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Vectors and Physics
MATHematics MAGnification™

Dr. Ralph deLaubenfels

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Teacher-Scholar Institute

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VECTORS and PHYSICS MAGNIFICATION

This is one of a series of very short books on math, statistics, and physics called "Math Magnifications." The "magnification" refers to focusing on a particular topic that is pivotal in or emblematic of mathematics.

OUTLINE

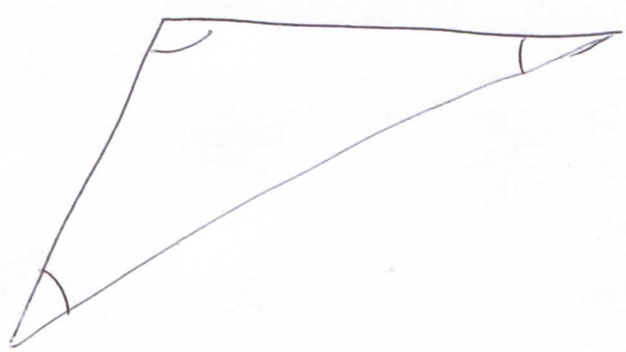
A vector has both magnitude and direction and is represented by an arrow. Some examples in physics of vectors are velocity and force. Speed and energy, having magnitude but not direction, are not vectors.

After defining (informally) and drawing vectors and their operations, we will use vectors to discuss swimming in the ocean with a current, bicycle stability, and friction.

This magnification will illustrate a common and useful theme in mathematics, the intertwining of algebra and geometry.

For this magnification, students should be familiar with arithmetic, the Cartesian plane and equations of lines in said plane, and should know about angles formed by two line segments meeting at a point, as in the interior angles in a triangle (see drawing below). Any additional geometry needed will be summarized in Chapter II.

Prerequisites for this magnification are first-year high school algebra, such as may be found in [6]. Students should have access to a calculator that can calculate square roots.

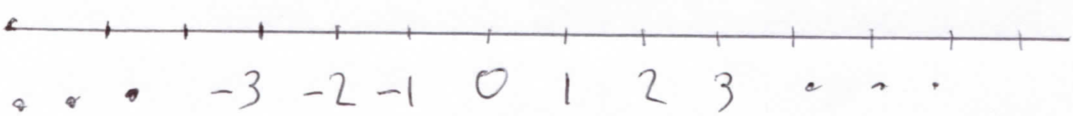


I. VECTORS

Suppose I'm traveling on the north-south freeway Interstate 3.14, and I tell you that my speed is 60 miles per hour. There is important missing information; namely, am I going north or south?

This illustrates that, even in one dimension, in addition to *magnitude* (60 miles per hour), there is also *direction* (north or south).

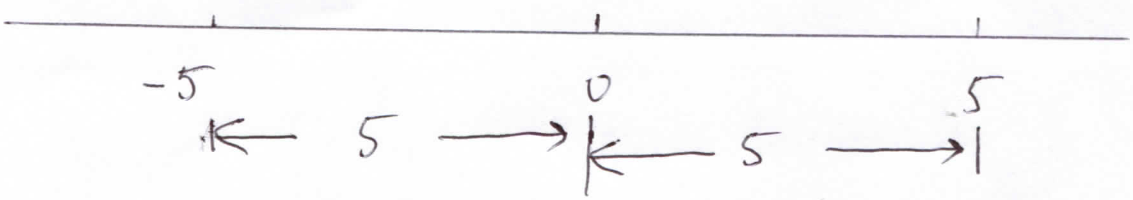
In the famous (real) *number line*, moving to the right corresponds to increasing numbers, to the left decreasing numbers.



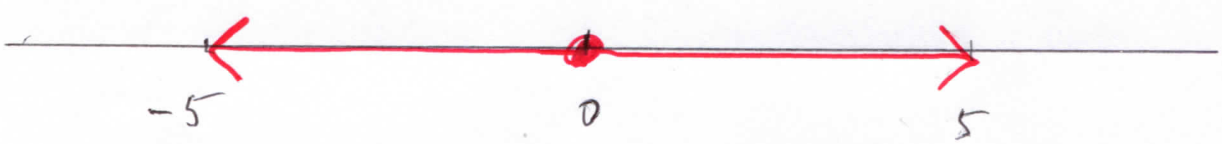
For example, 5 is different than (-5), although their magnitudes

$$|5| = 5 = |-5|$$

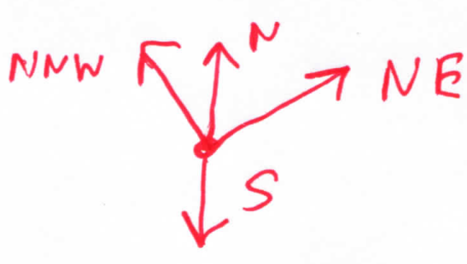
are equal.



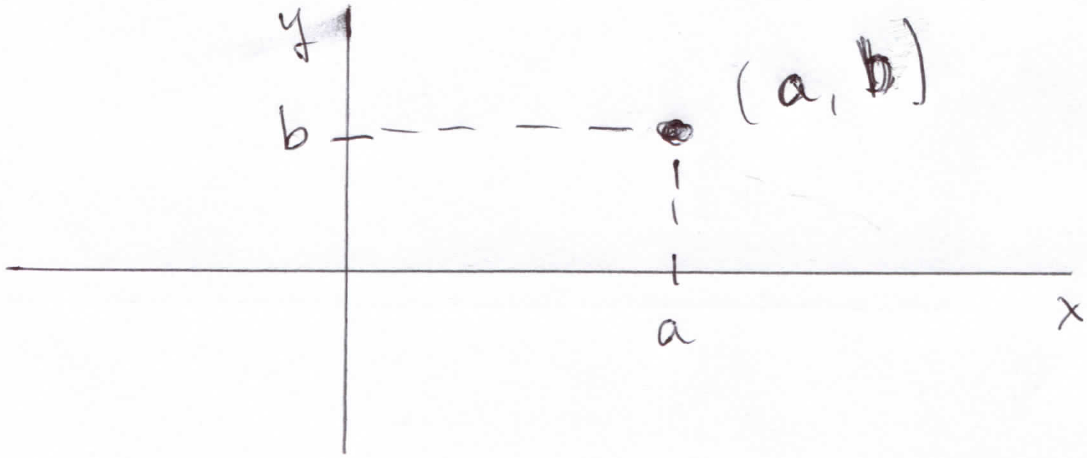
We can emphasize this difference by drawing arrows, first from 0 to (-5), then from 0 to 5; these arrows illustrate the different directions of 5 and (-5), and may be considered *one-dimensional vectors.*



In one dimension we have only two directions, north versus south, to the right versus to the left, etc. In two dimensions, meaning a plane, we have infinitely many directions: north, south, northeast, north by northwest (famous movie), all the points of the compass.



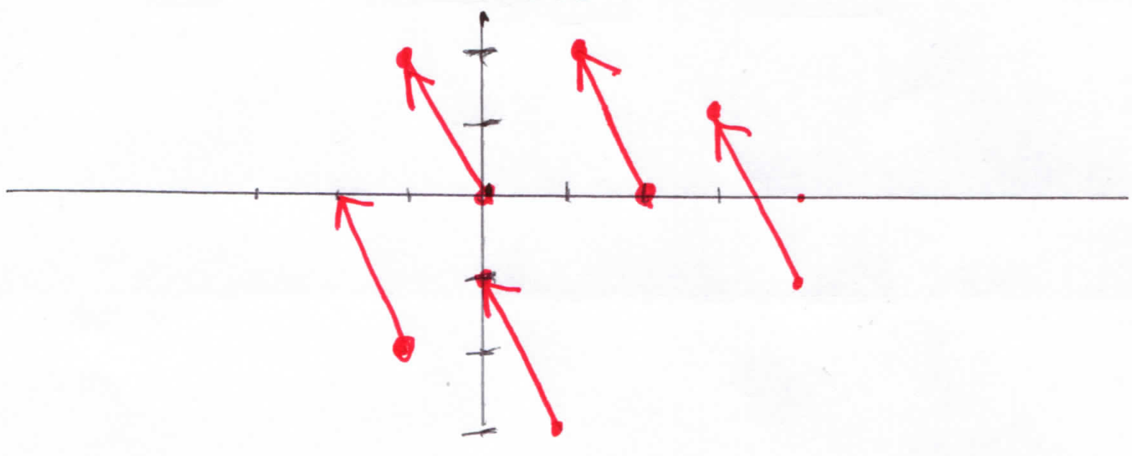
We wish to dwell in two dimensions for the rest of this magnification. We assume familiarity with the *Cartesian plane*, hereafter denoted *the plane*, where, for any real a, b, the ordered pair (a, b) represents the point a units to the right of the origin and b units above the origin. a is called the **x coordinate** of (a, b), b is called the **y coordinate** of (a, b).



Definition 1.1. A (two-dimensional) **vector** is (more precisely, is **represented by**) a directed line segment in the plane, with the direction indicated by a fat dot as the starting point, an arrowhead as the finishing point. Two directed line segments represent the same vector if they have the same length and direction.

In the drawing below, each arrow represents the same vector.

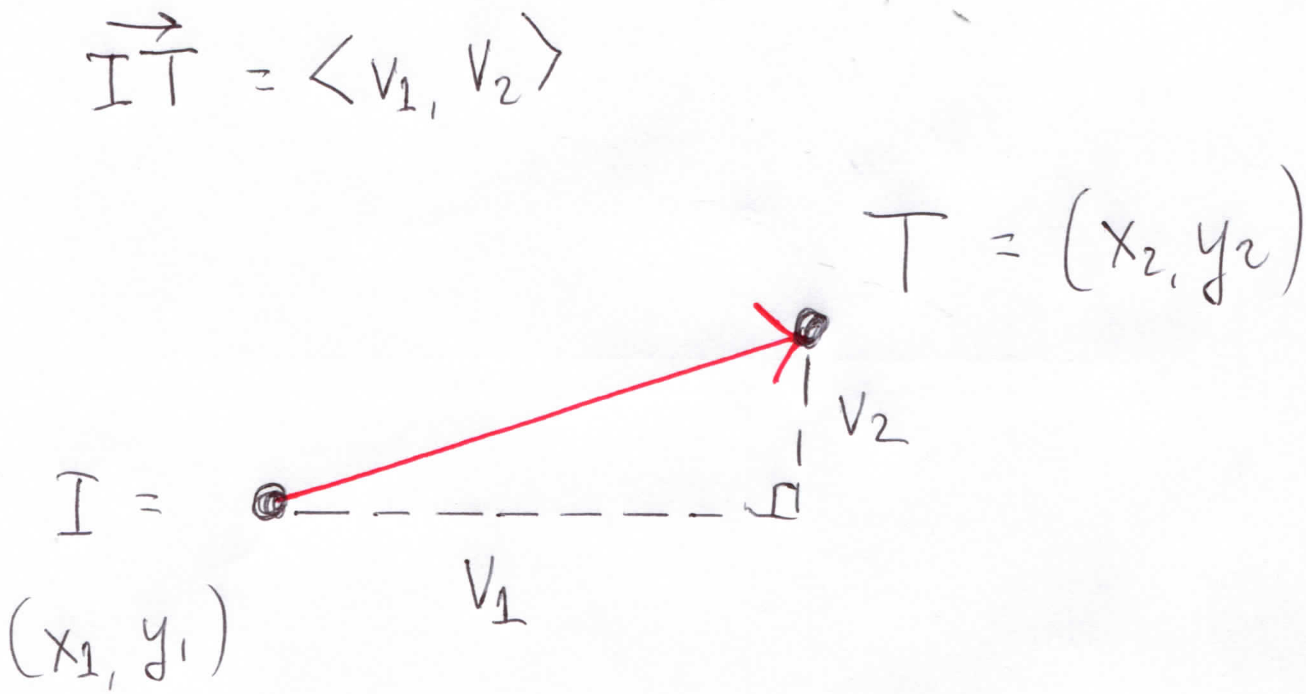
DRAWING 1.2



See [3] or [4] for much more about vectors, including more precise definitions. References [1] and [2] have informal introductions to vectors.

We need to make "length" and "direction" more explicit. Here is a close-up of a vector:

DRAWING 1.3



Definitions 1.4. I is the **initial point** of the vector drawn directly above in DRAWING 1.3, T is the **terminal point** of the vector, and the vector is denoted $\overrightarrow{IT}$. If $I = (x_1, y_1)$ and $T = (x_2, y_2)$, then the **components** of $\overrightarrow{IT}$ are $< (x_2 - x_1), (y_2 - y_1) >$; $v_1 \equiv (x_2 - x_1)$ is the **x component** of $\overrightarrow{IT}$ and $v_2 \equiv (y_2 - y_1)$ is the **y component** of $\overrightarrow{IT}$.

