TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI

Bees and Hexagons
MATHematics MAGnification™

Dr. Ralph deLaubenfels

TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI

Teacher-Scholar Institute

Columbus, Ohio

2019

BEES and HEXAGONS MAGNIFICATION

This is one of a series of very short books on math, statistics, and physics called "Math Magnifications." The "magnification" refers to focusing on a particular topic that is pivotal in or emblematic of mathematics.

OUTLINE

Cross sections of a honeycomb are regular (see Definition 1.4) hexagons, placed so that there are no gaps or overlaps. This is an example of a *tiling* or *tessellation.* In this Magnification, we will discover all possible tilings with copies of a fixed regular polygon and derive that this tiling with hexagons is ideal, in a certain surprising sense, among all such tilings.

Prerequisites for this Magnification are some high school geometry and first-year algebra including the Pythagorean theorem and the definitions of convex polygon, vertices of polygons, perimeter, and angle. Reference [3] is more than sufficient for both the geometry and algebra needed.

1. INTRODUCTION.

To paraphrase Leibniz (see also Dr. Pangloss, in [4]), hexagons make the best of all possible honeycombs. It is a fact that honeycombs (to be precise, cross sections of a honeycomb) are made of hexagons. For this Magnification, we will assume we don't know this fact and are too squeamish, or nonviolent, or afraid of being stung by bees, to physically discover it by slicing open a honeycomb. We will instead discover it purely as a thought experiment, armed only with faith that the universe is not only rational, but optimal.

First we need some terminology.

Definitions 1.1. A **tiling** or **tessellation** is an arrangement of flat things that fit together, without gaps or overlap, to cover all or some of the plane.

For many beautiful tessellations by M.C. Escher, go to

www.josleys.com/galleries.php?catid=6

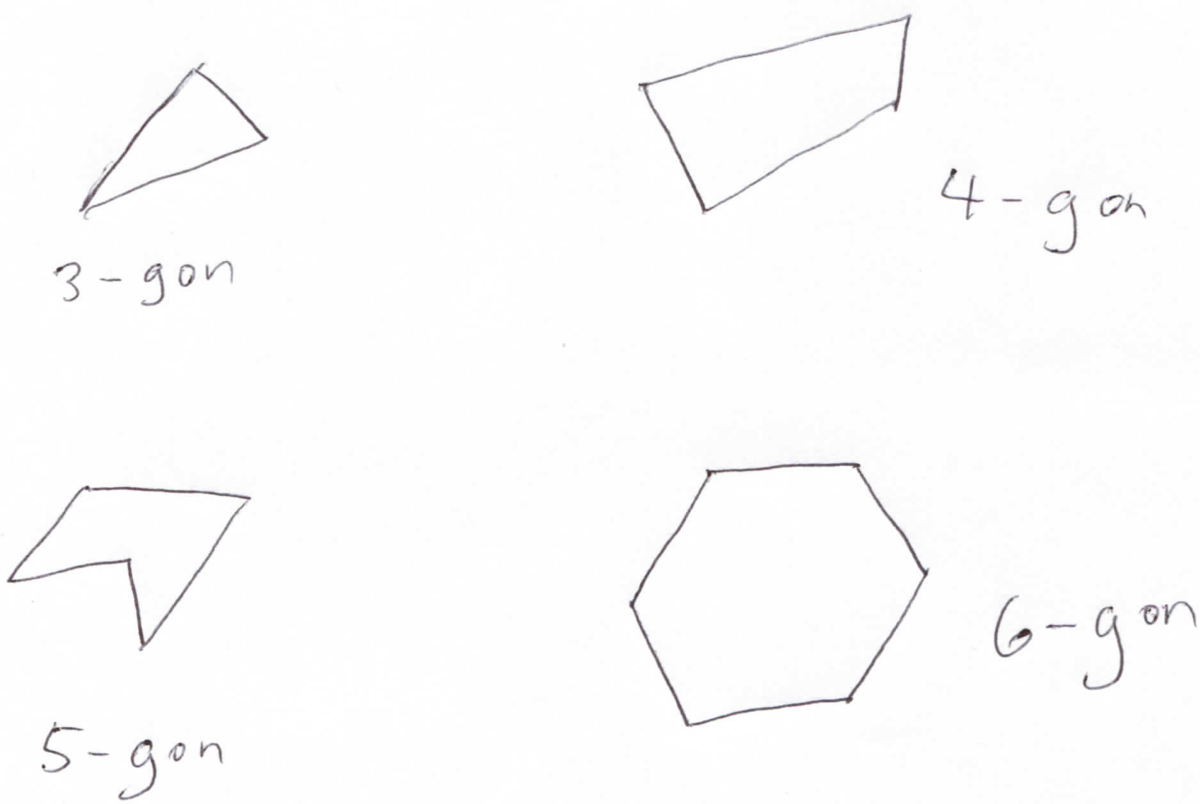
and click "Escher tilings".

