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Vectors and Perpendicular Geometry
MATHematics MAGnification™

Dr. Ralph deLaubenfels

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VECTORS and PERPENDICULAR GEOMETRY MAGNIFICATION

This is one of a series of very short books on math, statistics, and physics called "Math Magnifications." The "magnification" refers to focusing on a particular topic that is pivotal in or emblematic of mathematics.

OUTLINE

We will define (two-dimensional) vectors and their operations both geometrically and algebraically. Our particular goal is to characterize perpendicular vectors algebraically. We will use this to give quick and easy proofs of some results in geometry involving right angles. The intuition of minimizing distance by "dropping a perpendicular" will be made explicit.

This magnification will introduce the reader to what is arguably the most successful theme in mathematics, the synthesis of algebra (calculation) and geometry (pictures). The algebra gives us precision and the geometry intuition. Proofs are also introduced, with the algebra making them half computation and the geometry giving a visual direction.

We conclude with some detective work, using our results to fill in missing sides and right angles in triangles and parallelograms.

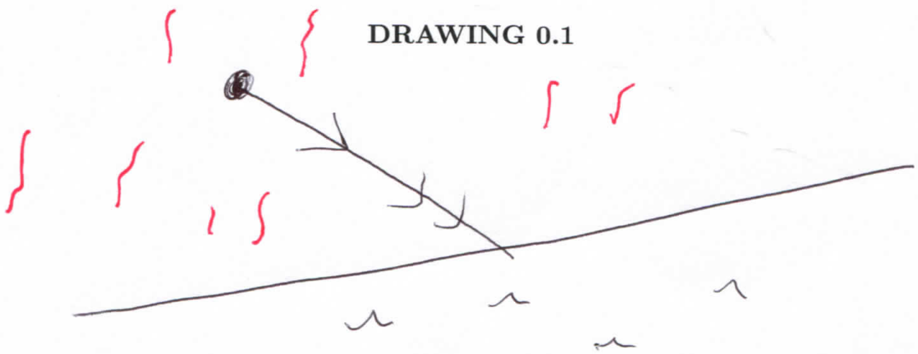
For this magnification, students should be familiar with first-year high school algebra, the definition of a polygon and its vertices, and the Pythagorean theorem. Reference [4] is more than sufficient. For the proofs, students should also be familiar with the language of logic: "if," "only if," "necessary," "sufficient," "converse." The double arrow " $\Longleftrightarrow$ " means "if and only if," or necessary and sufficient, or equivalent.

See [2] for much more on vectors in the setting of linear algebra. A much more complete treatment of trigonometry and geometry via vectors and complex numbers will appear in a future book ([3]).

INTRODUCTION.

By way of urgent motivation, imagine yourself on a beach with hot sand bordering a cool ocean. If you're barefoot, you want to cool off as quickly as possible by taking the shortest path to the ocean. Below (DRAWING 0.1) is a picture of your path to the ocean.

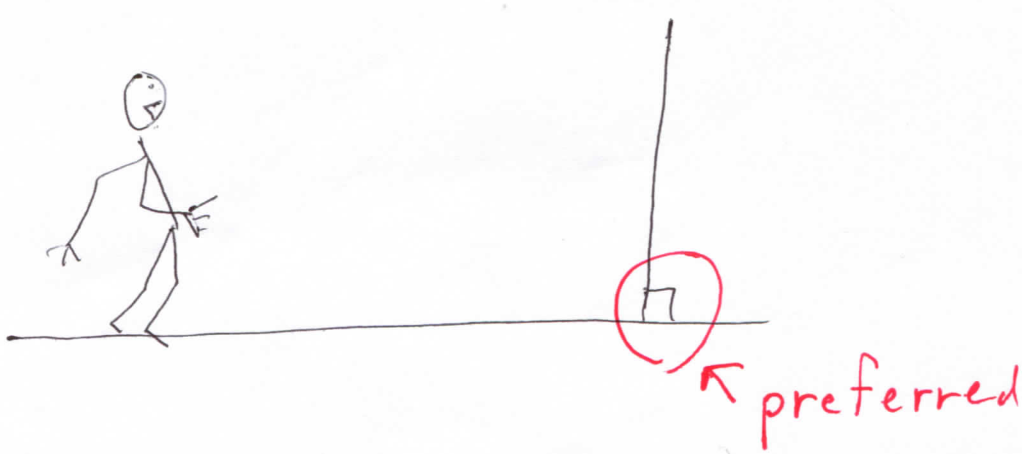
DRAWING 0.1



If this exposition were spoken, with an audience, we would now ask the audience "What's wrong with this picture?" In my experience, many members of said audience would answer "Not the shortest path." When I respond "Why not?", I would hear "not making a right angle with the shoreline." See DRAWING 0.2 for this shortest path, with right angle drawn in.

DRAWING 0.2

Other physical motivations for being perpendicular abound. Housebuilders try to make walls perpendicular to the ground, and good posture suggests that we try to make our bodies perpendicular to the ground, unless we're lying down.



CHAPTER I: VECTORS.

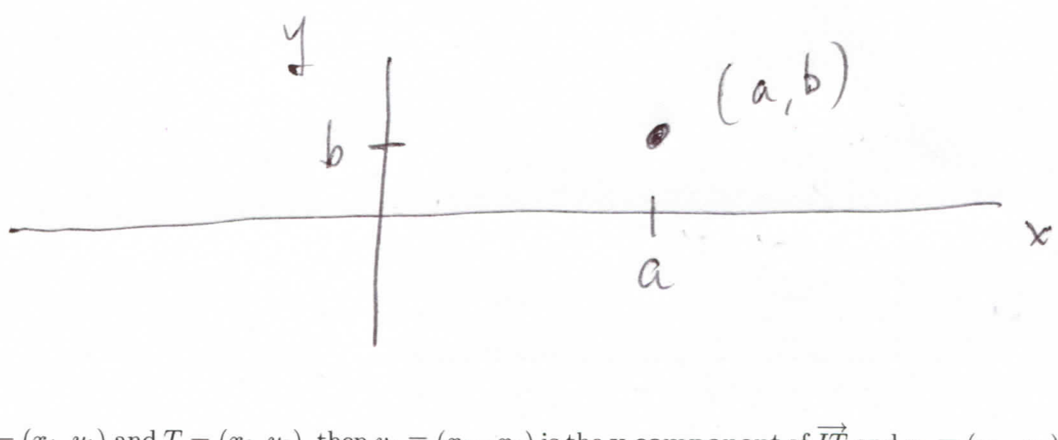
We would like an algebraic characterization of right angles, in the setting of *vectors* (Definitions 1.1).

Definitions 1.1. A **vector** is represented by a directed line segment, meaning an arrow. The **initial point**, denoted in DRAWING 1.2 by the letter I, is marked with a fat dot, and the **terminal point** denoted in DRAWING 1.2 by the letter T, is marked with an arrow head; the directed line segment is then denoted $\overrightarrow{IT}$. Two directed line segments represent the same vector if they have the same length and direction.

DRAWING 1.2

For two-dimensional vectors, we place ourselves in the **Cartesian plane** $\mathbf{R}^2 \equiv \{(x, y) \mid x, y \text{ are real}\}$. A point (a, b) in $\mathbf{R}^2$ is denoted by a dot a units to the right of the **origin** $(0, 0)$, b units above the origin, as in DRAWING 1.3. The number a is the **x coordinate** of (a, b) and b is the **y coordinate** of (a, b).

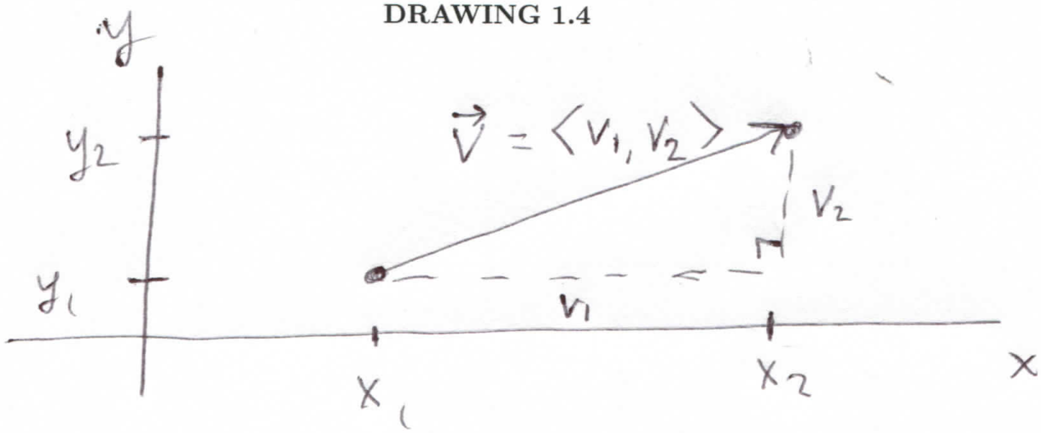
DRAWING 1.3



If $I = (x_1, y_1)$ and $T = (x_2, y_2)$, then $v_1 \equiv (x_2 - x_1)$ is the **x component** of $\overrightarrow{IT}$ and $v_2 \equiv (y_2 - y_1)$ is the **y component** of $\overrightarrow{IT}$. Two directed line segments represent the same vector if and only if said line segments have the same components; the vector with x component v_1 and y component v_2 is then denoted