Thus the vector drawn (repeatedly) in DRAWING 1.2 is $< -1, 2 >$. Referring to specific arrows in DRAWING 1.2, $< -1, 2 >=< (-1-0), (2-0) >=< (0-1), (2-0) >=< (1-2), (-1-(-3)) >$, etc.

Neither the initial point nor the terminal point, in isolation, describes a vector; a vector is the *displacement*, that is, change, in traveling from the initial point to the terminal point. The x component is the displacement in the x coordinate and the y component is the displacement in the y coordinate.

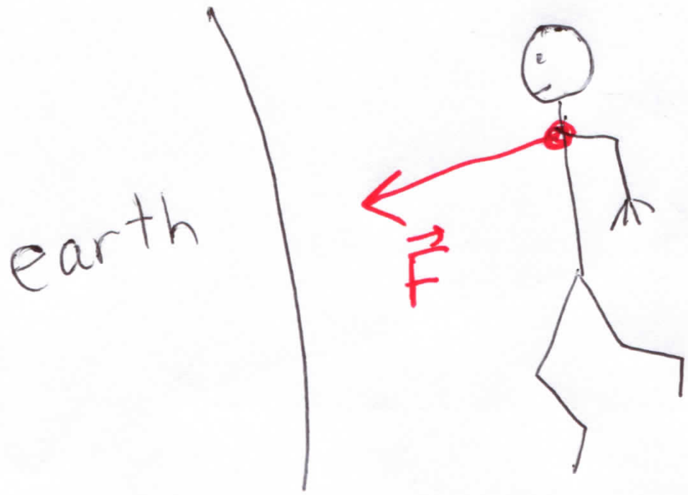
More generally, here is an algebraic characterization of two directed line segments representing the same vector.

Proposition 1.5. Two directed line segments represent the same vector if and only if their components are the same.

Terminology 1.6. If a vector $\vec{v}$ has x component v_1 and y component v_2, we denote

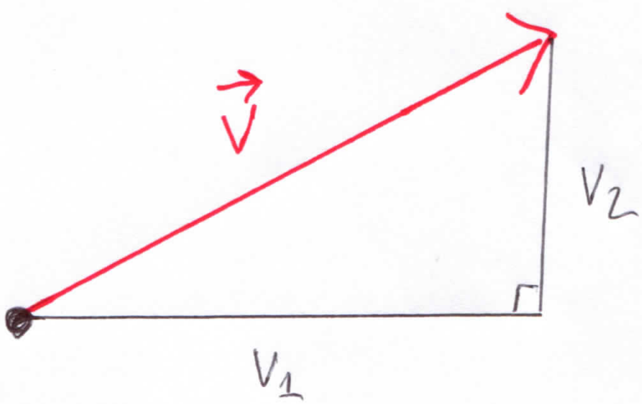
$$\vec{v} =< v_1, v_2 > .$$

It is very convenient to be able to move vectors around, as in DRAWING 1.2. For example, if you're in space, close enough to earth to be affected by the pull of earth's gravity, it is natural to place the initial point of that force on your body, because that is where you *feel* the force pulling you to earth.



Definitions 1.7. The **norm** or **magnitude** of a vector $\vec{v} \equiv < v_1, v_2 >$ is

$$\|\vec{v}\| \equiv \sqrt{v_1^2 + v_2^2}.$$



Notice that the norm of $\vec{v}$ is the length of an arrow (directed line segment) representing $\vec{v}$.

A vector of norm one is called a **unit vector**. By standardizing the length to be one, we may focus on the *direction* of a vector.

Examples 1.8. (a) If $\vec{v} \equiv < 3, 4 >$, then $\|\vec{v}\| = \sqrt{3^2 + 4^2} = 5$.

We will leave it to the reader to calculate that the norm of $< \frac{3}{5}, \frac{4}{5} >$ is one. Thus $< \frac{3}{5}, \frac{4}{5} >$ is a unit vector pointing in the same direction as $\vec{v}$. See Definitions 1.11.

(b) Velocity is a vector, while speed is the norm of velocity.

Definitions 1.9: Vector Algebra. We may add vectors:

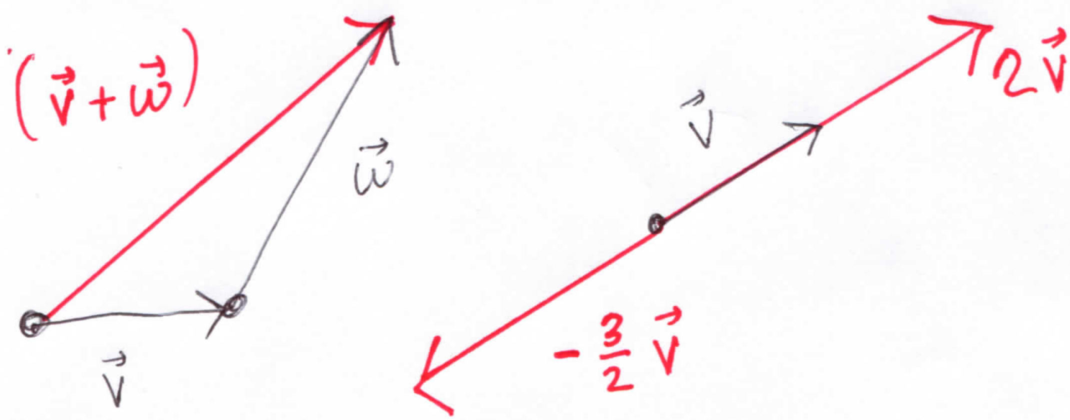
$$< v_1, v_2 > + < w_1, w_2 > \equiv < (v_1 + w_1), (v_2 + w_2) >,$$

and multiply vectors by real numbers:

$$\alpha < v_1, v_2 > \equiv < \alpha v_1, \alpha v_2 > .$$

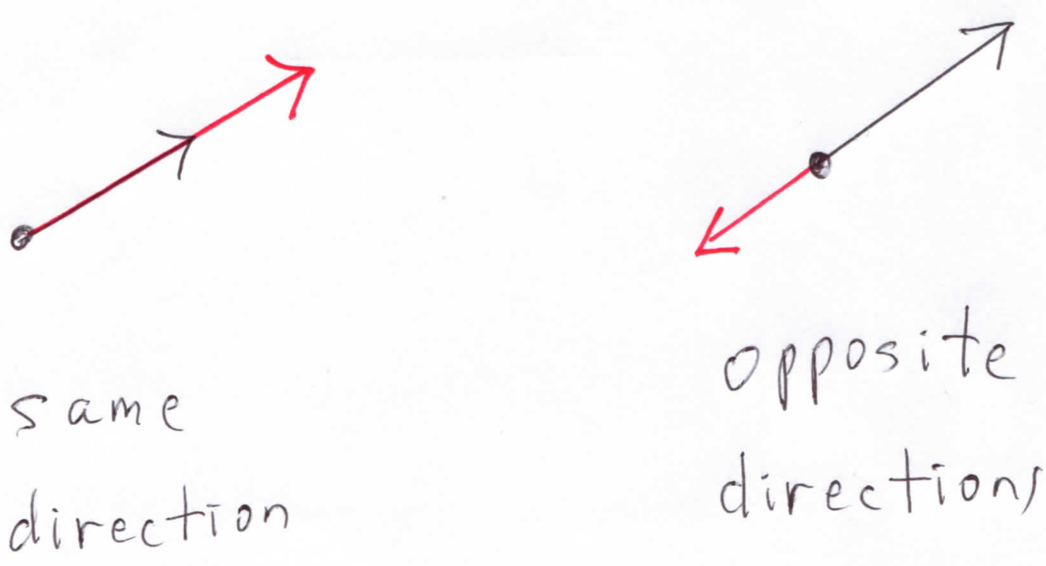
Examples 1.10. $< 1, 2 > + < 3, -4 > = < 4, -2 >$; $(-2) < 3, -1 > = < -6, 2 >$.

Definitions 1.11: Geometry of Vector Algebra. Here are the pictures of the directed line segments representing vector algebra.



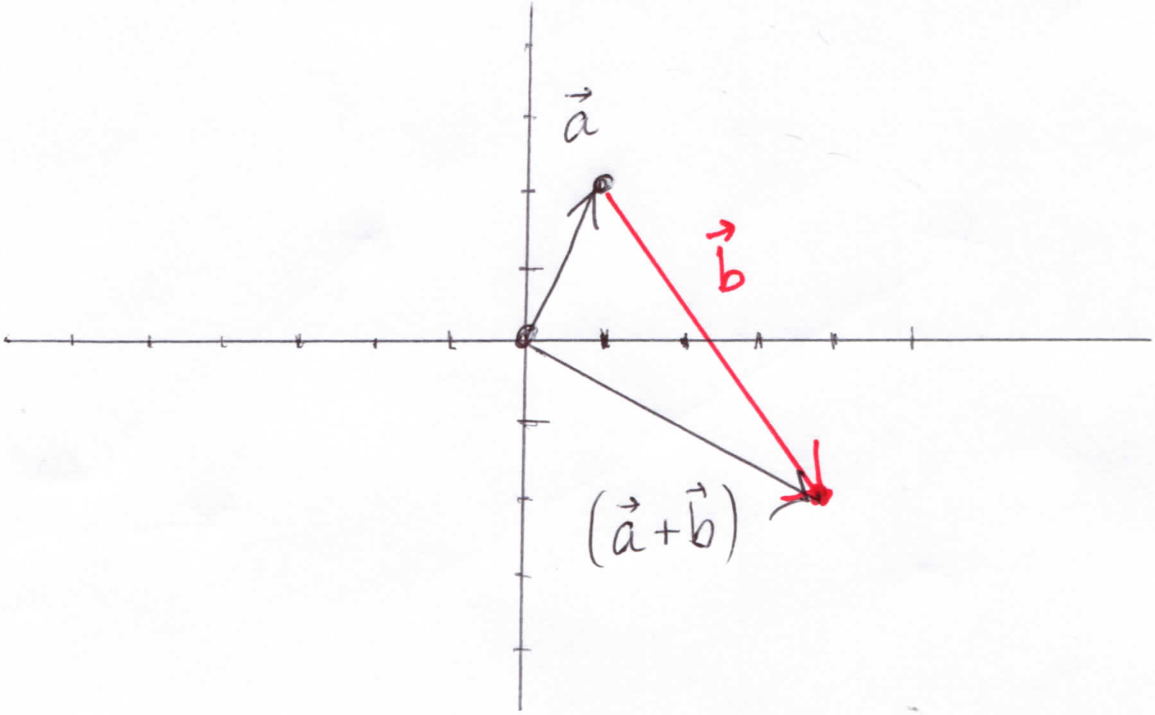
The picture of multiplying a number times a vector suggests the following definitions.