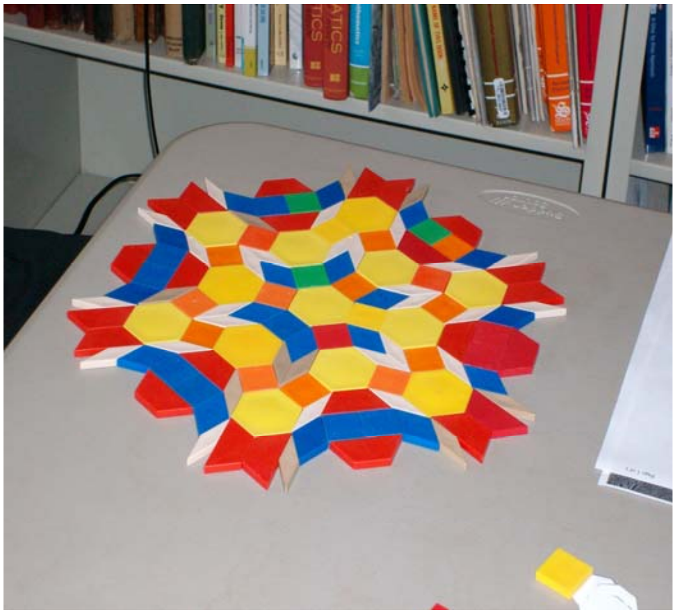
The flat things we would like to tile with are polygons, of a particular kind. On the next page, we have photos of some nice tessellations of this kind, made by students at our tessellation workshops.

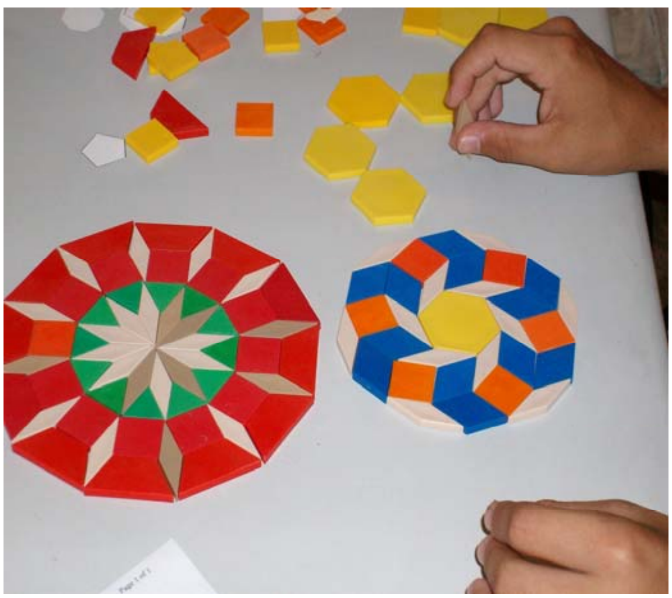
Definition 1.2. We assume, for this Magnification, that the reader knows the definition of a *polygon.* Here is awkward-sounding terminology that specifies the number of sides of a polygon.

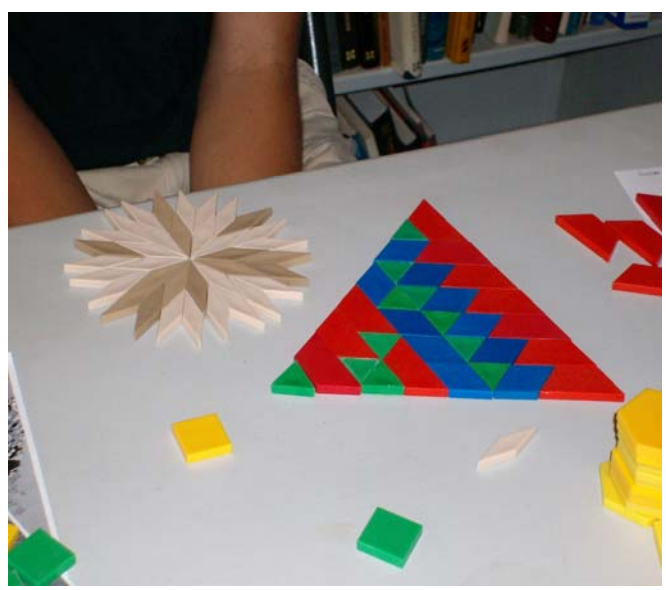
For $n = 3, 4, 5, \ldots$, an **n-gon** is a polygon with n sides.

For example, a 3-gon is a triangle, a 4-gon is a quadrilateral, a 5-gon is a pentagon, and a 6-gon is a hexagon.