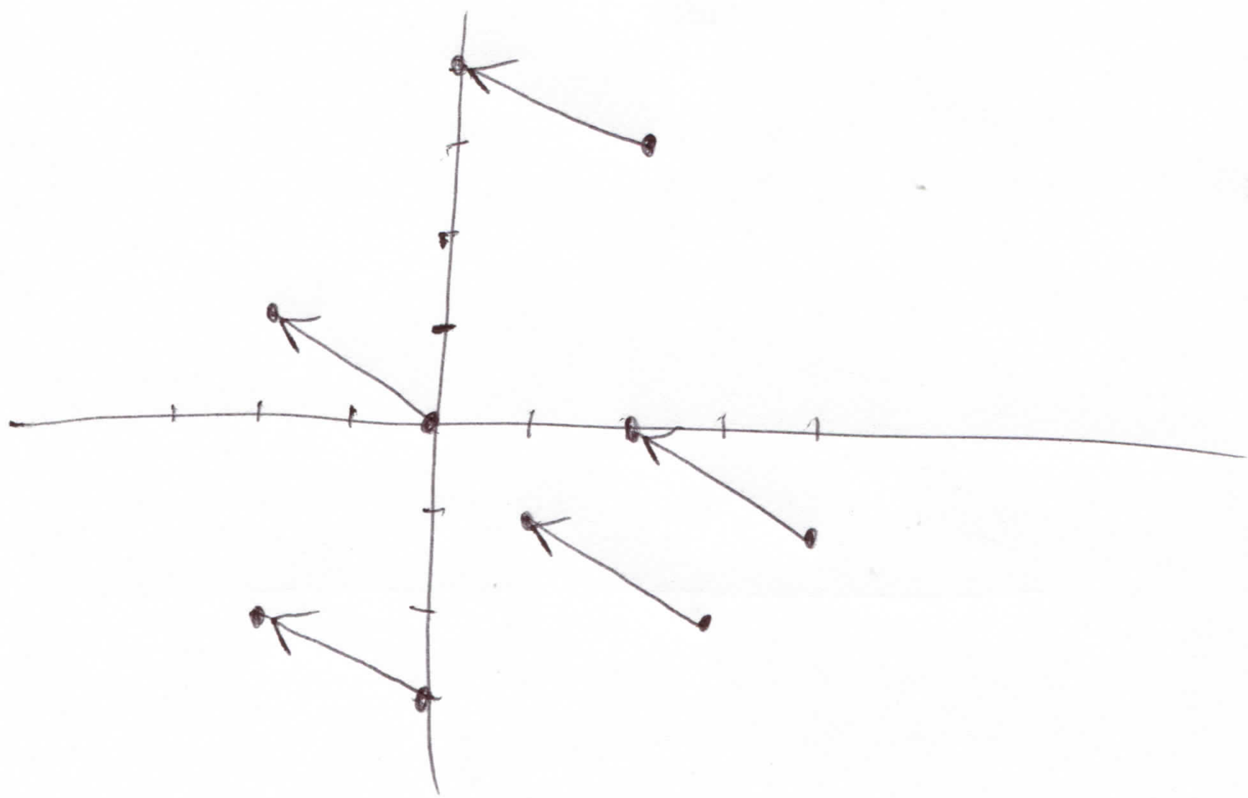
$$\vec{v} \equiv < v_1, v_2 > .$$

DRAWING 1.4



In DRAWING 1.5, each directed line segment (arrow) represents the vector $< -2, 1 >$.

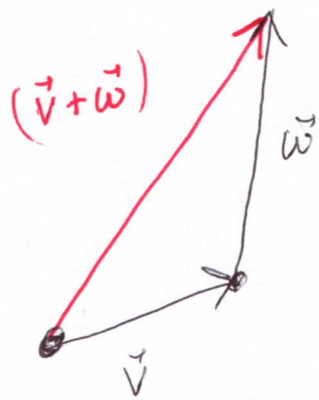
DRAWING 1.5

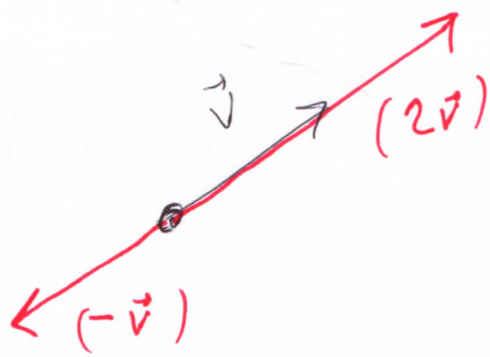


Definitions 1.6. We may add vectors and multiply vectors by real numbers. If $\vec{v} \equiv < v_1, v_2 >$, $\vec{w} \equiv < w_1, w_2 >$, and c is a real number, then

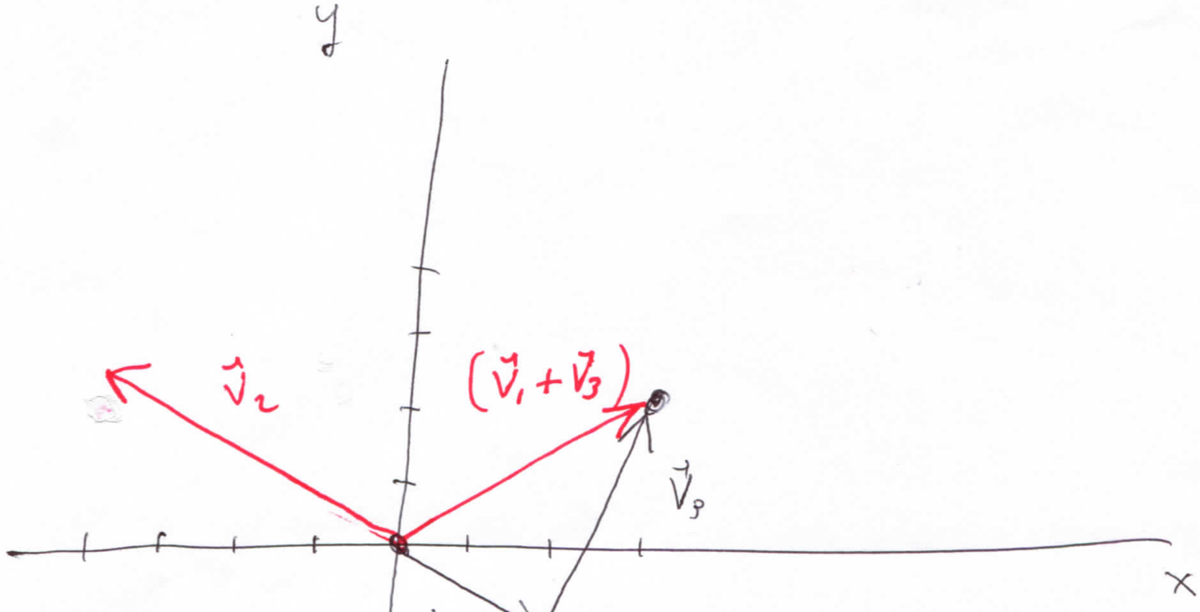
$$(\vec{v} + \vec{w}) = < v_1 + w_1, v_2 + w_2 > \quad \text{and} \quad c\vec{v} = < cv_1, cv_2 > .$$

Here are the corresponding directed line segments. Note that (the corresponding line segment for) $c\vec{v}$ is parallel to (the corresponding line segment for) $\vec{v}$.





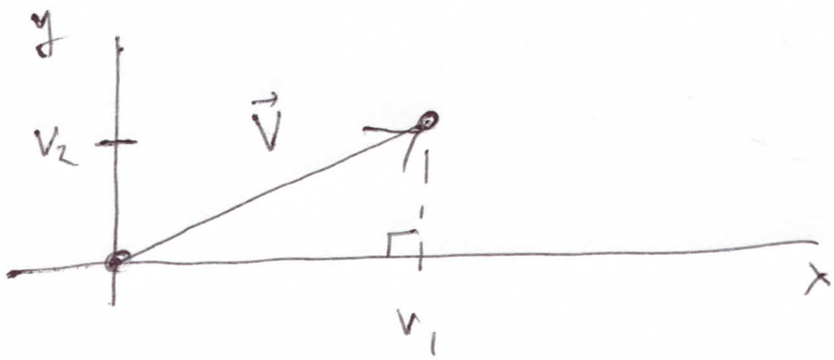
Examples 1.7. Let $\vec{v}_1 \equiv < 2, -1 >, \vec{v}_2 \equiv < -4, 2 >$, and $\vec{v}_3 \equiv < 1, 3 >$. See below for drawings of the directed line segments for $\vec{v}_1, \vec{v}_2 = -2\vec{v}_1, \vec{v}_3$, and $(\vec{v}_1 + \vec{v}_3)$.



Definition 1.8. The **norm** of a vector $\vec{v} \equiv < v_1, v_2 >$ is

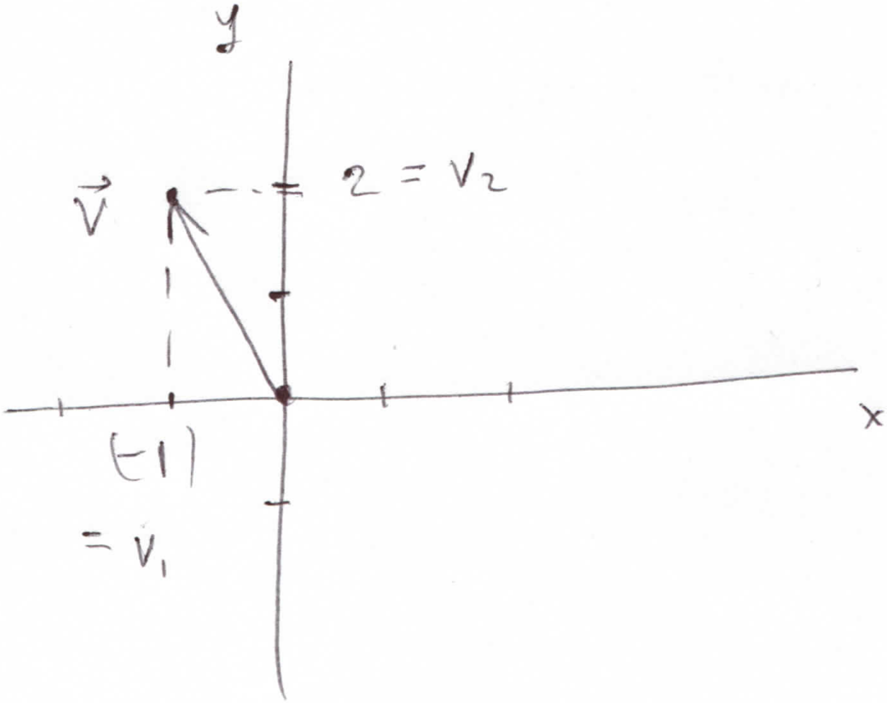
$$\|\vec{v}\| \equiv \sqrt{v_1^2 + v_2^2}.$$

Note that, by the Pythagorean theorem, $\|\vec{v}\|$ is the length of a directed line segment representing $\vec{v}$.



Example 1.9.

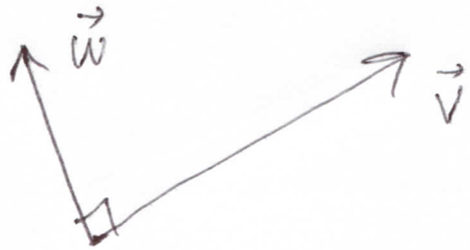
$$\vec{v} \equiv \langle -1, 2 \rangle \to \|\vec{v}\| = \sqrt{(-1)^2 + 2^2}$$
$$= \sqrt{5}.$$



CHAPTER II: ORTHOGONALITY and DOT PRODUCT.

Discussion 2.1. To characterize two vectors $\vec{v} \equiv < v_1, v_2 >, \vec{w} \equiv < w_1, w_2 >$ being perpendicular, also known as *orthogonal*, as in DRAWING 2.2,

DRAWING 2.2



we will use the Pythagorean theorem: $\|\vec{v} + \vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2$ *IF* $\vec{v}$ and $\vec{w}$ are perpendicular (see the picture of the triangle formed by $\vec{v}, \vec{w}$, and $(\vec{v} + \vec{w})$ in Definitions 1.6).

For arbitrary $\vec{v} \equiv < v_1, v_2 >, \vec{w} \equiv < w_1, w_2 >$,

$$\|(\vec{v}+\vec{w})\|^2 = \| < v_1+w_1, v_2+w_2 > \|^2 = (v_1+w_1)^2+(v_2+w_2)^2 = (v_1^2+2v_1w_1+w_1^2)+(v_2^2+2v_2w_2+w_2^2)$$
$$= (v_1^2 + v_2^2) + (w_1^2 + w_2^2) + 2(v_1w_1 + v_2w_2) = \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2(v_1w_1 + v_2w_2).$$

That last term in parentheses is our only barrier to the Pythagorean theorem, thus it deserves a name.