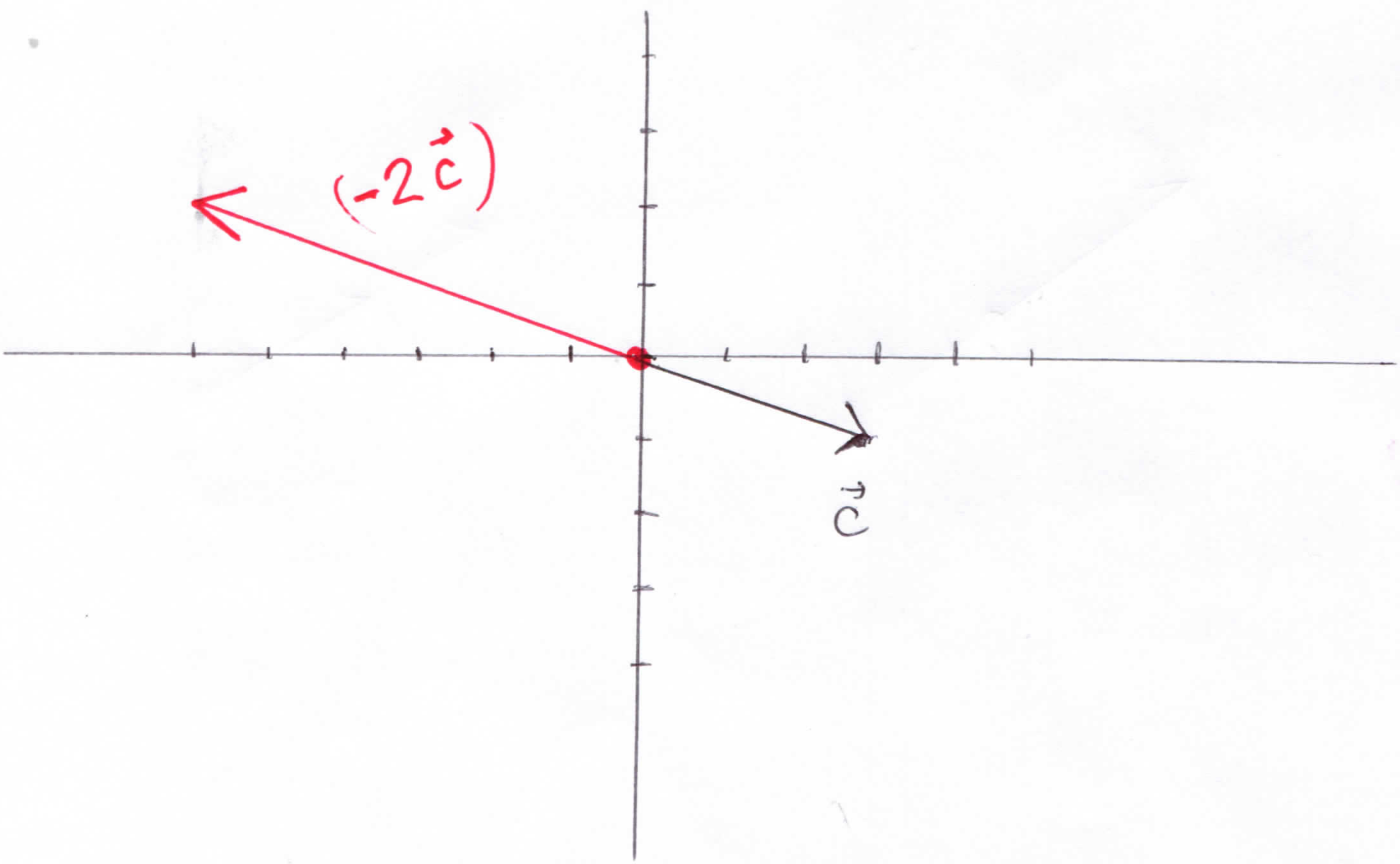
Two vectors $\vec{v}, \vec{w}$ are **parallel** if one is a real multiple of the other: $\vec{v} = \alpha\vec{w}$ or $\vec{w} = \alpha\vec{v}$.
If $\alpha > 0$, we say that $\vec{v}$ and $\vec{w}$ **point in the same direction.** If $\alpha < 0$, we say that $\vec{v}$ and $\vec{w}$ **point in opposite directions.**



Examples 1.12. Drawn below is the geometry of the operations in Examples 1.10, with

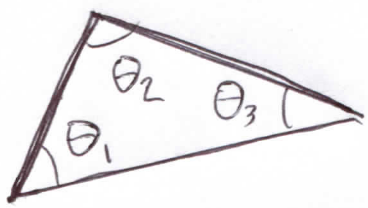
$$\vec{a} \equiv < 1, 2 >, \quad \vec{b} \equiv < 3, -4 >, \quad \text{and} \quad \vec{c} \equiv < 3, -1 > .$$





II. GEOMETRY ASSUMPTIONS

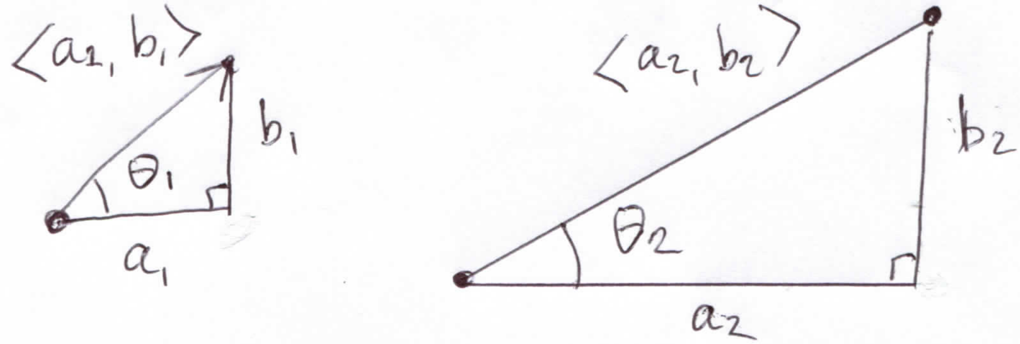
Geometry Assumption 2.1. The interior angles in a triangle add up to 180 degrees.



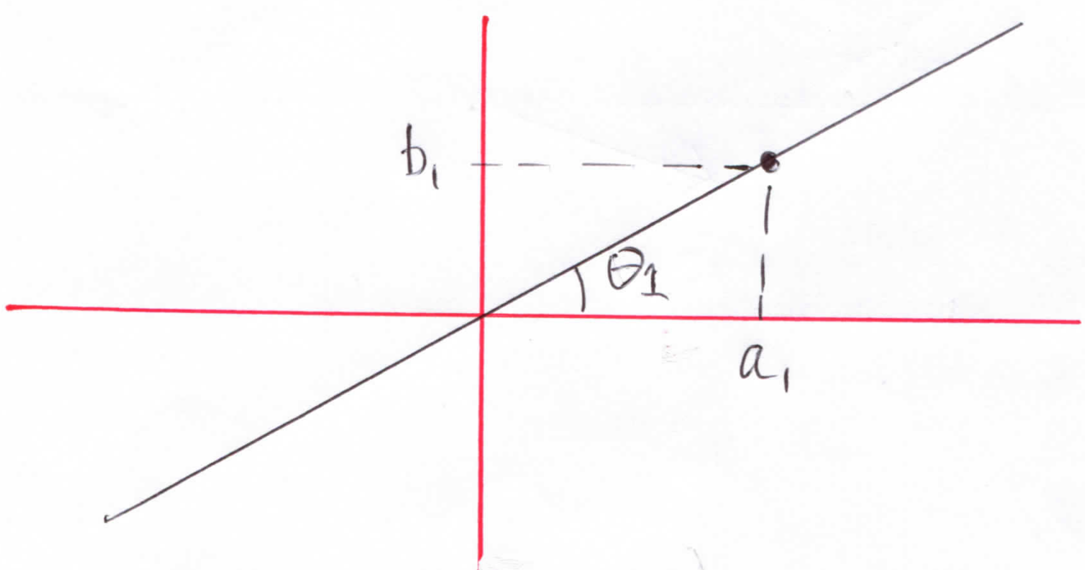
$(\theta_1 + \theta_2 + \theta_3) =$ 180 degrees

Geometry Assumption 2.2. In the two right triangles (meaning they each have a right angle, that is, an angle of 90 degrees) below,

$$(a)\ \theta_1 = \theta_2 \text{ if and only if } \frac{b_1}{a_1} = \frac{b_2}{a_2} \quad \text{and} \quad (b)\ \theta_1 > \theta_2 \text{ if and only if } \frac{b_1}{a_1} > \frac{b_2}{a_2}\ .$$



Remarks 2.3. Notice that $\frac{b_1}{a_1}$ is the *slope* of the line through $(0,0)$ and (a_1, b_1). In trigonometry, $\frac{b_1}{a_1}$ is the *tangent* of the angle θ_1 in the right triangle above.



III. VELOCITY and ANGULAR MOMENTUM

Examples 3.1: Swimming. (a) I swim NorthEast at 5 mph (miles per hour); that is, if there were no current, that would be my speed and direction. What I don't know is that there is an ocean current East at 3 mph; that is, if I stopped swimming, I would float East at 3 mph.

After two hours, where will I be and how far will I have traveled?

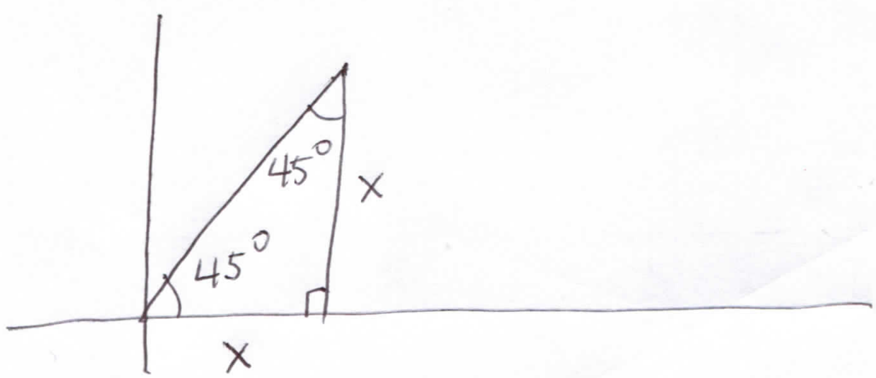
(b) Another swimmer plans ahead: he will swim NorthWest, and needs to know at what speed he should swim, so that he will be traveling directly north.

Solutions. (a) We need two vectors, one for my swimming, the other for the ocean current. "East" means a multiple of $< 1, 0 >$. Since the speed (defined to be the norm of the velocity) is 3, the velocity vector for the current is

$$3 < 1, 0 > = < 3, 0 > \text{ mph.}$$

For my NorthEast swimming, I need a vector pointing as drawn below.

By Geometry Assumptions 2.1 and 2.2, the two sides in the right triangle drawn below are of equal length.



Use the Pythagorean theorem to get that length:

$$5^2 = x^2 + x^2 = 2x^2 \to x = \frac{5}{\sqrt{2}}.$$

Thus my swimming vector is

$$< \frac{5}{\sqrt{2}}, \frac{5}{\sqrt{2}} > \text{ mph.}$$

We could also have gotten that swimming vector in the following way. Start with $<1,1>$ as a vector pointing NorthEast, then divide by its norm, to get

$$\frac{1}{\sqrt{2}}<1,1>=<\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}>$$

as a *unit* vector pointing NorthEast; to give it a speed of 5, still pointing NorthEast, multiply the unit vector by 5:

$$5<\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}>=<\frac{5}{\sqrt{2}},\frac{5}{\sqrt{2}}>.$$

For the combined effect of swimming and current, add those two vectors to get my velocity relative to where I started:

$$\left[<\frac{5}{\sqrt{2}},\frac{5}{\sqrt{2}}>+<3,0>\right]=<\frac{5}{\sqrt{2}}+3,\frac{5}{\sqrt{2}}>\sim<6.54,3.54>\text{ mph.}$$

After two hours, my location is

$$2<\frac{5}{\sqrt{2}}+3,\frac{5}{\sqrt{2}}>=<(5\sqrt{2}+6),5\sqrt{2}>\sim<13.07,7.07>$$

miles relative to where I started, a distance of

$$\|<(5\sqrt{2}+6),5\sqrt{2}>\|=\sqrt{(5\sqrt{2}+6)^2+(5\sqrt{2})^2}=\sqrt{(50+60\sqrt{2}+36)+50}=\sqrt{136+60\sqrt{2}}\sim 14.86$$

miles from where I started.