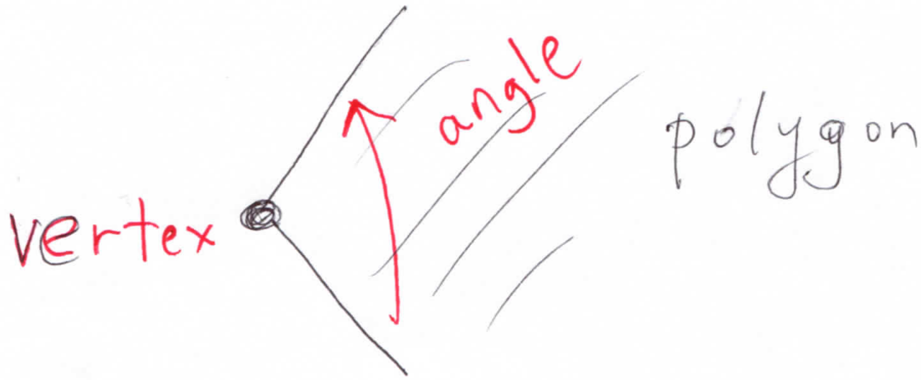






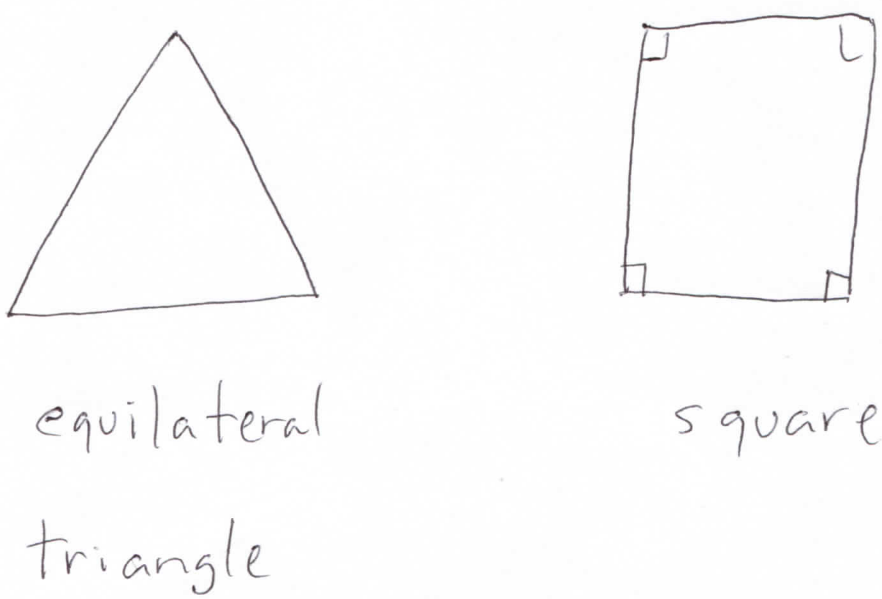


Definition 1.3. The **interior angle** of a convex polygon at a vertex of said polygon is the angle of smaller measure formed by the two sides meeting at said vertex.



Definition 1.4. A **regular polygon** is a convex polygon with sides of equal length and interior angles of equal measure.

A regular 3-gon is also called an **equilateral triangle**; a regular 4-gon is also called a **square**. A regular pentagon is related to the *golden ratio* ϕ in surprising ways; see [1, Remarks 1.7].



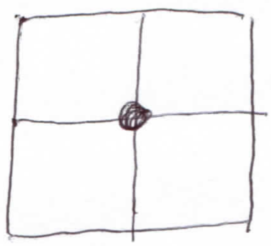
2. TILING with COPIES of a FIXED REGULAR POLYGON

In this section, we will address the following question.

Big Question 2.1. For what n can we tessellate with copies of the same regular n-gon?

Discussion 2.2. Consider tessellating with copies of a regular 4-gon, that is, a square. After sliding copies around, if necessary, we get the arrangement of streets in the downtown area of an organized city, as drawn below.

Notice that groups of 4 contiguous squares then meet at the same point; that is, the 4 squares share a vertex, as drawn below.



Recall that a complete rotation measures 360 degrees. In order that 4 squares fit together as drawn above, it must be that each interior angle of a square measures

$$\frac{360}{4} \text{ degrees } = 90 \text{ degrees }.$$

The same is true for any tiling with copies of a fixed regular polygon: the interior angles in said polygon must measure, for some integer k, $\frac{360}{k}$, so that k copies of an interior angle have measures adding up to an angle of measure 360 degrees, implying that k copies of the regular polygon fit together at a shared vertex. See Table 2.3 on the next page, listing all possible interior angles for tiling with a fixed regular polygon.