Definition 2.3. The **dot product** or **inner product** of $\vec{v} \equiv < v_1, v_2 >$ and $\vec{w} \equiv < w_1, w_2 >$ is

$$(\vec{v} \cdot \vec{w}) \equiv (v_1w_1 + v_2w_2).$$

Example 2.4. $< 1, -2 > \cdot < 3, 4 >= 1 \cdot 3 + (-2) \cdot 4 = 3 - 8 = -5.$

Properties of dot product 2.5. Suppose $\vec{v}, \vec{w}$, and $\vec{u}$ are vectors and c is a real number.

(a) $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$.

(b) $(c\vec{v}) \cdot \vec{w} = c\,(\vec{v} \cdot \vec{w})$.

(c) $(\vec{v} + \vec{w}) \cdot \vec{u} = (\vec{v} \cdot \vec{u}) + (\vec{w} \cdot \vec{u})$.

(d) $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$.

(e) $\|\vec{v} + \vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2(\vec{v} \cdot \vec{w})$.

(f) (directed line segments for) $\vec{v}$ and $\vec{w}$ are **orthogonal**, meaning perpendicular, as in DRAWING 2.2, denoted $\vec{v} \perp \vec{w}$, if and only if $\vec{v} \cdot \vec{w} = 0$.

Example 2.6. Let $\vec{v}_1$ and $\vec{v}_3$ be as in Examples 1.7. Are they perpendicular? They *look* sort of perpendicular, but perhaps I drew them poorly; I trust neither my artistic skills nor my eyesight to answer such an important question. But the dot product gives us supernatural precision, completely beyond the physical limitations alluded to in the previous sentence:

$$\vec{v}_1 \cdot \vec{v}_3 = 2 \cdot 1 + (-1) \cdot 3 = -1 \neq 0,$$

thus the answer is *no*, they are not perpendicular.

Definition 2.7. It is time to realize the hot-sand and cool-ocean picture of the Introduction.

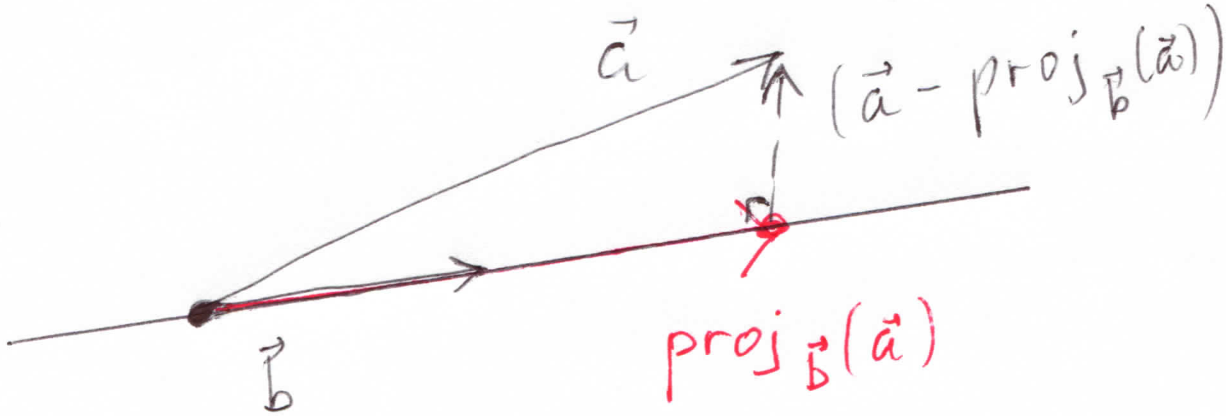
If $\vec{a}$ and $\vec{b}$ are vectors, the **(orthogonal) projection of $\vec{a}$ onto $\vec{b}$**, denoted

$$\mathrm{proj}_{\vec{b}}(\vec{a}),$$

is a real multiple of $\vec{b}$ such that

$$(\vec{a} - \mathrm{proj}_{\vec{b}}(\vec{a})) \perp \vec{b}.$$

DRAWING 2.8.

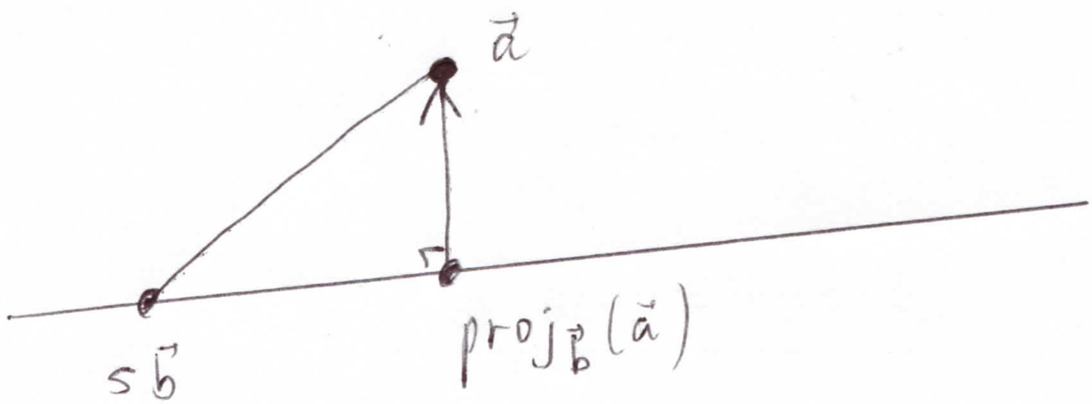


Here is a precise statement of the projection of $\vec{a}$ onto $\vec{b}$ giving us the point on the line through the origin and $\vec{b}$ that is closest to $\vec{a}$.

Proposition 2.9. For any real s, vectors $\vec{a}$ and $\vec{b}$,

$$\|\vec{a} - \mathrm{proj}_{\vec{b}}(\vec{a})\| \leq \|\vec{a} - s\vec{b}\|.$$

DRAWING 2.10.



Proof: By orthogonality,

$$\|\vec{a}-s\vec{b}\|^2 = \|\ (\vec{a} - \mathrm{proj}_{\vec{b}}(\vec{a}))+\left(\mathrm{proj}_{\vec{b}}(\vec{a}) - s\vec{b}\right)\ \|^2 = \|\ ((\vec{a} - \mathrm{proj}_{\vec{b}}(\vec{a}))\ \|^2+\|\ \left(\mathrm{proj}_{\vec{b}}(\vec{a}) - s\vec{b}\right)\ \|^2 \geq \|\ ((\vec{a} - \mathrm{proj}_{\vec{b}}(\vec{a}))\ \|^2.$$

Discussion 2.11. We don't yet actually know if $\text{proj}_{\vec{b}}(\vec{a})$ exists, for any $\vec{a}, \vec{b}$. If it does, we would like the precision and certainty of an algebraic formula for the projection, to supplement and interact with, the intuition of the purely geometric definition given.

By Definition 2.7, $\text{proj}_{\vec{b}}(\vec{a})$ (if it exists) equals $t\vec{b}$, for some real t. We can figure out what t must be:

$$0 = (\vec{a} - t\vec{b}) \cdot \vec{b} = \vec{a} \cdot \vec{b} - t(\vec{b} \cdot \vec{b}),$$

thus

$$t = \frac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2},$$

so that $\text{proj}_{\vec{b}}(\vec{a}) = \left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2}\right)\vec{b}$.

Proposition 2.12. For any vectors $\vec{a}, \vec{b}$, $\text{proj}_{\vec{b}}(\vec{a})$ exists, and equals

$$\left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2}\right)\vec{b}.$$

Proof: This is running backwards through Discussion 2.11:

$$\left(\vec{a} - \left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2}\right)\vec{b}\right) \cdot \vec{b} = \vec{a} \cdot \vec{b} - \left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2}\right)(\vec{b} \cdot \vec{b}) = \vec{a} \cdot \vec{b} - \left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2}\right)\|\vec{b}\|^2 = \vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{b} = 0.$$

Example 2.13. Get each of the following projections.

(a) $\text{proj}_{<1,2>}(< 0, 1 >)$.

(b) $\text{proj}_{<0,1>}(< 1, 2 >)$.

(c) $\text{proj}_{<1,-2>}(< -2, 4 >)$.

(d) $\text{proj}_{<1,-2>}(< 2, 1 >)$.

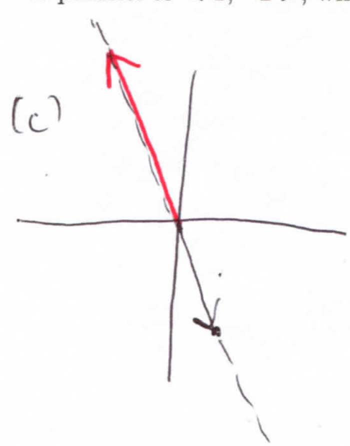
Solutions. (a) $\frac{<0,1>\cdot<1,2>}{\|<1,2>\|^2} < 1, 2 >= \frac{2}{5} < 1, 2 >$.

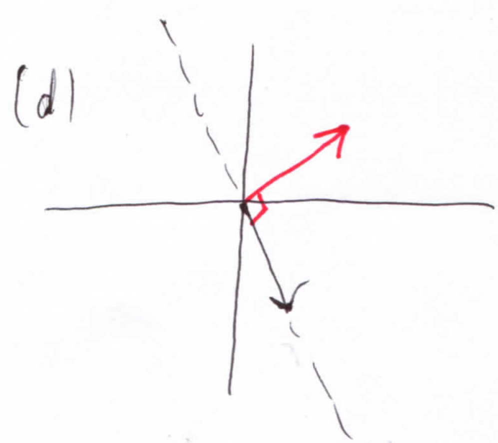
(b) $\frac{<1,2>\cdot<0,1>}{\|<0,1>\|^2} < 0, 1 >= 2 < 0, 1 >$.

(c) $\frac{<-2,4>\cdot<1,-2>}{\|<1,-2>\|^2} < 1, -2 >= \frac{-10}{5} < 1, -2 >=< -2, 4 >$.

(d) $\frac{<2,1>\cdot<1,-2>}{\|<1,-2>\|^2} < 1, -2 >=< 0, 0 >$.

Note that (equating vectors with directed line segments representing vectors), in (c), $< -2, 4 >$ is parallel to $< 1, -2 >$, while, in (d), $< 2, 1 >$ is perpendicular to $< 1, -2 >$.



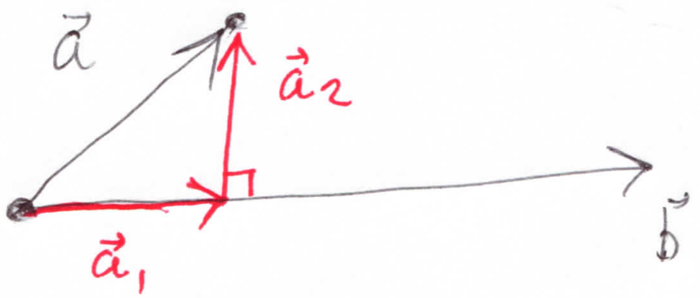


Notice that projection allows us to make the following decompositions.