Alternatively, we could have first gotten the speed (from both swimming and current) at which I move away from where I started:

$$\|<\frac{5}{\sqrt{2}}+3,\frac{5}{\sqrt{2}}>\|=\sqrt{(\frac{5}{\sqrt{2}}+3)^2+(\frac{5}{\sqrt{2}})^2}=\sqrt{34+\frac{30}{\sqrt{2}}}\sim 7.43\,\text{mph},$$

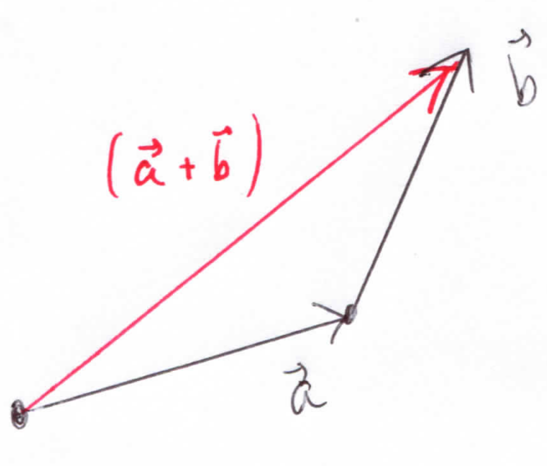
then multiplied by 2 hours:

$$2\sqrt{34+\frac{30}{\sqrt{2}}}\sim 14.86\,\text{miles}.$$

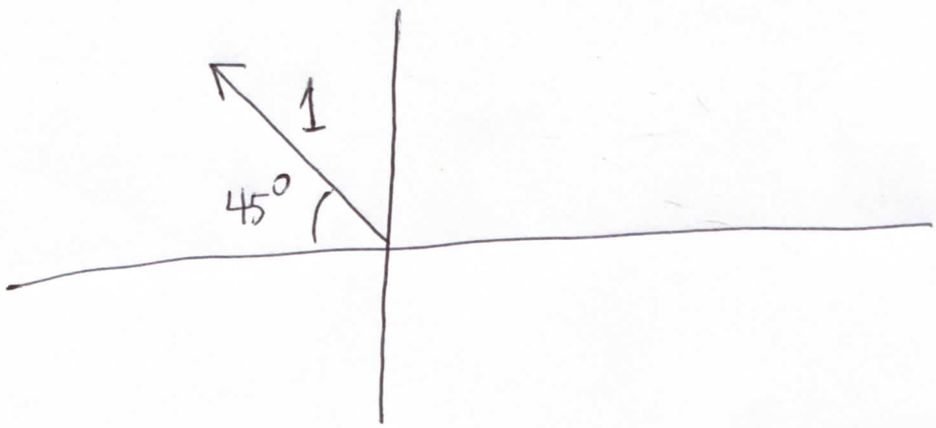
Notice that the net speed, of my swimming added to the current: ~ 7.43 mph, is less than the sum of the speeds of swimming and current: $5+3=8$ mph. Equality would occur only if I swam in the direction of the current. This is an example of the *triangle inequality:*

$$\|\vec{a}+\vec{b}\|\leq\|\vec{a}\|+\|\vec{b}\|$$

with equality only when $\vec{a}$ and $\vec{b}$ point in the same direction. The triangle inequality is stating that the shortest path between two points is the line segment from one point to the other.



(b) Since we don't yet know the speed, for our swimming contribution we should begin with a unit vector pointing NorthWest:



As with the solution of (a), this unit vector will be

$$< -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} >;$$

denoting s for our speed, this means our swimming vector will be

$$s < -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} >=< -\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}} > .$$

When we add the swimming vector to the current vector $< 3, 0 >$, we get a vector pointing north:

$$< 3 - \frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}} >=< 3, 0 > + < -\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}} >=< 0, \cdots >,$$

so that

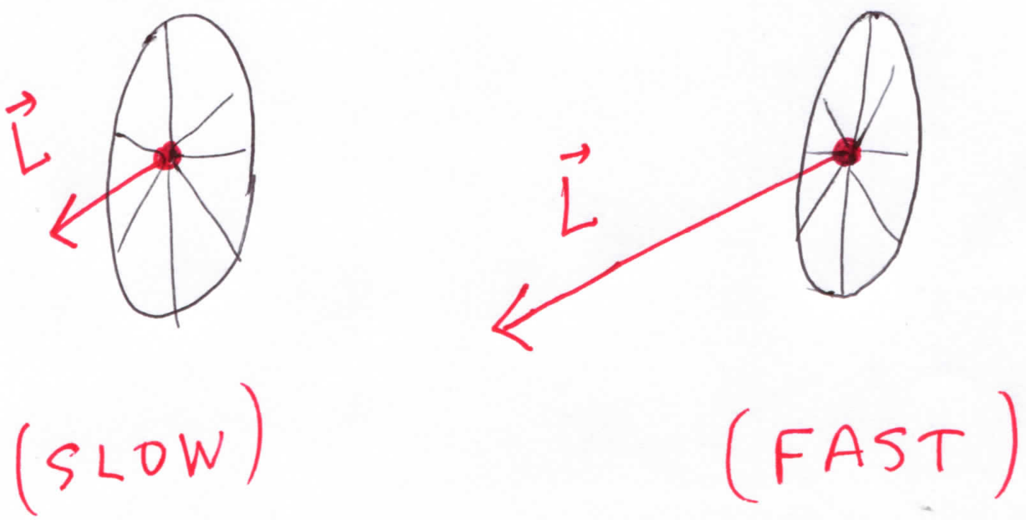
$$3 - \frac{s}{\sqrt{2}} = 0,$$

thus $s = 3\sqrt{2} \sim 4.24$ mph.

Examples 3.2. The **momentum** of an object is its mass times its velocity; this is a number times a vector, thus momentum is a vector. *Angular* momentum, and its relationship to angular velocity, we do not wish to go into very thoroughly; a more complete treatment of linear algebra, as in [3], than this magnification is needed. See [5] for physics that uses linear algebra and calculus.

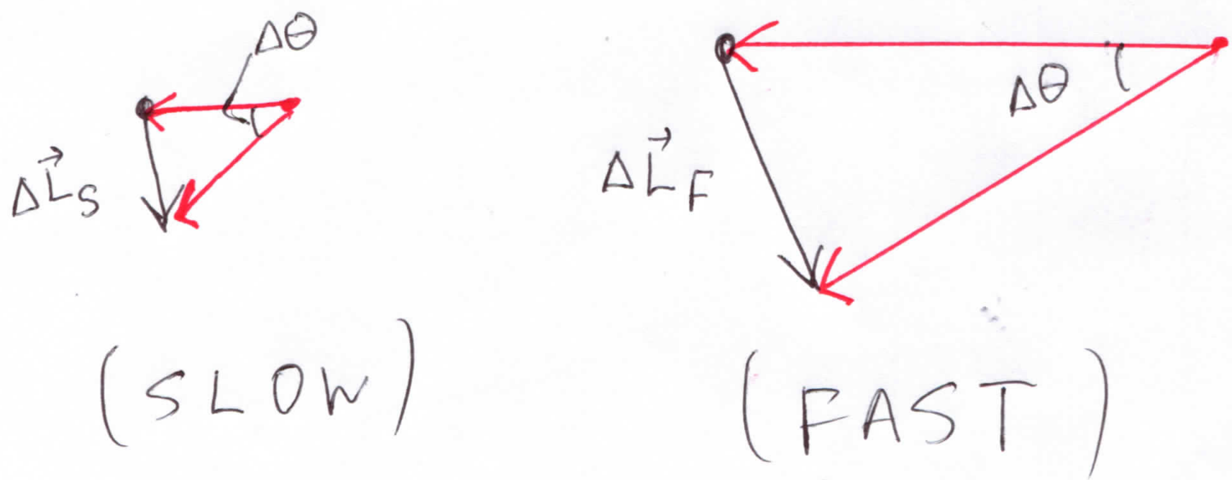
We will focus in this example on a bicycle in motion. We will assume the following two facts about the angular momentum, denoted $\vec{L}$, of each bicycle wheel:

1. $\vec{L}$ is a vector perpendicular to the plane of the wheel; and
2. as the speed of the bicycle increases, $\|\vec{L}\|$, the magnitude of the angular momentum, increases.

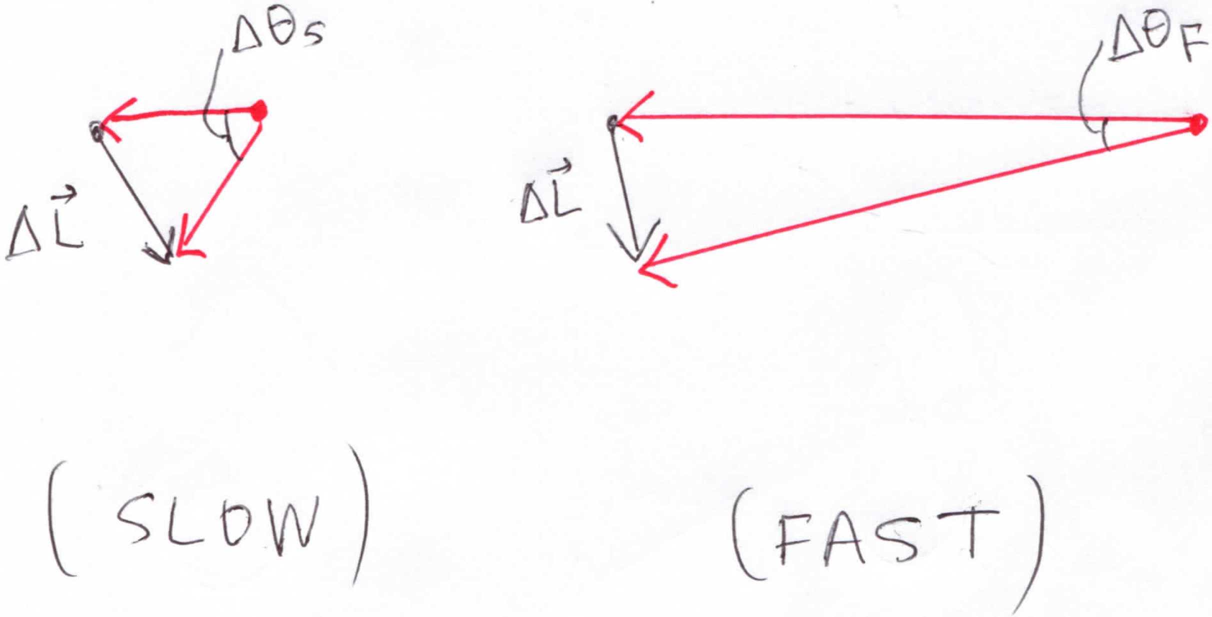


We will now show that these two facts are enough to explain why a bicycle is more stable at higher speeds, something that the author always found hard to believe.

Denote by $\Delta\theta$ the deviation from 90 degrees that the bicycle's angle with the ground is making ("Δ" is the Greek letter Delta, for "difference" or "change"), by $\Delta\vec{L}$ the change in $\vec{L}$ resulting from this deviation, as drawn below, with $\vec{L}$ drawn in red. $\Delta\theta$ we may take as a quantification of instability: the larger $\Delta\theta$, the less stable the bicycle. Notice that the same $\Delta\theta$ creates a bigger $\|\Delta\vec{L}\|$, at higher speeds.



The vector $\Delta\vec{L}$ is called *torque*, in this case possibly the result of fear-induced thrashing.
Looked at another way, the same fear-created $\Delta\vec{L}$ will produce a smaller instability $\Delta\theta$ when the bicycle is going faster.



The analysis above also explains why tightrope walkers carry poles; the longer the pole, the more stable the tightrope walker.

The same vector physics occurs with *rifling*, making the path of a bullet more stable by giving it angular momentum.

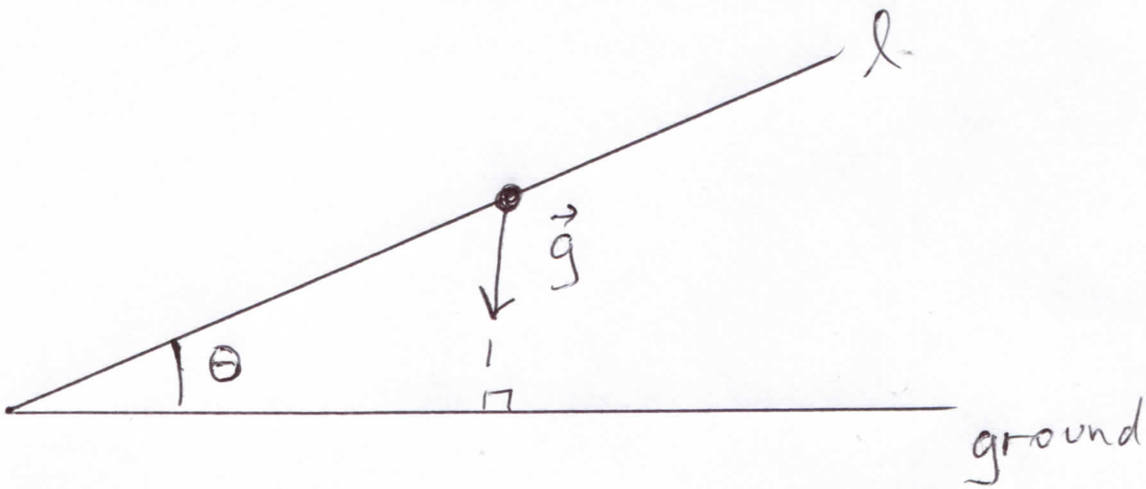
IV. FRICTION and MOTION on an INCLINED PLANE

Many vector force problems similar to the vector velocity problems of Examples 3.1 abound; see, for example, [5] and [7]. We will focus on a particular case of force vectors, those that cause motion on an inclined plane, and apply them to calculating friction.

