line segment from B to C, with the former two-thirds the length of the latter.



2.

(a)



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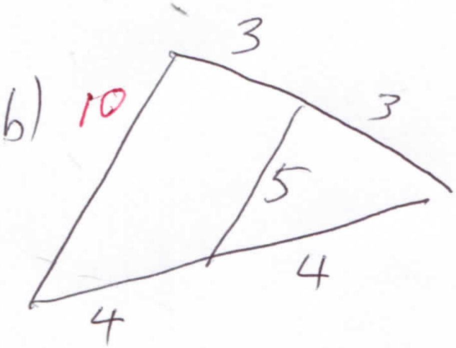

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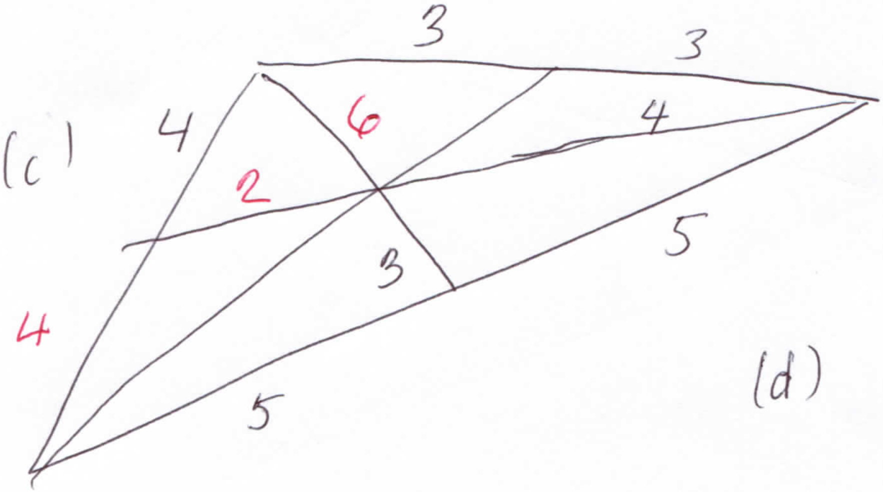

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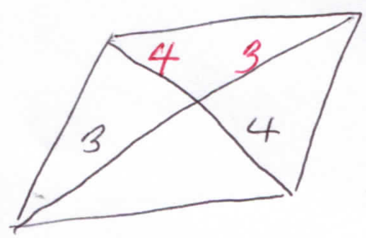

(e)