Decomposition 2.14. If $\vec{a}$ and $\vec{b}$ are nontrivial (that is, not equal to $< 0, 0 >$) vectors, then there exist vectors $\vec{a}_1$ and $\vec{a}_2$ such that $\vec{a}_1$ is parallel to $\vec{b}$, $\vec{a}_2$ is perpendicular to $\vec{b}$, and

$$\vec{a} = \vec{a}_1 + \vec{a}_2.$$

DRAWING 2.15.



Proof: Let $\vec{a}_1 \equiv \text{proj}_{\vec{b}}(\vec{a})$, $\vec{a}_2 \equiv (\vec{a} - \vec{a}_1)$.

Example 2.16. Write $< 1, -2 >$ as a sum $\vec{a}_1 + \vec{a}_2$, with $\vec{a}_1$ parallel to $< 3, 2 >$ and $\vec{a}_2$ perpendicular to $< 3, 2 >$.

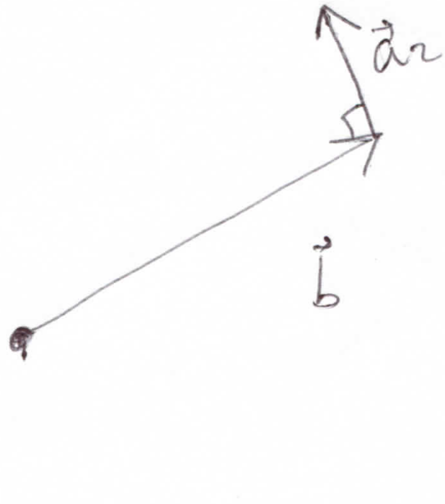
Solution: $\text{proj}_{<3,2>}(< 1, -2 >) = \left(\frac{<1,-2>\cdot<3,2>}{\|<3,2>\|^2}\right) < 3, 2 >= \left(\frac{-1}{13}\right) < 3, 2 >$, so define

$$\vec{a}_1 \equiv < \frac{-3}{13}, \frac{-2}{13} >, \quad \vec{a}_2 \equiv \left(< 1, -2 > - < \frac{-3}{13}, \frac{-2}{13} >\right) =< \frac{16}{13}, \frac{-24}{13} > .$$

Discussion 2.17. This decomposition is of particular interest when $\vec{a}$ is a force or wind and $\vec{b}$ is displacement; all we care about then is $\vec{a}_1$, from Decomposition 2.14. For example, if $\vec{a}$ is the wind and $\vec{b}$ is the direction of a path on which you are bicycling, $\vec{a}_2$ neither speeds you up nor slows you down; it only tries to bowl you over.

For example, the absolute value of the work done in making that displacement $\vec{b}$ under the force $\vec{a}$ is then $\|\vec{a}_1\|\|\vec{b}\|$, which the reader should show is $|\vec{a} \cdot \vec{b}|$.

When $\vec{a}$ is a force acting on a door $\vec{b}$, hinged at the initial point of $\vec{b}$, then only $\vec{a}_2$ is relevant to the rotation of that door.



CHAPTER III: SOME ORTHOGONAL (PERPENDICULAR) GEOMETRY.

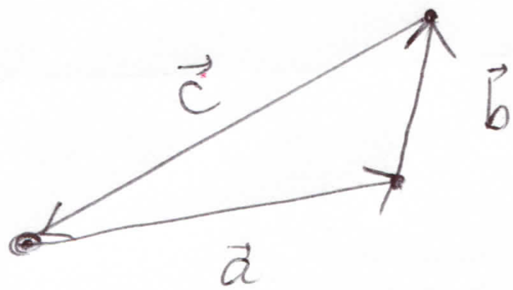
Besides the existence of orthogonal projections guaranteed by Proposition 2.12, all we need for this chapter is the dot product of Chapter II (Definition 2.3), or *anything* called a dot product, with the properties of 2.5.

Discussion 3.1. A **triangle** is a three-sided polygon and a quadrilateral is a four-sided polygon.

A triangle is determined by three vectors $\vec{a}, \vec{b}, \vec{c}$, with

$$\vec{a} + \vec{b} + \vec{c} = \vec{0} \equiv < 0, 0 > .$$

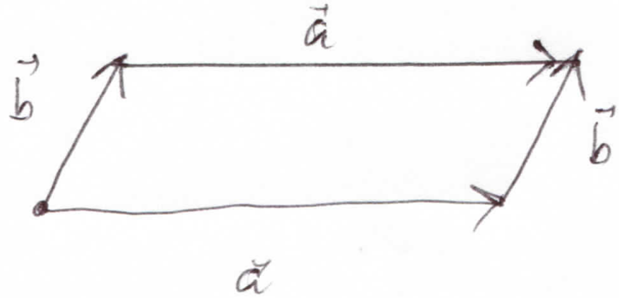
DRAWING 3.2.



See also the Definitions 1.6 picture.

We may similarly characterize a quadrilateral ([1, Vector definitions and Drawing 6]), but wish to focus on a **parallelogram**, which is a quadrilateral with nonconsecutive sides parallel. We showed in [1, Geometry Theorem 9] that nonconsecutive sides of a parallelogram are automatically of equal length, thus any parallelogram may be described with two vectors $\vec{a}, \vec{b}$ as follows.

DRAWING 3.3.



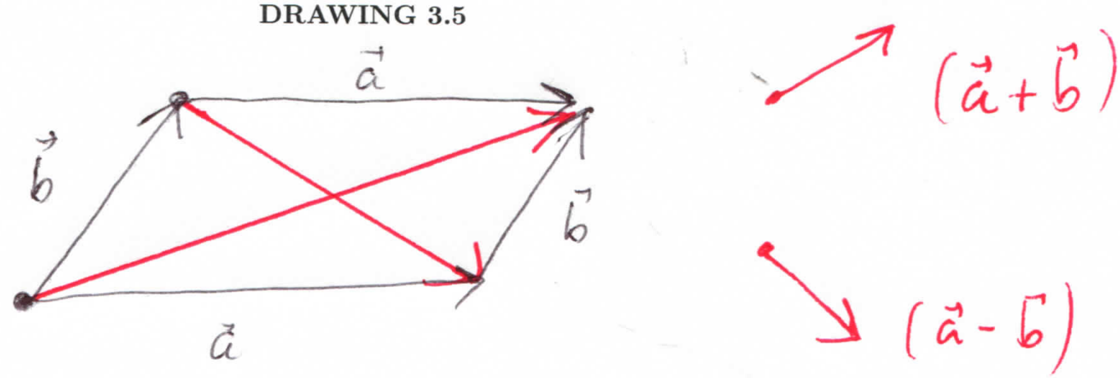
Geometry Theorem 3.4. The diagonals of a parallelogram are perpendicular $\iff$ all sides of the parallelogram have equal length (such a parallelogram is called a **rhombus**).

Proof: Let the parallelogram be as in DRAWING 3.3. The diagonals are $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ (see DRAWING 3.5 "below"); they are perpendicular if and only if

$$0 = (\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) \iff 0 = \vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} - \vec{b} \cdot \vec{b} \iff 0 = \|\vec{a}\|^2 - \|\vec{b}\|^2,$$

which is equivalent to $\|\vec{a}\| = \|\vec{b}\|$.

DRAWING 3.5



As an analogue of 3.4, we have the following.

Geometry Theorem 3.6. The diagonals of a parallelogram have equal length $\iff$ all interior angles of the parallelogram are right angles (such a parallelogram is called a **rectangle**).

Proof is left for homework. We recommend starting, analogous to the proof of 3.4, with $\|\vec{a}+\vec{b}\|^2 = \|\vec{a}-\vec{b}\|^2 \iff \ldots$

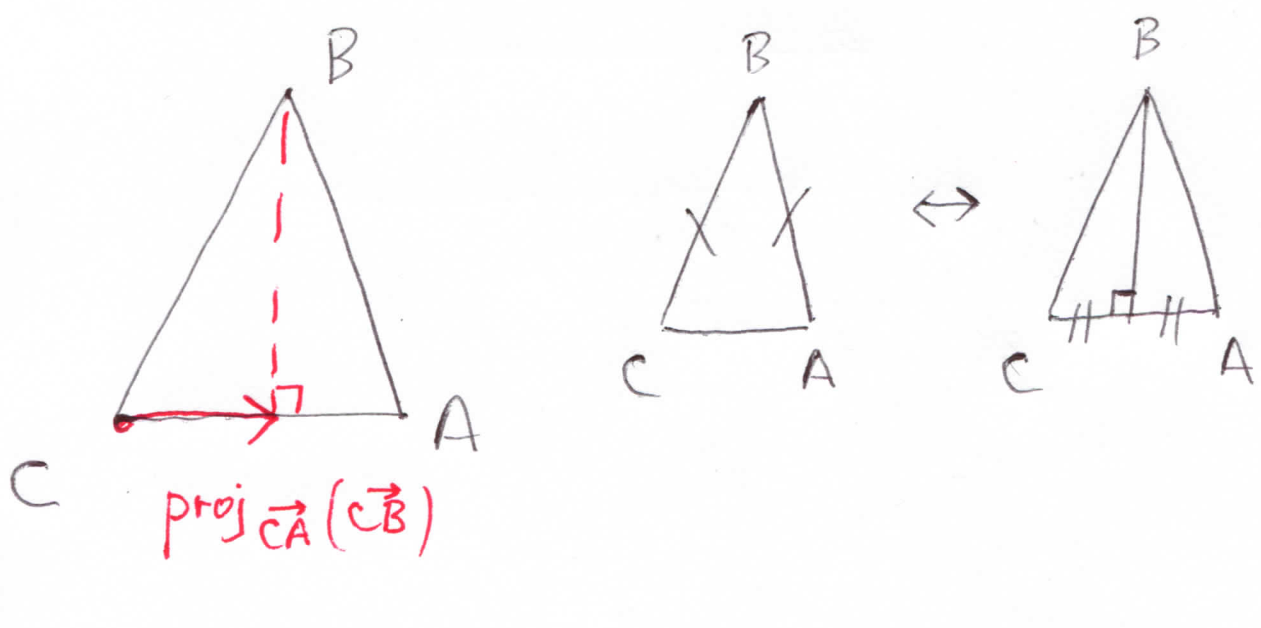
Geometry Theorem 3.7. (parallelogram law) In a parallelogram, the sum of the squares of the lengths of the diagonals equals the sum of the squares of the lengths of the sides.

Proof is left for homework. NOTE that, in DRAWINGS 3.3 and 3.5, "the sum of the squares of the lengths of the diagonals" is $\|\vec{a}+\vec{b}\|^2 + \|\vec{a}-\vec{b}\|^2$.

Geometry Theorem 3.8. Suppose a triangle has vertices A, B, and C (see drawing below). Then $\|\overrightarrow{BC}\| = \|\overrightarrow{BA}\| \iff$ the projection of $\overrightarrow{CB}$ onto $\overrightarrow{CA}$ is the midpoint of $\overrightarrow{CA}$.