Anyone familiar with cats knows that the force required to slide a cat along a surface depends, not only on the weight of said cat, but the *friction* (including instinctive resistance by the cat) the cat makes with the surface. For example, a glass surface would require less force than a rug; this is what it means to say that the friction between the cat and the glass is less than the friction between the cat and the rug. The *coefficient of static friction* (Definition 4.3), denoted μ_s (μ is written as "mu" and pronounced "mew"), assigns a number, to a stationary object on a surface, that measures its friction. A cat on glass has a lower coefficient of static friction than a cat on a rug.

In this chapter, we will use vectors to calculate the coefficient of static friction of an object on a plane (a flat surface) by tilting the plane counterclockwise from the ground until the object starts to slide.

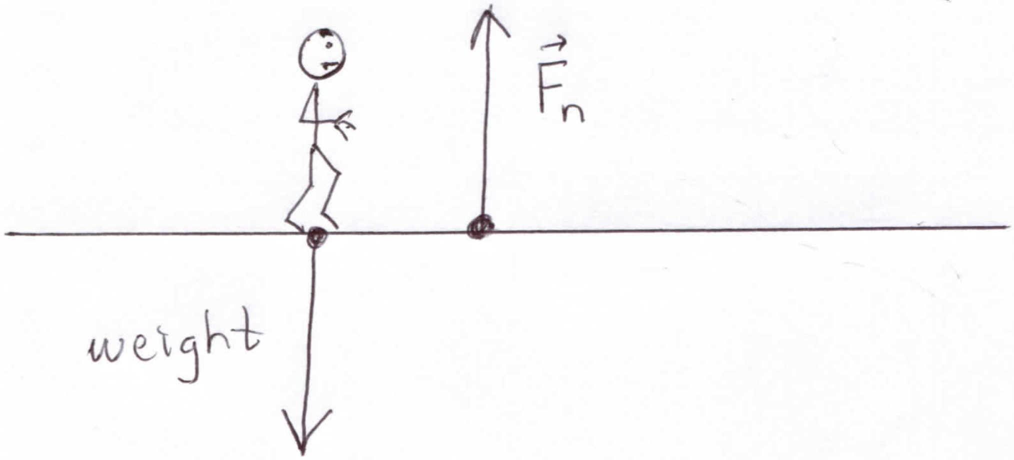
Throughout this chapter, pictures such as the one below will be a side view of a plane, such as a bread board, tilted up from the ground. Both the plane and the ground are coming out of the page of this magnification. The angle θ is the angle of inclination, of the plane from the ground. The vector $\vec{g}$ is gravity, perpendicular to the ground. We denote by ℓ the side view of the plane; that is, ℓ is the intersection of the plane with the surface of this page.



To quantify friction we need the idea of a *normal force* (Definition 4.1 and DRAWING 4.2).

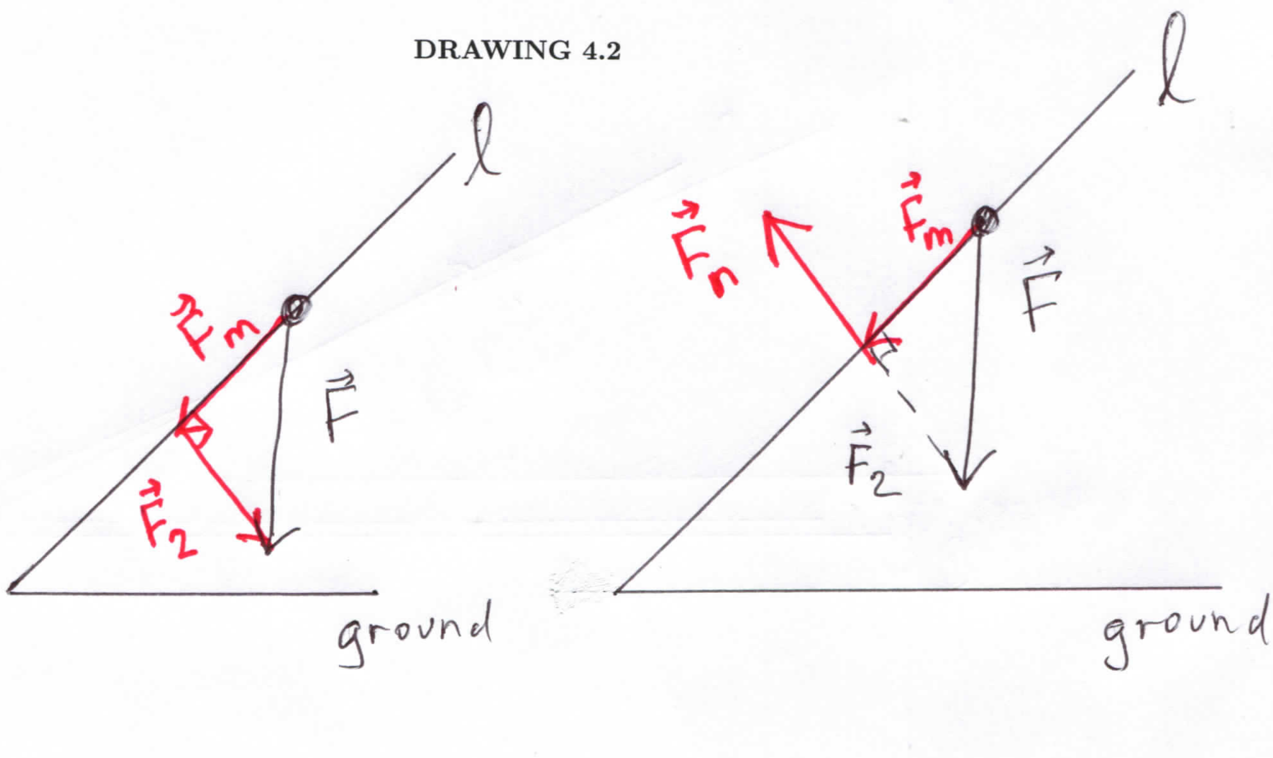
Definition 4.1. Suppose an object is on a surface. Unless the surface is vertical or said object has no mass, gravity would make said object crash through the surface, *unless* there were another force cancelling out gravity's pull. This force, perpendicular to the surface, is called the **normal force** on said object.

On a horizontal surface, the normal force $\vec{F}_n$ on an object is said object's weight (mass times gravity), in the opposite direction.



When the surface is tilted, to make what is called an inclined plane, normal force becomes more interesting. Let $\vec{F}$ be the force on the object created by gravity. Write $\vec{F}$ as the sum of two vectors, one, call it $\vec{F}_m$, parallel to the inclined plane, and the other, call it $\vec{F}_2$, perpendicular to the inclined plane. Then the normal force $\vec{F}_n = -\vec{F}_2$. The "m" in $\vec{F}_m$ stands for motion, since $\vec{F}_m$ is the force trying to move the object down the slope.

DRAWING 4.2



In DRAWING 4.2, note that, when the plane (side view ℓ) is parallel to the ground, $\vec{F}_2$ then equals $\vec{F}$, and our definition of normal force is the same as when previously defined on a horizontal surface.

Definition 4.3. The **coefficient of static friction** for an object at rest on a surface is

$$\mu_s \equiv \frac{\text{(magnitude of the maximum force perpendicular to the normal force that does not start the object moving)}}{\text{(magnitude of the normal force)}}.$$

That is, if F_m is the magnitude of the force perpendicular to the normal force and F_n is the magnitude of the normal force, then

$$\text{the object does not move} \quad \text{if } F_m \leq \mu_s F_n$$

and

$$\text{the object starts to move} \quad \text{if } F_m > \mu_s F_n.$$

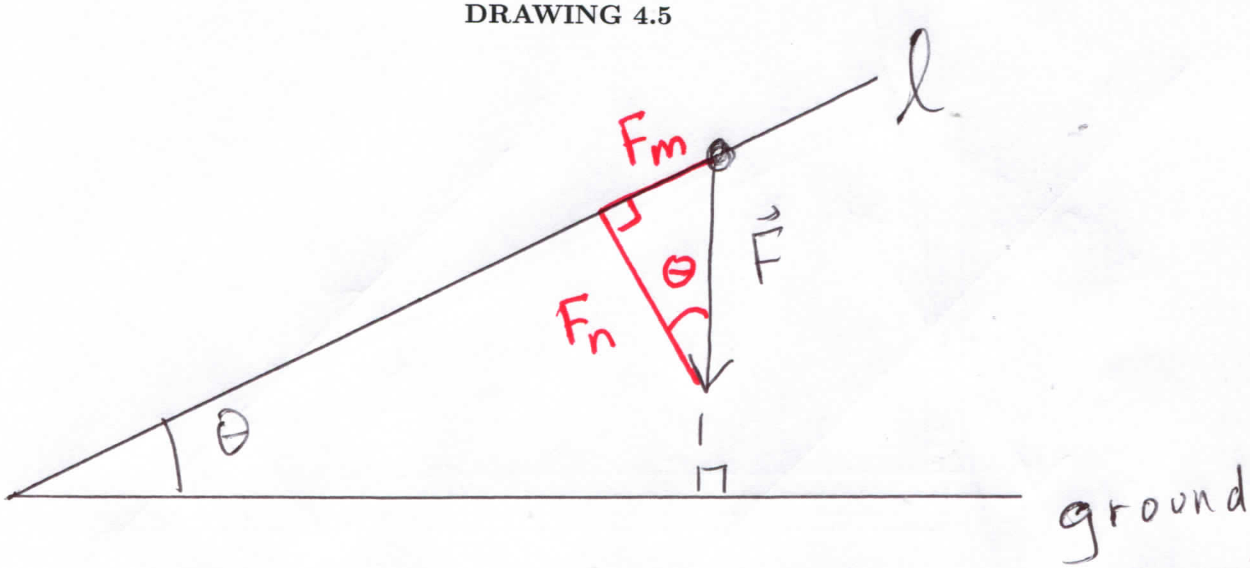
Examples 4.4. (i) Suppose a 10 pound cat is on a horizontal surface, with $\mu_s = \frac{1}{2}$. Then a force of magnitude greater than 5 pounds is required to start moving the cat.

(ii) If $\mu_s = 3$ in (i), then a force of magnitude greater than 30 pounds is needed.

In general, μ_s large corresponds to a rough, scratchy surface and/or object, while μ_s small corresponds to a slippery situation, like traveling on ice.

In DRAWING 4.2 we would like to add the angle of inclination, denoted θ, between the inclined plane and the ground. We will denote $F_m \equiv \|\vec{F}_m\|$ and $F_n \equiv \|\vec{F}_n\|$. The additional appearance of θ in DRAWING 4.5, drawn in red, follows from Geometry Assumption 2.1.

DRAWING 4.5



Proposition 4.6. In DRAWINGS 4.2 and 4.5, denote $F_m \equiv \|\vec{F}_m\|$ and $F_n \equiv \|\vec{F}_n\|$ and let μ_s be the coefficient of static friction for an object on the inclined plane with side view ℓ.

(a) The object starts sliding down the inclined plane if and only if $\frac{F_m}{F_n} > \mu_s$.

(b) μ_s equals the largest possible value of $\frac{F_m}{F_n}$ that does not make the object move.

(c) If we start with ℓ horizontal, then rotate it counterclockwise, μ_s equals the value of $\frac{F_m}{F_n}$ above which the object starts moving.

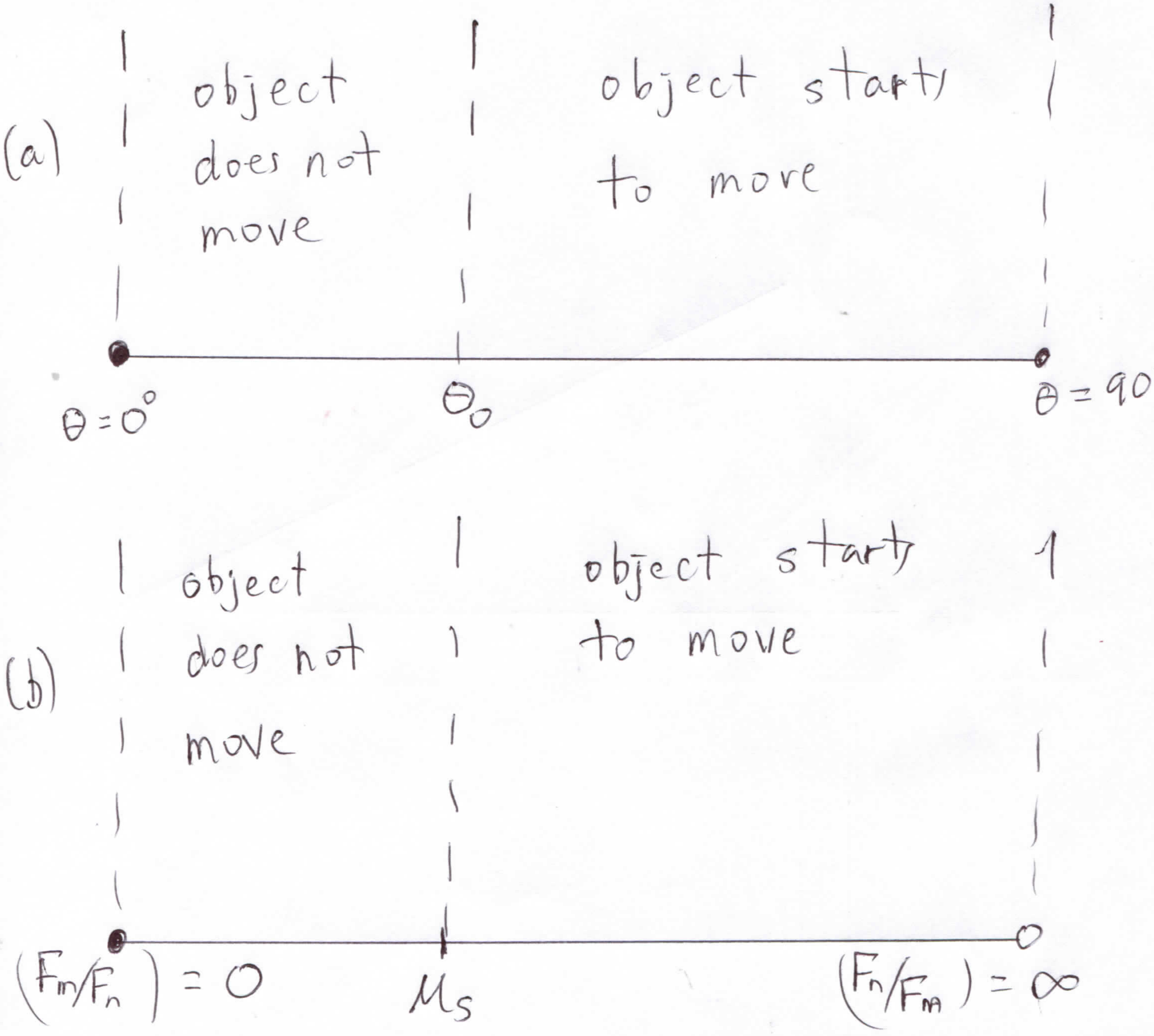
Proof: (a) is merely a restatement of the definition of μ_s.

As ℓ in DRAWING 4.5 is rotated counterclockwise, equivalent to the angle of inclination θ increasing, $\frac{F_m}{F_n}$ increases, by Geometry Assumption 2.2(b). Thus (a) implies there is a particular value of θ, call it θ_0, such that, for $\theta < \theta_0$, the object does not move, while for $\theta > \theta_0$, the object starts to move; see DRAWING 4.7(a) below.

In DRAWING 4.5, each value of θ corresponds to a unique value of $\frac{F_m}{F_n}$; under that correspondence, θ_0 corresponds to μ_s, giving us (a)–(c). See DRAWING 4.7(a) and (b) below.

DRAWING 4.7

BOARD ROTATES COUNTERCLOCKWISE; ANGLE OF INCLINATION θ INCREASES
as we move from left to right in pictures below



The vector magnitudes F_m and F_n in Proposition 4.6 are not easy to measure. We need an analogue of Proposition 4.6 that involves only the line ℓ in DRAWINGS 4.2 and 4.5.

Proposition 4.8. In DRAWINGS 4.2 and 4.5, the slope of ℓ equals $\frac{\|\vec{F}_m\|}{\|\vec{F}_n\|} = \frac{F_m}{F_n}$.

Proof: Apply Geometry Assumptions 2.2(a) and see Remarks 2.3 to DRAWING 4.5.

Proposition 4.8 allows us to restate Proposition 4.6 entirely in terms of the slope of ℓ in DRAWINGS 4.2 and 4.5.

Inclined Plane Friction Theorem 4.9. In DRAWINGS 4.2 and 4.5, let μ_s be the coefficient of static friction for an object on the inclined plane with side view ℓ.

(a) The object starts sliding down the inclined plane if and only if

$$[\text{slope of } \ell] > \mu_s.$$

(b) μ_s equals the largest possible slope of ℓ that does not make the object move.

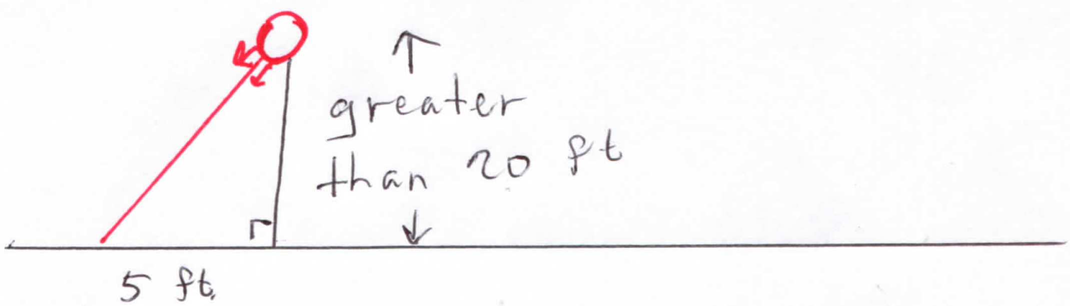
(c) If we start with ℓ horizontal, then rotate it counterclockwise, μ_s equals the slope of ℓ above which the object starts moving.

Examples 4.10. In each part, μ_s is the coefficient of static friction for the object and plane of that part.

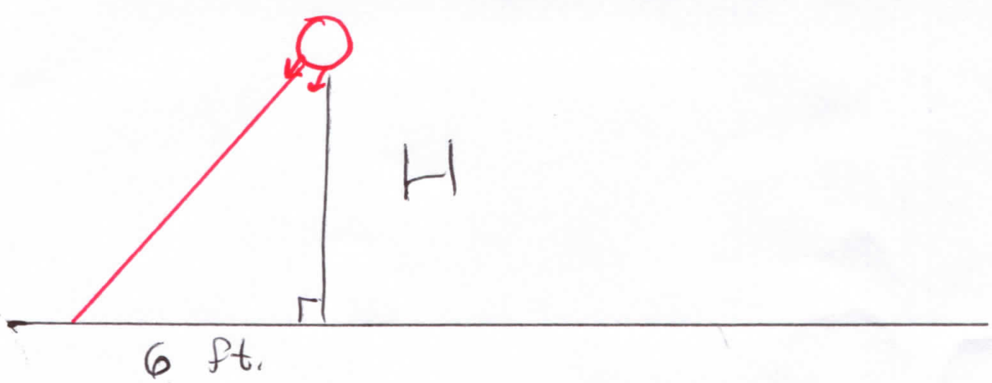
(a) If $\mu_s = 3$, how large must the slope of the plane be for the object on the plane to start moving?

(b) If we start tilting a plane counterclockwise from horizontal, and the object on the plane starts sliding when the slope of the plane is greater than ten, what is μ_s?

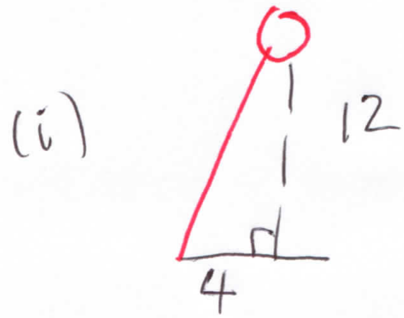
(c) The object on the plane starts moving when the plane is as drawn below. What is μ_s?

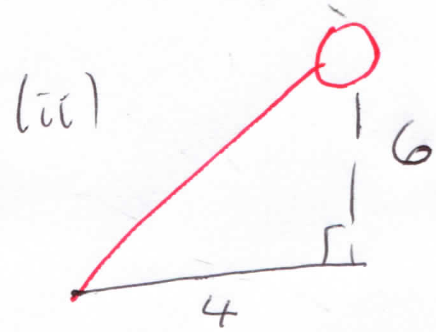


(d) Suppose $\mu_s = 5$. Let H be the height of the plane above the ground in the drawing below. If the object starts to move, what can be said about H?

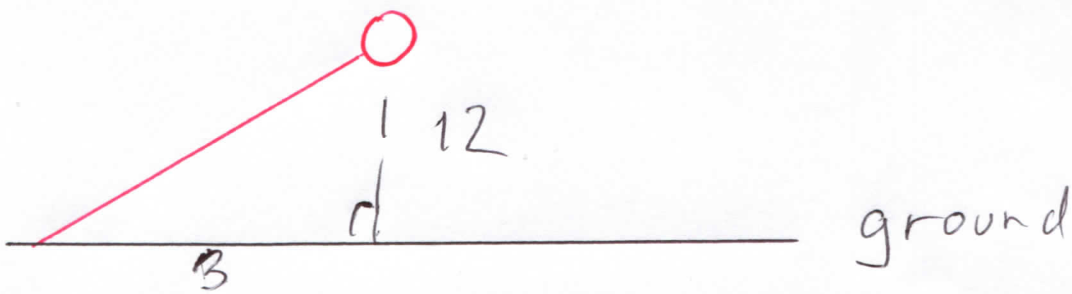


(e) Suppose $\mu_s = 2$. In which of the following drawings of the object and plane, in red, below, will the object start to move?

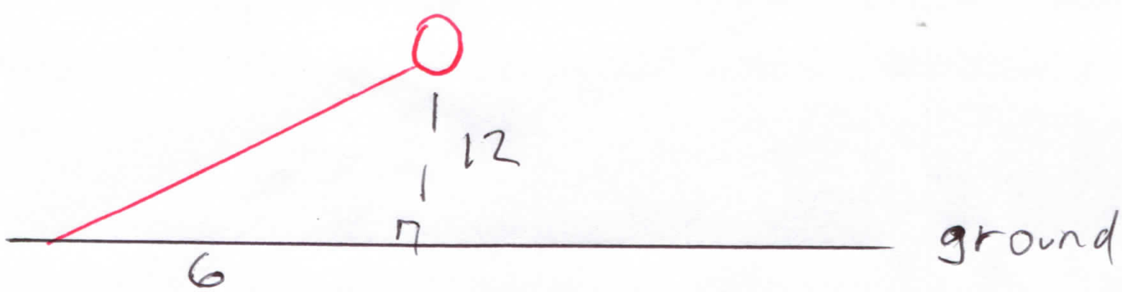




(f) Suppose the object in the picture below starts to move. What can be said about μ_s?



(g) Suppose the object in the picture below does not start to move. What can be said about μ_s?



Solutions. (a) greater than 3. (b) ten (c) $\frac{20\text{ft.}}{5\text{ ft.}} = 4$. (d) $\frac{\text{H}}{6\text{ ft.}} > 5 \to H > 30$ ft.

(e) (i) only, since $\frac{12}{4} > 2$ and $\frac{6}{4} \leq 2$. (f) $\mu_s < \frac{12}{3} = 4$. (g) $\mu_s \geq \frac{12}{6} = 2$.

Remarks 4.11. (a) The **coefficient of kinetic friction** for an object moving on a surface is

$$\mu_k \equiv \frac{\text{(magnitude of the minimum force perpendicular to the normal force that keeps the object moving)}}{\text{(magnitude of the normal force)}}.$$

An interesting physics factoid is that, usually, $\mu_k < \mu_s$. Thus, once a force starts the object moving, it may be kept moving with a diminished force.

(b) In DRAWINGS 4.2 and 4.5, the angle of inclination θ may be obtained from the line ℓ, with the trigonometric function *tangent*, nicknamed *tan*:

$$\tan(\theta) = [\text{slope of } \ell]\,.$$

For example, in Examples 4.10(f), the angle of inclination may be obtained as follows, on a calculator: get $\tan^{-1}(4)$ (reads "inverse tangent of 4"), about 76 degrees.