Proof: Pythagorean theorem (Proposition 2.12 was needed to ensure that the projection $\text{proj}_{\overrightarrow{CA}}(\overrightarrow{CB})$ existed).

The triangle we have just characterized is called an **isosceles** triangle.

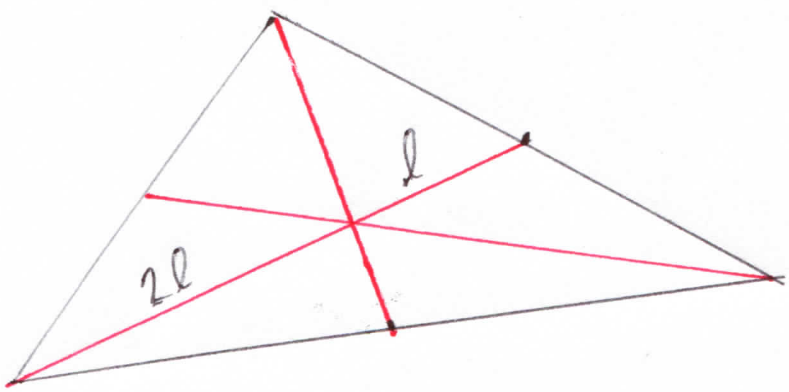


Discussion 3.9. An **equilateral triangle** is a triangle with all sides of equal length.

We reproduce here (see DRAWING 3.10) the following from the proof of [1, Geometry Theorem 12]:

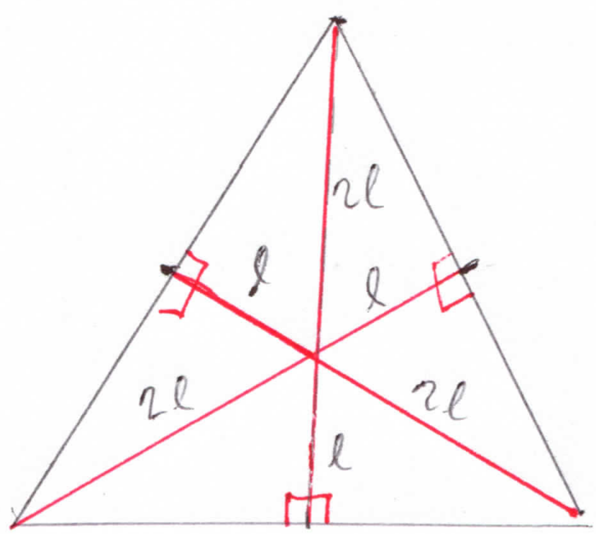
In any triangle, the lines from vertices to midpoints of opposite sides all intersect at the same point, two thirds of the way from vertex to opposite side.

DRAWING 3.10.



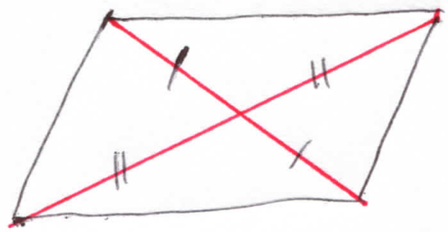
Combining Geometry Theorem 3.8 above with DRAWING 3.10 gives an extraordinary amount of symmetry for equilateral triangles, as drawn in DRAWING 3.11.

DRAWING 3.11.

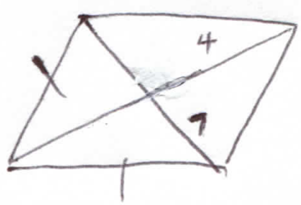


Examples 3.12. In each of the following, fill in lengths of sides and right angles, where possible. Assume all quadrilaterals are parallelograms.

Use the results of this chapter, including DRAWINGS 3.3, 3.5, and 3.10, combined with the following ([1, Geometry Theorem 18]): diagonals of a parallelogram bisect each other.



(a)



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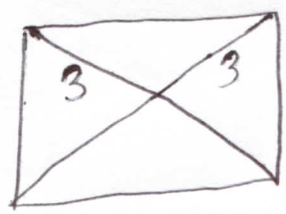

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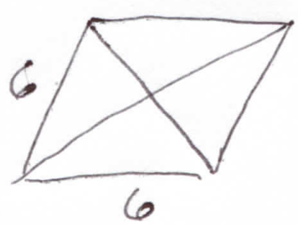

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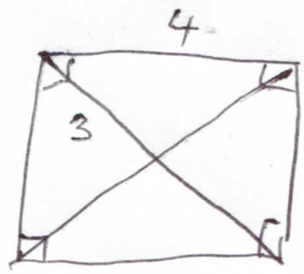

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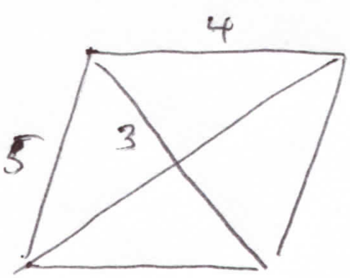

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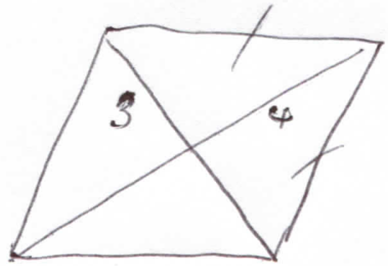

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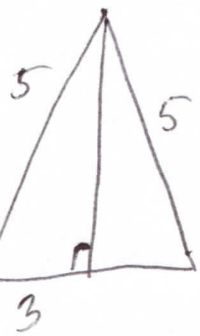

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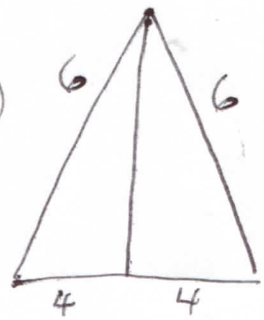

(i)

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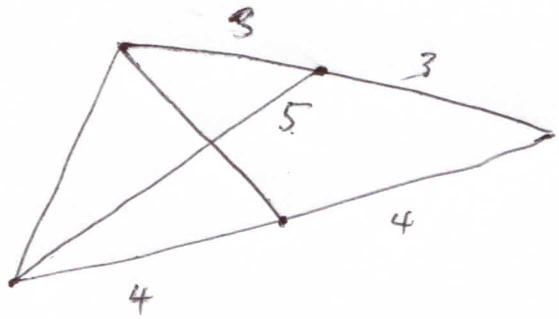

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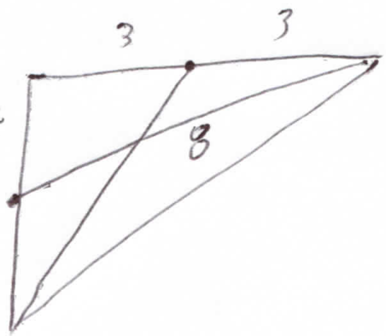

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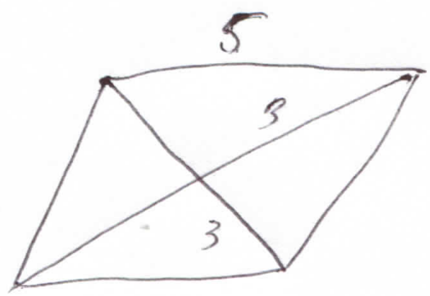

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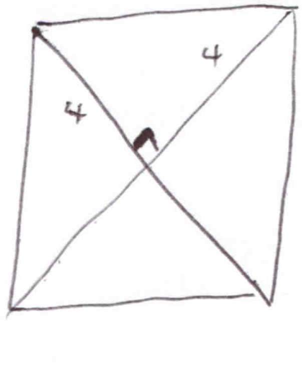

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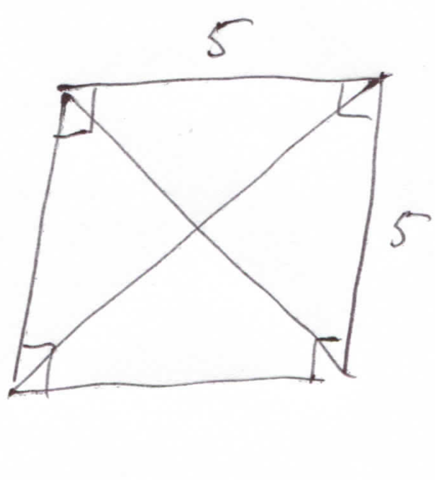

SOLUTIONS

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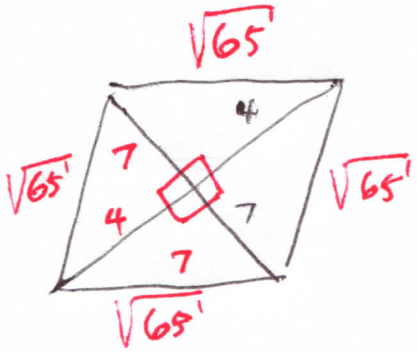
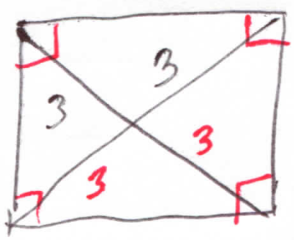


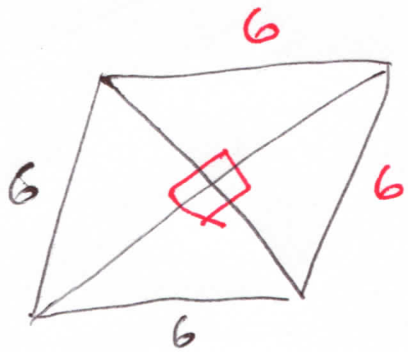
3.4, Pythag., [1, Thm. 18]

(b)



3.6, [1, Thm. 18]

(c)



3.3, 3.4

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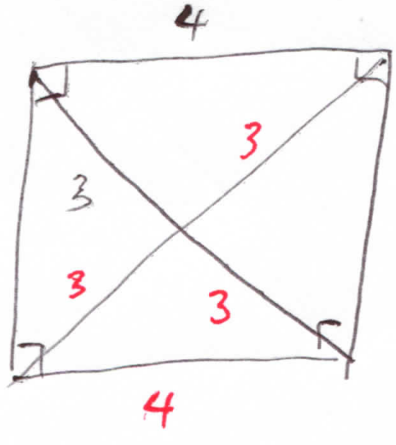
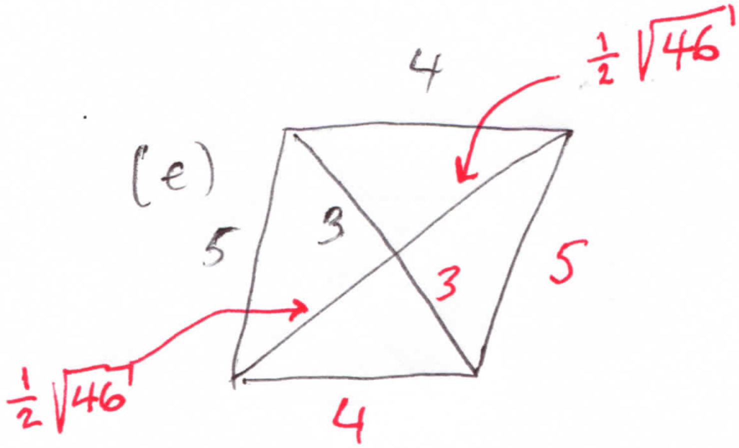


3.3, 3.6

(e)



3.3, 3.7

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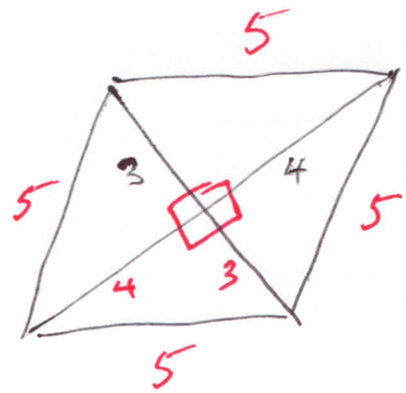
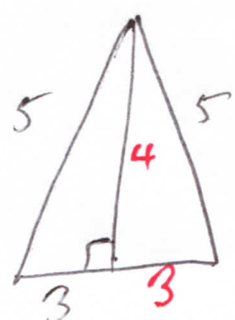


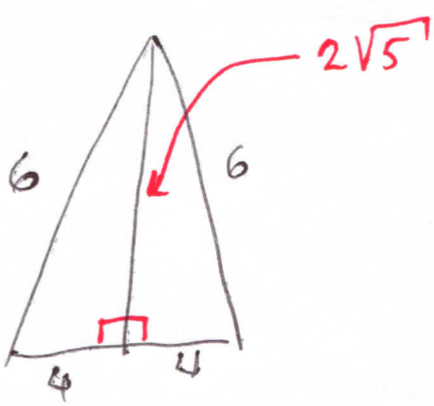
3.3, 3.4, Pythag.

(g)



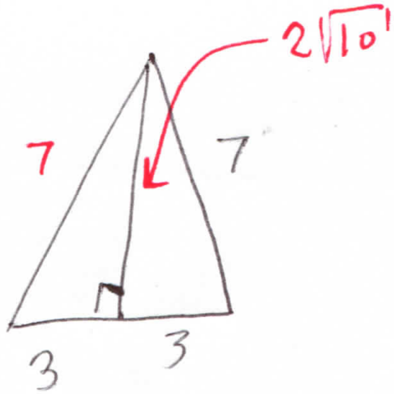
3.8, Pythag.

(h)



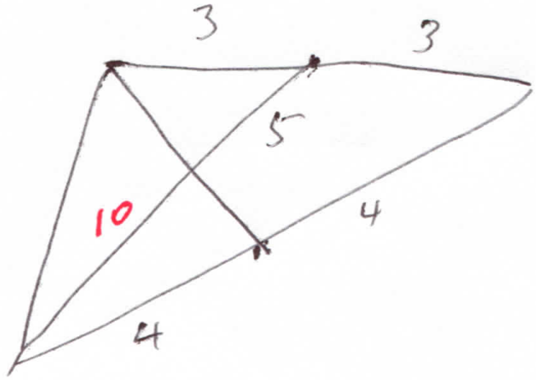
3.8, Pythag.

(i)



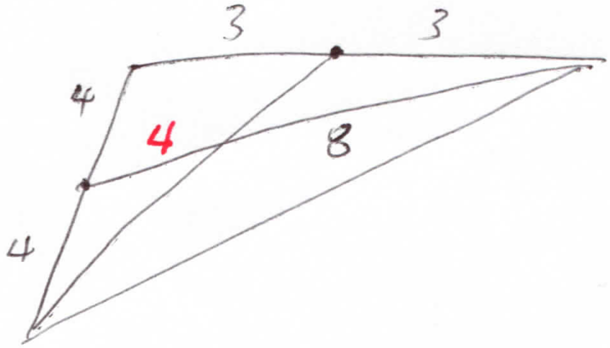
3.8, Pythag.

(j)



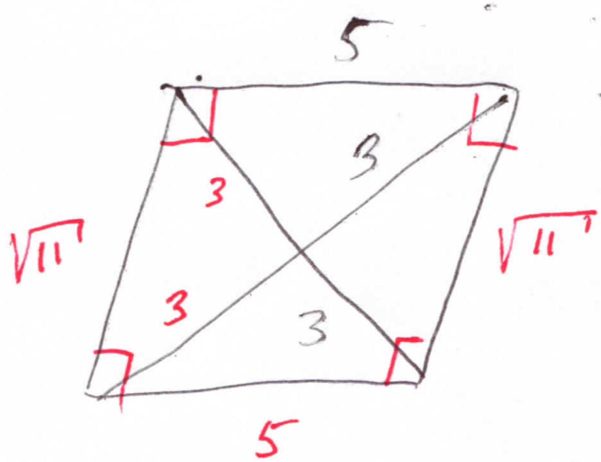
3.10

(k)



3.10

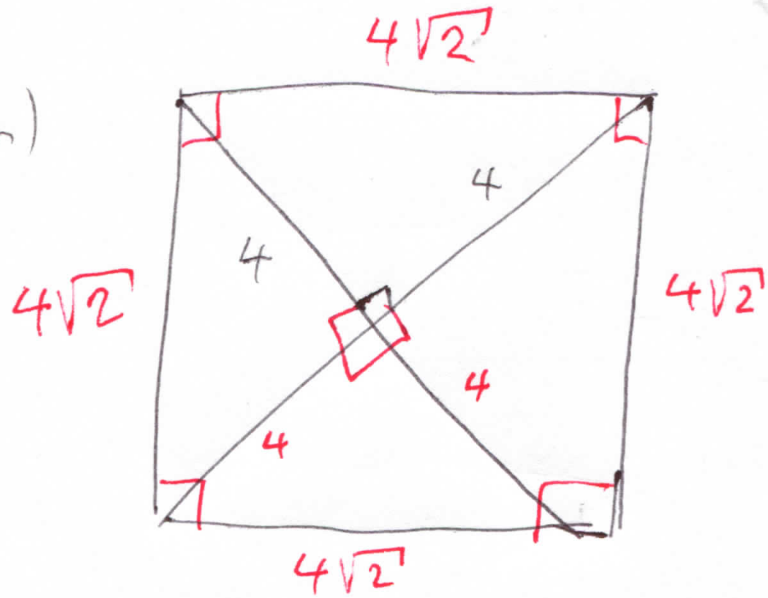
(l)



3.3, 3.6

Pythag.

(m)

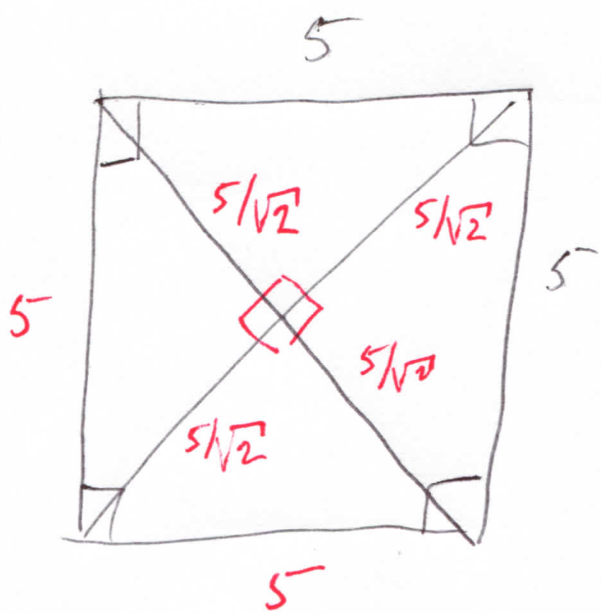


3.4, 3.6,

[1, Thm. 18],

Pythag.

(n)



3.3, 3.4,

3.6,

[1, Thm. 18]

HOMEWORK

1. Use vector methods, including DRAWINGS 3.3 and 3.5, to prove Geometry Theorem 3.6.

2. Use vector methods, including DRAWINGS 3.3 and 3.5, to prove Geometry Theorem 3.7.

3. Prove that $\|\vec{a} + t\vec{b}\| \geq \|\vec{a}\|$, for all real t if and only if $\vec{a} \perp \vec{b}$.

HINTS: Use 2.5(e) to show that $\vec{a} \perp \vec{b}$ is necessary for $\|\vec{a} + t\vec{b}\| \geq \|\vec{a}\|$, by plugging in $t = \left(-\frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\right)$ into $\|\vec{a} + t\vec{b}\|^2$. For $\vec{a} \perp \vec{b}$, use the Pythagorean theorem to show the inequality.

4. Let $\vec{a}$ and $\vec{b}$ be arbitrary nontrivial (not equal to $< 0, 0 >$) vectors.

(a) Use 2.5(e) or the Pythagorean theorem to show that

$$\|\vec{a}\|^2 = \|\text{proj}_{\vec{b}}(\vec{a})\|^2 + \|\vec{a} - \text{proj}_{\vec{b}}(\vec{a})\|^2.$$

(b) Use (a) to show that

$$\|\vec{a}\| \geq \|\text{proj}_{\vec{b}}(\vec{a})\|.$$

(c) Use (b) and Proposition 2.12 to prove the *Cauchy inequality*:

$$|\vec{a} \cdot \vec{b}| \leq \|\vec{a}\|\|\vec{b}\|.$$

(d) Use (c) and 2.5(e) to prove the *triangle inequality*:

$$\|\vec{a} + \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|.$$

See the drawing of vector sums in Definitions 1.6 for the geometry of the triangle inequality.

5. Get each of the following projections.

(a) $\text{proj}_{<2,-3>}(< 1, 0 >)$.

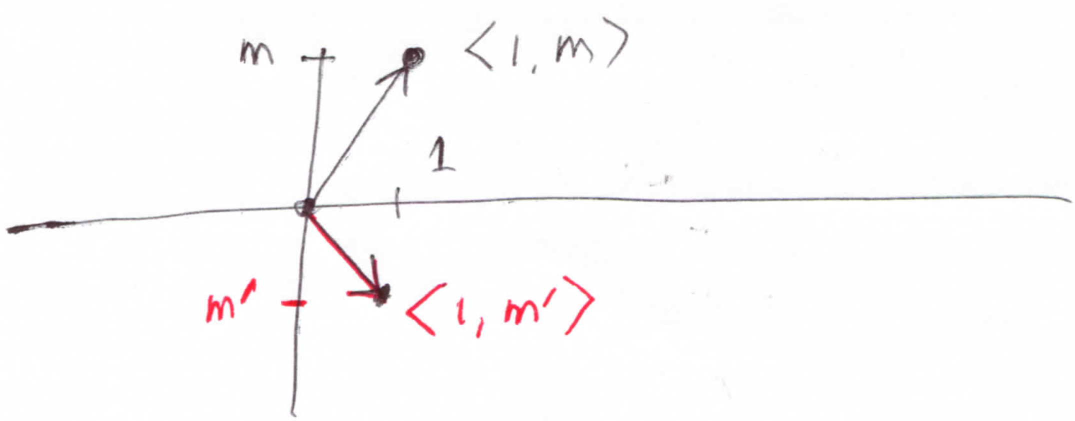
(b) $\text{proj}_{<1,0>}(< 2, -3 >)$.