(c) The definition of coefficient of static friction is sometimes subtly modified from Definition 4.3 as follows.

$$\mu_s \equiv \frac{\text{(magnitude of the force perpendicular to the normal force required to start the object moving)}}{\text{(magnitude of the normal force)}}.$$

Equivalently,

$$\mu_s \equiv \frac{\text{(magnitude of the minimum force perpendicular to the normal force that starts the object moving)}}{\text{(magnitude of the normal force)}}.$$

That is, if F_m is the magnitude of the force perpendicular to the normal force and F_n is the magnitude of the normal force as in Definition 4.3, then

$$\text{the object does not move} \quad \text{if } F_m < \mu_s F_n$$

and

$$\text{the object starts to move} \quad \text{if } F_m \geq \mu_s F_n.$$

The only difference between this definition and Definition 4.3 is when $F_m = \mu_s F_n$; then Definition 4.3 says there is no motion whereas this modified definition says there will be motion.

For example, in Example 4.4(i), if a force of 5 pounds is applied to the cat, Definition 4.3 says the cat will not move, but our modified definition says the cat will start to move.

In practice, there is no difference between Definition 4.3 and this modified definition, because force is continuous, thus, for any number c, it is not possible for force to be exactly c pounds; it is only the falsification of rounding that creates that illusion. The only possible assertions about continuous parameters is that they are *between* two numbers. For example, if I say I am 67 inches tall, I really mean that my height in inches is greater than or equal to 66.5 and is less than 67.5.

Getting back to the definition of the coefficient of static friction μ_s, it is not possible for F_m to equal exactly $\mu_s F_n$, thus the difference between Definition 4.3 and our modified definition will not occur. For example, in Example 4.4(i), it is not possible to apply a force of exactly 5 pounds to the cat, thus our rival definitions of μ_s cannot be different.

Notice the similarity between this alternative definition of μ_s and the definition of μ_k.

HOMEWORK

1. Suppose $\vec{a} \equiv < 1, 2 >, \vec{b} \equiv < -2, 1 >$.

a. Find $\|\vec{a}\|$, $\|\vec{b}\|$, and $\|\vec{a} + \vec{b}\|$.

b. Draw a triangle whose sides are $\vec{a}, \vec{b}, (\vec{a} + \vec{b})$.

c. Describe any interesting relationships in (a) or what appear to be interesting angles in (b).

2. Suppose $\vec{a} \equiv < 1, 2 >, \vec{b} \equiv < 2, 4 >$.

a. Find $\|\vec{a}\|$, $\|\vec{b}\|$, and $\|\vec{a} + \vec{b}\|$.

b. Draw $\vec{a}, \vec{b}, (\vec{a} + \vec{b})$, all with initial point $(0, 0)$.

c. Describe any interesting relationships in (a) or what appear to be interesting angles in (b).

3. Suppose $\vec{a} \equiv < 1, 2 >, \vec{b} \equiv < -1, -2 >$.

a. Find $\|\vec{a}\|$, $\|\vec{b}\|$, and $\|\vec{a} + \vec{b}\|$.

b. Draw $\vec{a}, \vec{b}, (\vec{a} + \vec{b})$, all with initial point $(0, 0)$.

c. Describe any interesting relationships in (a) or what appear to be interesting angles in (b).

4. Suppose $\vec{a} \equiv < 1, 2 >, \vec{b} \equiv < 2, 0 >$.

a. Find $\|\vec{a}\|$, $\|\vec{b}\|$, and $\|\vec{a} + \vec{b}\|$.

b. Draw a triangle whose sides are $\vec{a}, \vec{b}, (\vec{a} + \vec{b})$.

c. Draw $\vec{a}, \vec{b}, (\vec{a} + \vec{b})$, all with initial point $(0, 0)$.

5. Give an example of two unit (magnitude one) vectors whose sum has magnitude zero.

6. Give an example of two unit vectors whose sum has magnitude $\sqrt{2}$.

7. Suppose I swim in the ocean to the SouthEast (MEANING that, if there is no current, I'd be traveling to the SouthEast) while the ocean current is seven miles per hour to the North (MEANING that, if I were not swimming, I would float North at seven miles per hour).

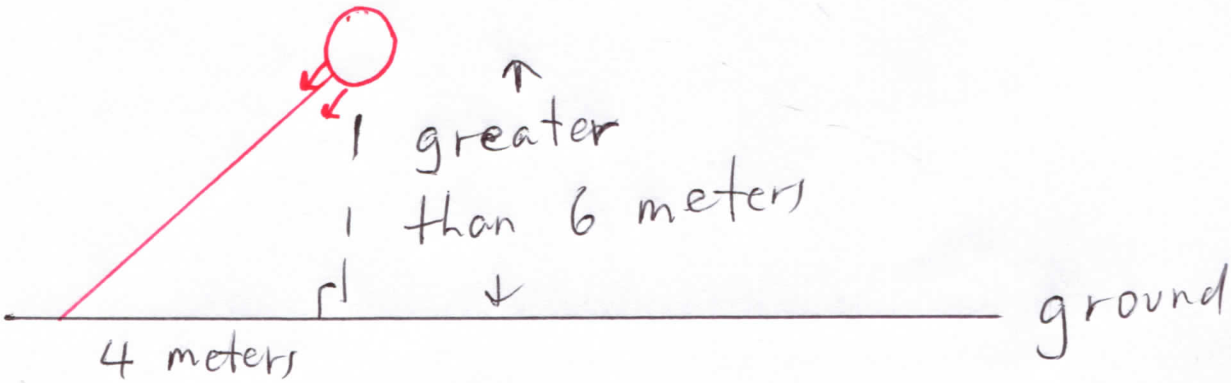
(a) If I swim at ten miles per hour, how far north of where I started will I be after a day, and how far will I have traveled?

(b) At what speed should I swim so that I don't go north or south?

8. In each part, μ_s is the coefficient of static friction for the object and plane of that part.

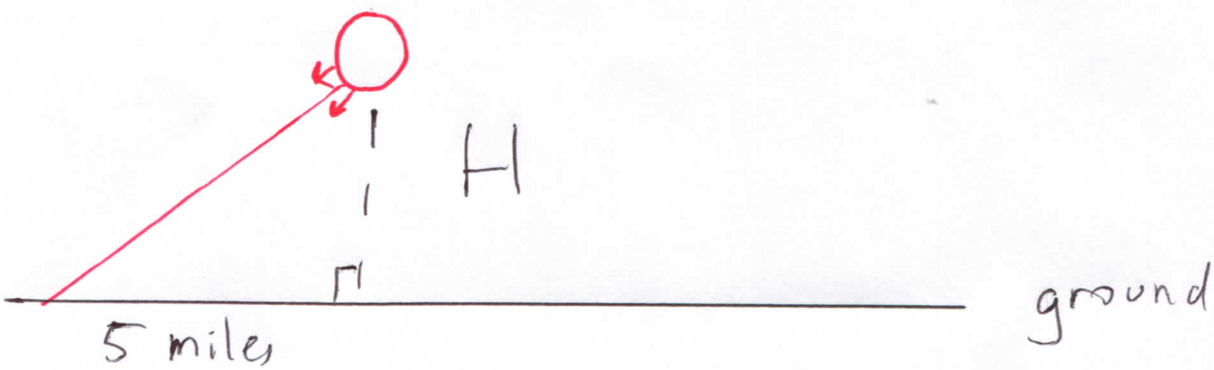
(a) If we start tilting a plane counterclockwise from horizontal, and the object on the plane starts sliding when the slope of the plane is greater than $\frac{1}{2}$, what is μ_s?

(b) The object on the plane starts moving when the plane is as drawn below. What is μ_s?

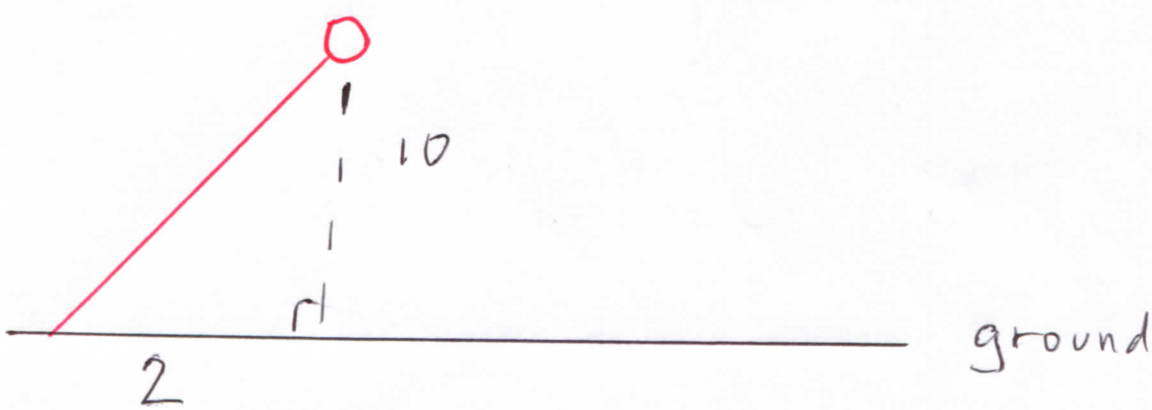


(c) If $\mu_s = 7$, how large must the slope of the plane be for the object on the plane to start moving?

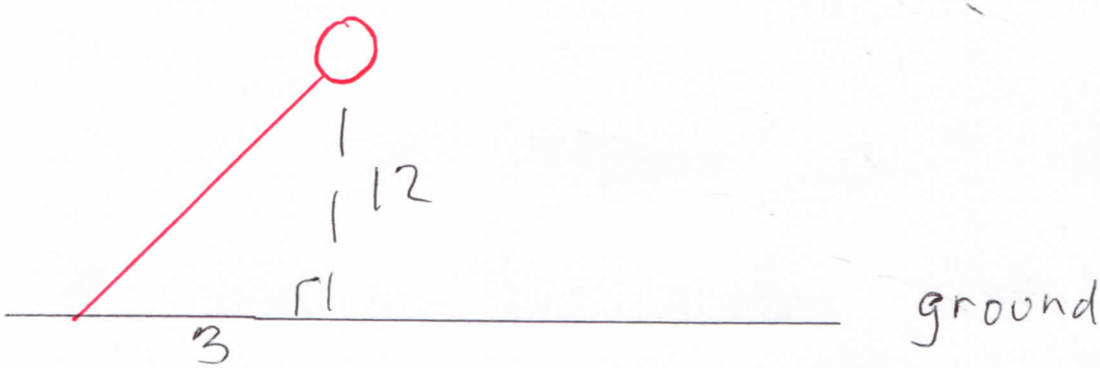
(d) Suppose $\mu_s = 2$. Let H be the height of the plane above the ground in the picture below. If the object on the plane starts moving, what can be said about H?



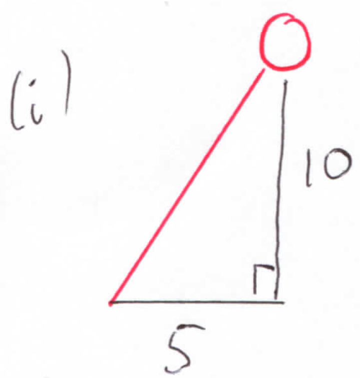
(e) Suppose the object in the picture below starts to move. What can be said about μ_s?

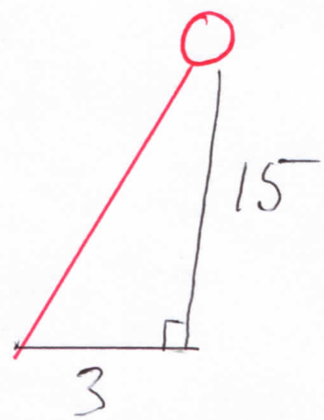


(f) Suppose the object in the picture below does not start to move. What can be said about μ_s?



(g) Suppose $\mu_s = 3$. In which of the following drawings of the object and plane, in red, below, will the object start to move?





9. EXPERIMENT. Given a plane and an object sitting on said plane, as in Chapter IV, two or more people can do the following to get μ_s, the coefficient of static friction, for the object on the plane.

Start with the plane on the ground, then have Person One slowly tilt it up. Stop when the object on the plane starts to move. While Person One holds the plane in that position that initiates motion, have Person Two measure the slope ("rise over run" in drawing on next page) of the plane relative to the ground. That slope is μ_s.

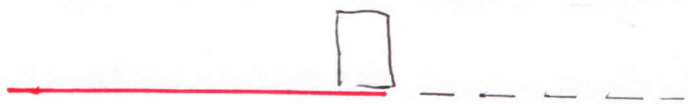
Some possible planes: breadboard, window screen, book cover, piece of lumber, etc.