(c) $\text{proj}_{<3,2>}(< 2, -3 >)$.

(d) $\text{proj}_{<6,-9>}(< 2, -3 >)$.

6. Suppose, for some arbitrary number m, $< 1, m' >$ is perpendicular to $< 1, m >$. Find m'.

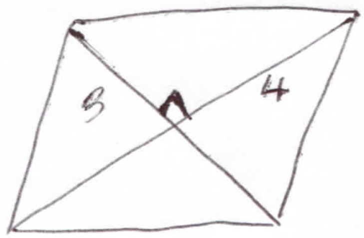
This can be used to get the slope of a line perpendicular to a specified line.



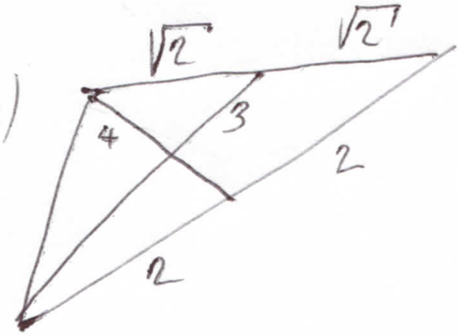
7. Write $< 1, -8 >$ as a sum $\vec{a}_1 + \vec{a}_2$, where $\vec{a}_1$ is parallel to $< 2, -1 >$ and $\vec{a}_2$ is perpendicular to $< 2, -1 >$.

8. In each of the following, fill in lengths of sides and right angles, where possible, as instructed in Examples 3.12. Assume all quadrilaterals are parallelograms.

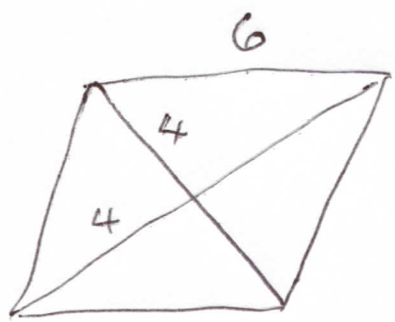
(a)



(b)



(c)



(d)

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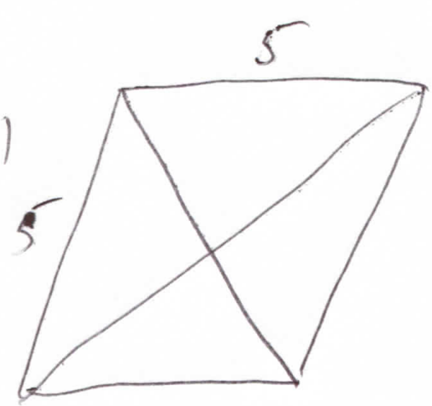
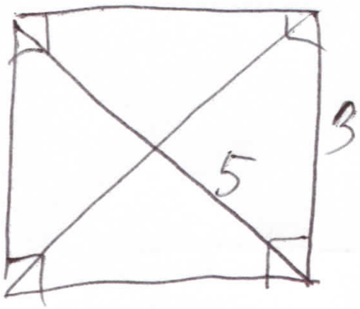


(f)

(g)



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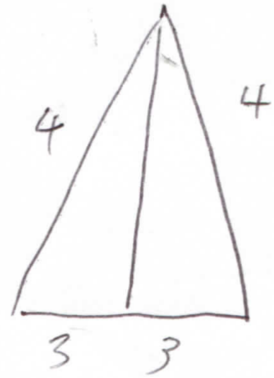

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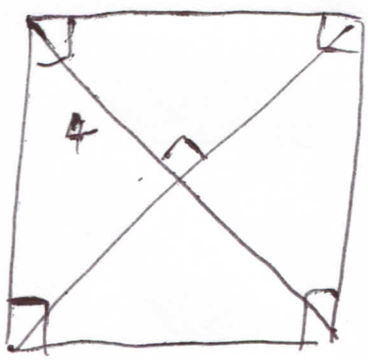

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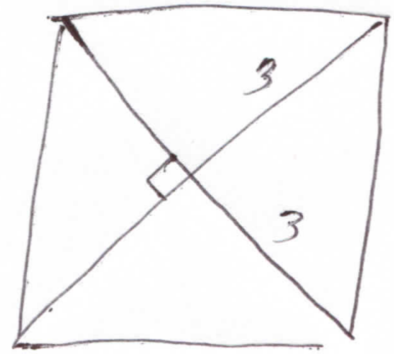
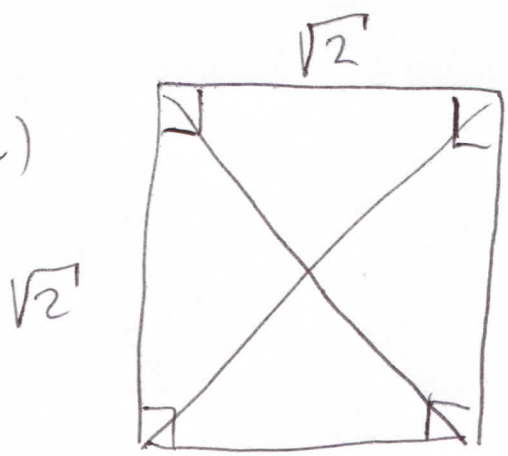


(k)



HOMEWORK ANSWERS

1. Let $\vec{a}$ and $\vec{b}$ be as in DRAWING 3.3 and 3.5. Then the diagonals of the parallelogram have equal length if and only if

$$\|\vec{a}+\vec{b}\|^2 = \|\vec{a}-\vec{b}\|^2 \iff (2.5(e))\|\vec{a}\|^2+\|\vec{b}\|^2+2\vec{a}\cdot\vec{b} = \|\vec{a}\|^2+\|-\vec{b}\|^2+2\vec{a}\cdot(-\vec{b}) \iff (2.5(b)\text{and}(d))$$

$$\vec{a}\cdot\vec{b}=0,$$

which is equivalent to the parallelogram being a rectangle.

2. Let $\vec{a}$ and $\vec{b}$ be as in DRAWING 3.3 and 3.5. The sum of the squares of the lengths of the diagonals equals

$$\|\vec{a}+\vec{b}\|^2+\|\vec{a}-\vec{b}\|^2 = (2.5(e))\left(\|\vec{a}\|^2+\|\vec{b}\|^2+2\vec{a}\cdot\vec{b}\right)+\left(\|\vec{a}\|^2+\|-\vec{b}\|^2+2\vec{a}\cdot(-\vec{b})\right) = (2.5(b)\text{and}(d))$$

$$2\|\vec{a}\|^2+2\|\vec{b}\|^2,$$

the sum of the squares of the lengths of the sides.

3. If $\vec{a}\perp\vec{b}$, then for any real t, by the Pythagorean theorem,

$$\|\vec{a}+t\vec{b}\|^2 = \|\vec{a}\|^2+\|t\vec{b}\|^2 \geq \|\vec{a}\|^2.$$

Conversely, if $\vec{a}$ is not perpendicular to $\vec{b}$, then $\vec{a}\cdot\vec{b}\neq 0$, thus, by 2.5(e) and (b), with $t=\left(-\frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\right)$,

$$\|\vec{a}+t\vec{b}\|^2 = \|\vec{a}\|^2+\|t\vec{b}\|^2+2\vec{a}\cdot(t\vec{b}) = \|\vec{a}\|^2+t^2\|\vec{b}\|^2+2t(\vec{a}\cdot\vec{b}) = \|\vec{a}\|^2+\left(-\frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\right)^2\|\vec{b}\|^2+2\left(-\frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\right)(\vec{a}\cdot\vec{b})$$

$$= \|\vec{a}\|^2-\left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|}\right)^2 < \|\vec{a}\|^2.$$

For those who have seen calculus, $t=\left(-\frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\right)$ is the value of t where the derivative of

$$F(t)\equiv\|\vec{a}\|^2+t^2\|\vec{b}\|^2+2t(\vec{a}\cdot\vec{b})$$

equals zero and the second derivative is positive, thus

$$F\left(-\frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\right)$$

is a minimum value for $F(t)$. For those who have seen the technique of completing the square,

$$F(t)\equiv\|\vec{a}\|^2+t^2\|\vec{b}\|^2+2t(\vec{a}\cdot\vec{b}) = \|\vec{a}\|^2+\|\vec{b}\|^2\left[\left(t+\frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\right)^2-\left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\right)^2\right],$$

with a minimum of

$$F\left(-\frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\right) = \|\vec{a}\|^2-\left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|}\right)^2.$$

4. (a) By the Pythagorean theorem and the geometric definition of $\text{proj}_{\vec{b}}(\vec{a})$,

$$\|\vec{a}\|^2 = \|\text{proj}_{\vec{b}}(\vec{a})+(\vec{a}-\text{proj}_{\vec{b}}(\vec{a}))\|^2 = \|\text{proj}_{\vec{b}}(\vec{a})\|^2+\|(\vec{a}-\text{proj}_{\vec{b}}(\vec{a}))\|^2.$$

(b)

$$\|\vec{a}\|^2 = \|\text{proj}_{\vec{b}}(\vec{a})\|^2+\|(\vec{a}-\text{proj}_{\vec{b}}(\vec{a}))\|^2 \geq \|\text{proj}_{\vec{b}}(\vec{a})\|^2.$$

(c) By (b), 2.5(b) and (d), and Proposition 2.12,

$$\|\vec{a}\|^2 \geq \|\text{proj}_{\vec{b}}(\vec{a})\|^2 = \| \left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2} \right) \vec{b}\|^2 = \left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|} \right)^2,$$

thus

$$\|\vec{a}\|\|\vec{b}\| \geq |\vec{a}\cdot\vec{b}|.$$

(d) By (c) and 2.5(e),

$$\|\vec{a}+\vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 + 2\vec{a}\cdot\vec{b} \leq \|\vec{a}\|^2 + \|\vec{b}\|^2 + 2\|\vec{a}\|\|\vec{b}\| = (\|\vec{a}\| + \|\vec{b}\|)^2.$$

5. (a) $\frac{<1,0>\cdot<2,-3>}{\|<2,-3>\|^2} < 2,-3 >= \frac{2}{13} < 2,-3 >=< \frac{4}{13}, -\frac{6}{13} >$.

(b) $\frac{<2,-3>\cdot<1,0>}{\|<1,0>\|^2} < 1,0 >= 2 < 1,0 >=< 2,0 >$.

(c) $\frac{<2,-3>\cdot<3,2>}{\|<3,2>\|^2} < 3,2 >=< 0,0 >$.