Some possible objects (to be placed on any of the possible planes): stuffed animal, dice, monopoly game figures, checkers piece, pencil, silverware, etc.

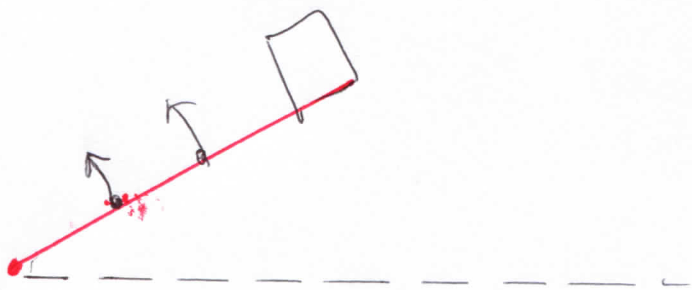
Competitions are possible: students guess, prior to experiment, what the coefficient of static friction is, or which pairs of object-on-plane will have a higher coefficient of static friction.

DRAWING OF EXPERIMENT (plane drawn in red, object is drawn as square)

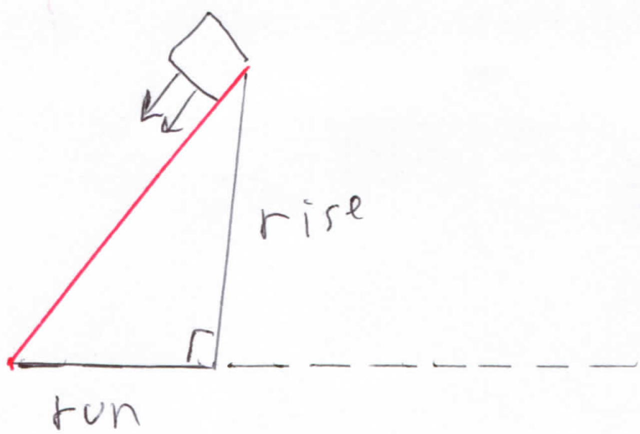
(1)



(2)



(3)

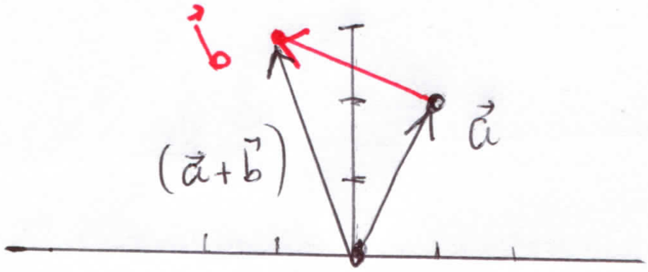


(4) $\mu_s = \frac{\text{rise}}{\text{run}}$

HOMEWORK SOLUTIONS

1. a. $\|\vec{a}\| = \sqrt{5} = \|\vec{b}\|, \|\vec{a}+\vec{b}\| = \| < -1, 3 > \| = \sqrt{10}$.

b.

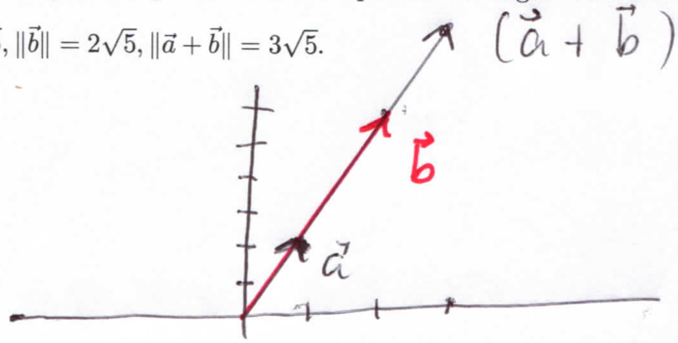


c. $\|\vec{a}+\vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2$; $\vec{a}$ looks perpendicular to $\vec{b}$. See [2], [3], or [4].

The other angles look equal; this follows from Geometry Assumption 2.2 and the fact that $\|\vec{a}\| = \|\vec{b}\|$. Geometry Assumption 2.1 then implies those angles each measure 45 degrees.

2. a. $\|\vec{a}\| = \sqrt{5}, \|\vec{b}\| = 2\sqrt{5}, \|\vec{a}+\vec{b}\| = 3\sqrt{5}$.

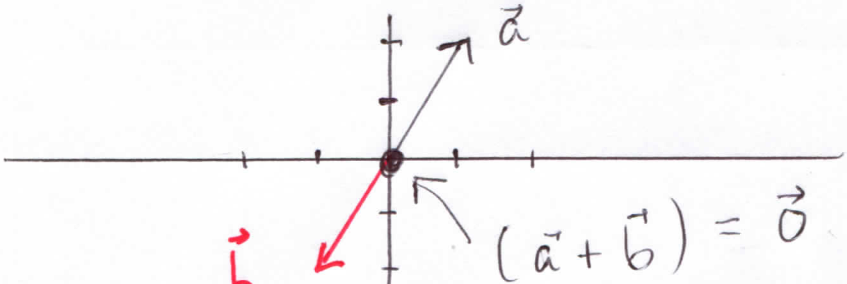
b.



c. $\|\vec{a}+\vec{b}\| = \|\vec{a}\| + \|\vec{b}\|$; $\vec{a}, \vec{b}$, and $(\vec{a}+\vec{b})$ point in the same direction. See Definitions 1.11.

3. a. $\|\vec{a}\| = \sqrt{5} = \|\vec{b}\|, \|\vec{a}+\vec{b}\| = \| < 0, 0 > \| = 0$.

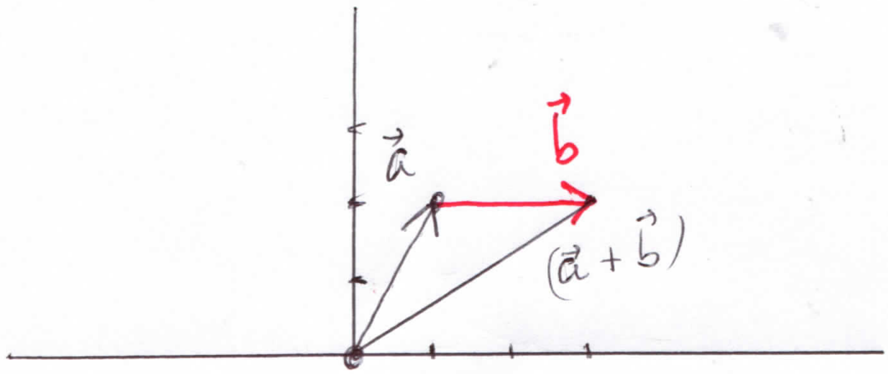
b.



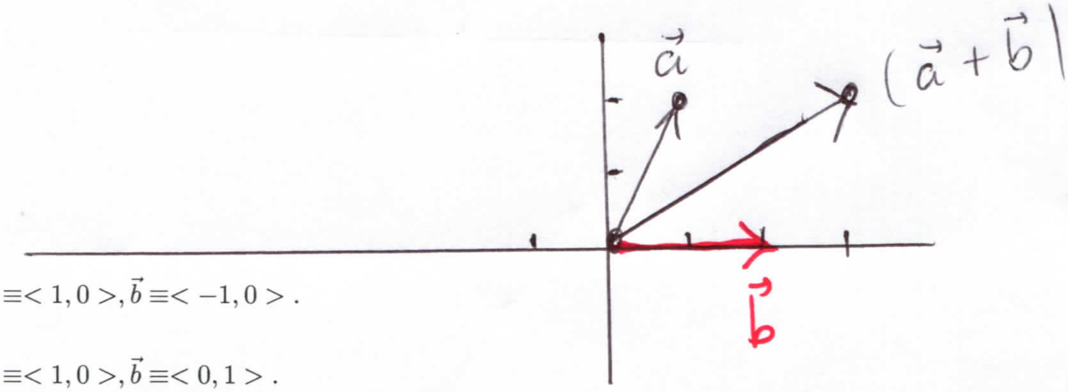
c. $\|\vec{a}+\vec{b}\| = 0$ seems "interesting." This cancellation of magnitude is possible because $\vec{a}$ and $\vec{b}$ point in opposite directions (see Definitions 1.11). Pointing in opposite directions does not necessarily mean the norm of the sum is zero; consider $\|\vec{a} + \frac{1}{2}\vec{b}\| = \|\frac{1}{2}\vec{a}\| = \frac{1}{2}\|\vec{a}\|$. *Some* of the norm of $\vec{a}$ got cancelled.

4. a. $\|\vec{a}\| = \sqrt{5}, \|\vec{b}\| = 2, \|\vec{a}+\vec{b}\| = \| < 3, 2 > \| = \sqrt{13}$.

b.



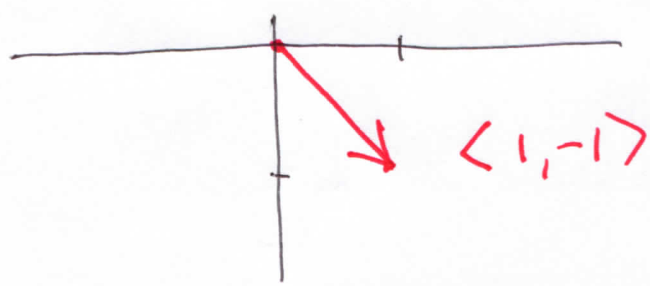
c.



5. $\vec{a} \equiv < 1, 0 >, \vec{b} \equiv < -1, 0 >$.

6. $\vec{a} \equiv < 1, 0 >, \vec{b} \equiv < 0, 1 >$.

7. A vector pointing SE (SouthEast) is $< 1, -1 >$; to make it a *unit* vector, divide it by its norm: $\frac{1}{\sqrt{2}} < 1, -1 >$. The current is $< 0, 7 >$.



a. Our swimming vector is $\frac{10}{\sqrt{2}} < 1, -1 >$. Add on the current

$$\left[\frac{10}{\sqrt{2}} < 1, -1 > + < 0, 7 >\right] \text{ miles per hour}$$

as our net velocity in the ocean.

For our displacement in a day, multiply by 24 (since there are 24 hours in a day):

$$24\left[\frac{10}{\sqrt{2}} < 1, -1 > + < 0, 7 >\right] \text{ miles;}$$

the displacement north is

$$24\left[\frac{10}{\sqrt{2}}(-1) + 7\right] \sim -1.71 \text{ miles north,}$$

or approximately 1.71 miles south.

The distance traveled is

$$\|24\left[\frac{10}{\sqrt{2}} < 1, -1 > + < 0, 7 >\right]\| = 24\sqrt{(\frac{10}{\sqrt{2}})^2 + (\frac{-10}{\sqrt{2}} + 7)^2} \sim 170 \text{ miles.}$$

b. Let s be the desired speed. Then our net velocity in the ocean is

$$\left[\frac{s}{\sqrt{2}} < 1, -1 > + < 0, 7 >\right] \text{ miles per hour;}$$

to avoid going north or south, we need the y component

$$\left[\frac{s}{\sqrt{2}}(-1) + 7\right]$$

to equal zero; solving for s gives us $s = 7\sqrt{2} \sim 9.90$ miles per hour.

8. a. $\frac{1}{2}$ b. $\frac{6}{4} = 1.5$ c. greater than 7 d. greater than 10 miles e. $\mu_s < \frac{10}{2} = 5$ f. $\mu_s \geq \frac{12}{3} = 4$

g. (ii) only, since $\frac{10}{5} \leq \mu_s < \frac{15}{3}$.

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REFERENCES

1. Ralph deLaubenfels, "Vectors and Geometry," http://teacherscholarinstitute.com/MathMagnificationsReadyToUse.html.

2. Ralph deLaubenfels, "Vectors and Perpendicular Geometry," http://teacherscholarinstitute.com/MathMagnificationsReadyToUse.html.

3. Ralph deLaubenfels, "Linear Algebra," or E Pluribus Unum, http://teacherscholarinstitute.com/FreeMathBooksHighschool.html (2017).

4. Ralph deLaubenfels, "From Vectors to Trigonometry to Geometry," to appear.

5. David Halliday, Robert Resnick, and Jearl Walker, "Fundamentals of Physics," Sixth Edition, John Wiley and Sons, Inc., 2001.

6. John Saxon, "Algebra 1. An Incremental Development," Second Edition, Saxon Publishers, Inc., 1990.

7. John Saxon, "Physics. An Incremental Development," Saxon Publishers, Inc., 1993.