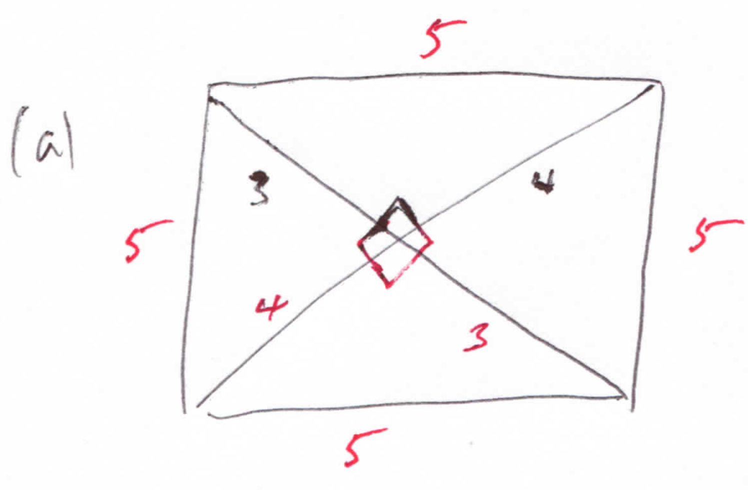
(d) $\frac{<2,-3>\cdot<6,-9>}{\|<6,-9>\|^2} < 6,-9 >=< 2,-3 >$.

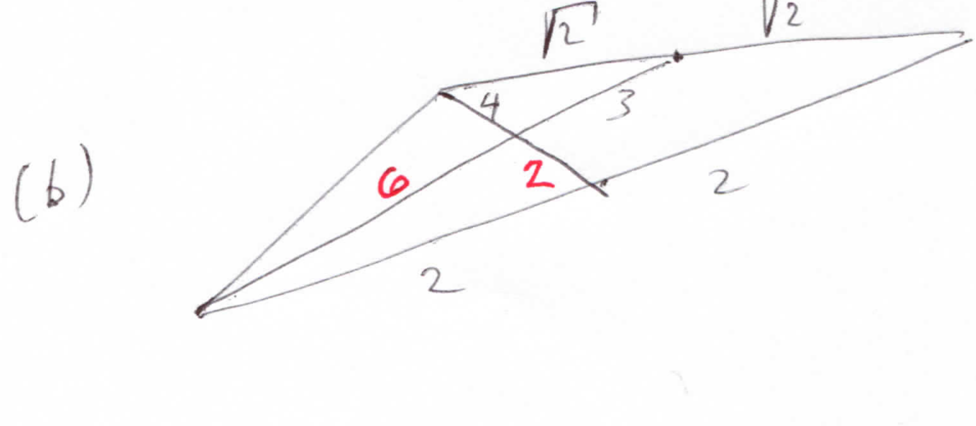
6. $0 =< 1,m > \cdot < 1,m' >= 1 + m\cdot m'$; solving for m' gives

$$m' = -\frac{1}{m}.$$

7. $\vec{a}_1 = \left(\frac{<1,-8>\cdot<2,-1>}{\|<2,-1>\|^2} \right) < 2,-1 >=< 4,-2 >,\ \vec{a}_2 =< 1,-8 > - < 4,-2 >=< -3,-6 >$.

8.





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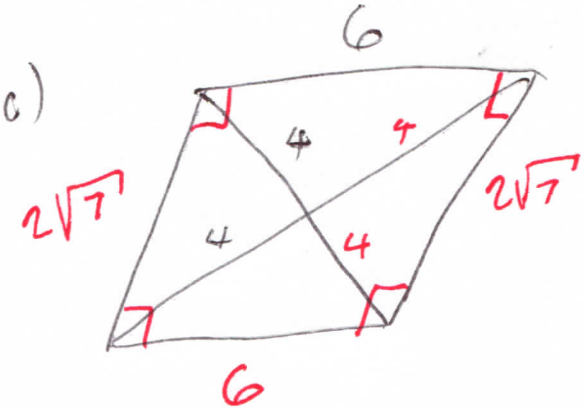

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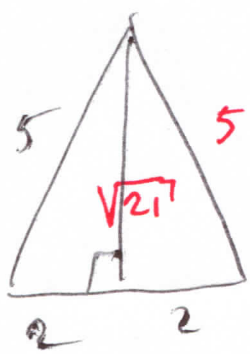

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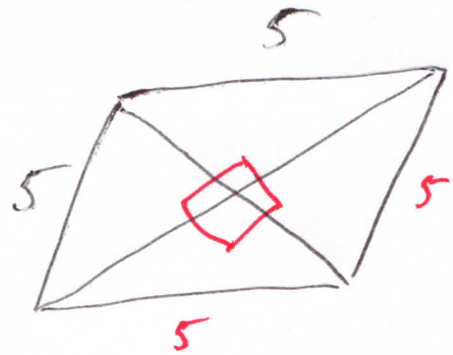

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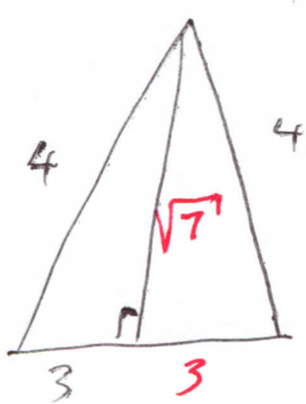

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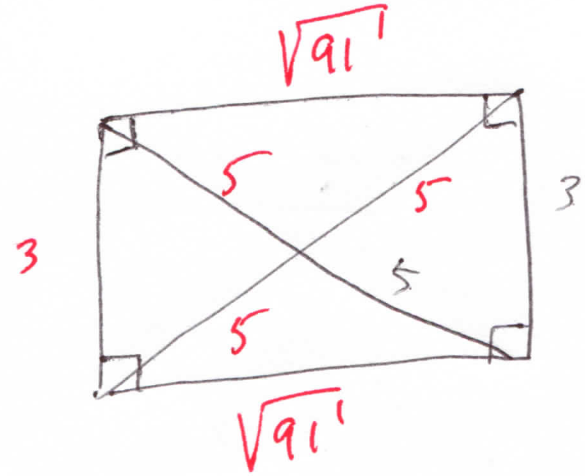

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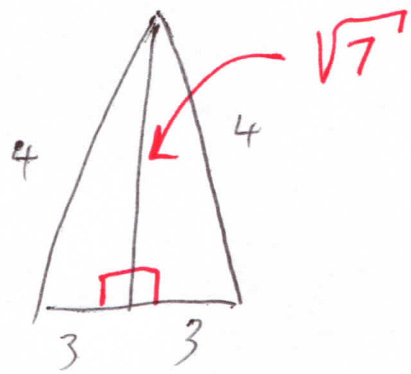

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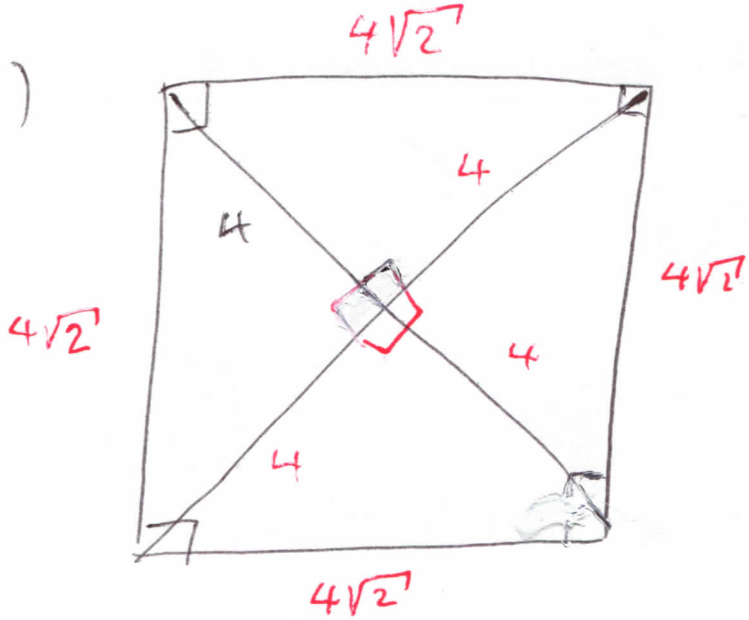

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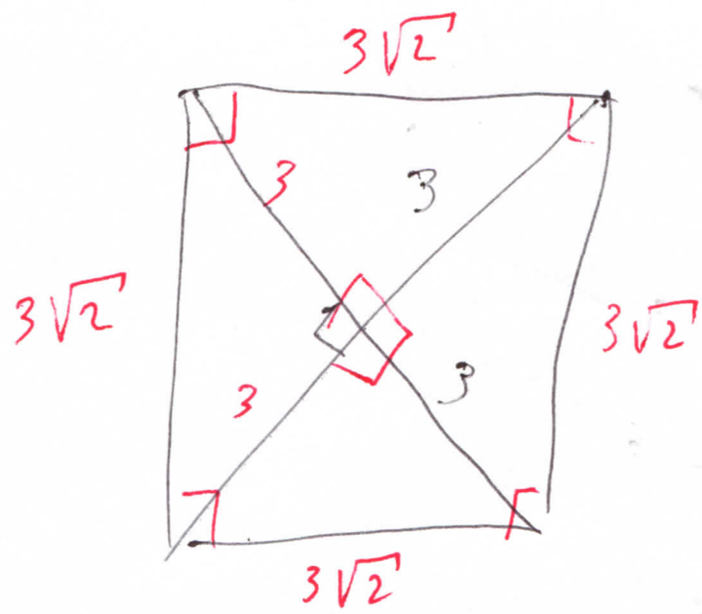

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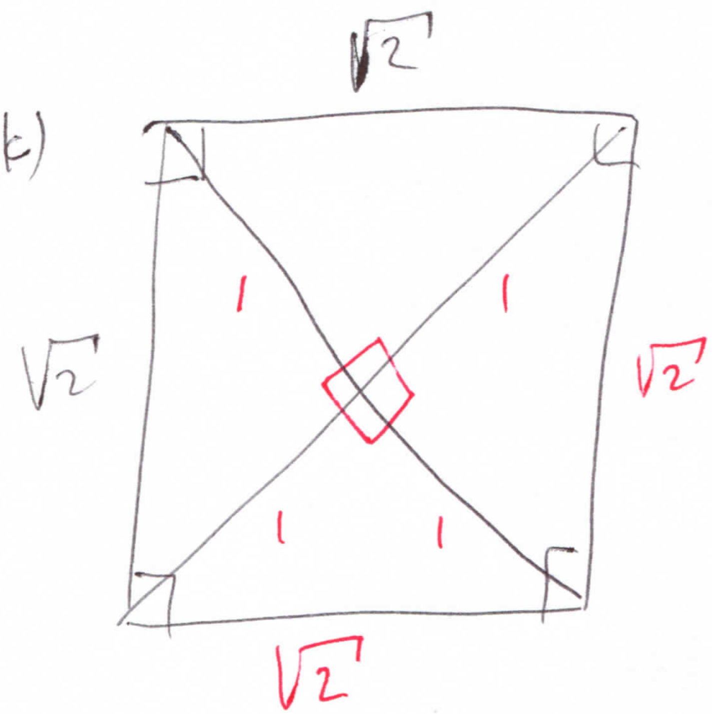

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