In the following, all angle measures are in degrees. The heading "number of copies" is shorthand for "number of copies of regular polygon fitting together at shared vertex." The fitting is sketched at the right of the table.

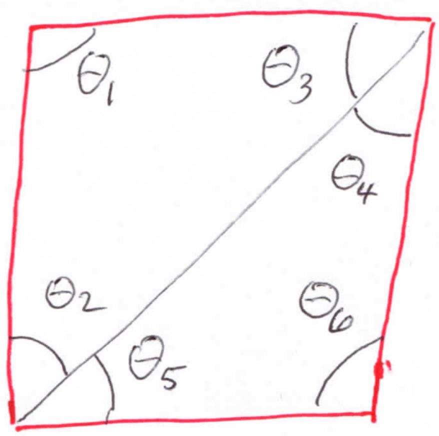
Table 2.3

Number of Copies	Measure of Interior Angles	
2	$\frac{360}{2} = 180$	
3	$\frac{360}{3} = 120$	
4	$\frac{360}{4} = 90$	
5	$\frac{360}{5} = 72$	
6	$\frac{360}{6} = 60$	
7	$\frac{360}{7} \sim 51$	
8	$\frac{360}{8} = 45$	
⋮	⋮ decreasing	

Discussion 2.4. We have shown (Discussion 2.2 and Table 2.3) that tiling with regular polygons requires the correct (listed in the second column of Table 2.3) interior angles. Let's worry about interior angles in a regular polygon.

You may have heard that the sum of the measures of the interior angles in a triangle is 180 degrees (see [3, C.C page 17]). Similar results hold for any polygon; for simplicity, let's restrict ourselves to regular n-gons.

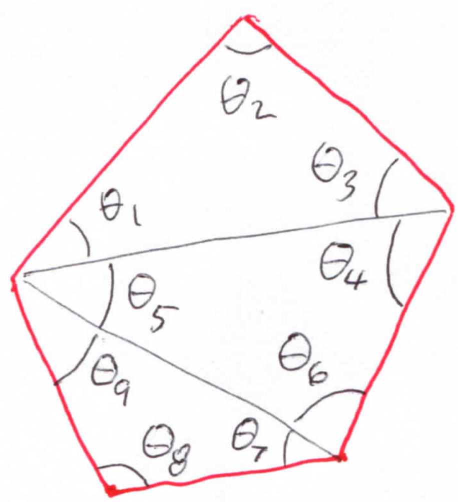
The sum of the measures of the interior angles in a square (a regular 4-gon) may be seen to be $360 = 2 \times 180$ degrees with the following picture



since

$$\text{(sum of measures of interior angles)} = \theta_6 + (\theta_4 + \theta_3) + \theta_1 + (\theta_2 + \theta_5) = (\theta_1 + \theta_2 + \theta_3) + (\theta_4 + \theta_5 + \theta_6)$$
$$= 180 + 180 = 2 \times 180 = 360 \text{ degrees.}$$

The sum of the measures of the interior angles in a regular 5-gon is similarly $540 = 3 \times 180$ degrees:



In general (see [2, Corollary 3.10] for details), for n $= 3, 4, 5, \ldots$, the sum of the measures of the interior angles in a regular n-gon is $(n-2) \times 180$ degrees, thus the measure of each interior angle in a regular n-gon is

$$\frac{(n-2)}{n} 180 \text{ degrees.}$$

See Table 2.5 on the next page. All angle measures are in degrees.

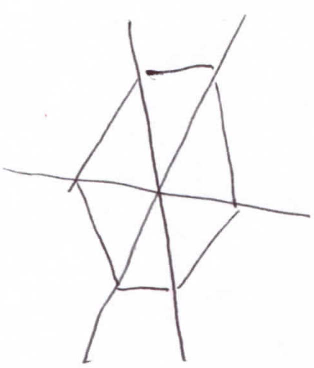
Table 2.5

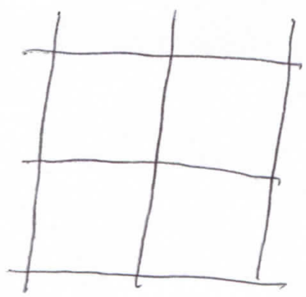
n = Number of Sides	Number of Triangles	Sum of Measures of Interior Angles	Measure of Interior Angles
3	1	$1 \times 180 = 180$	$\frac{180}{3} = 60$
4	2	$2 \times 180 = 360$	$\frac{360}{4} = 90$
5	3	$3 \times 180 = 540$	$\frac{540}{5} = 108$
6	4	$4 \times 180 = 720$	$\frac{720}{6} = 120$
7	5	$5 \times 180 = 900$	$\frac{900}{7} \sim 129$
8	6	$6 \times 180 = 1,080$	$\frac{1080}{8} = 135$
⋮	⋮	⋮	⋮ increasing

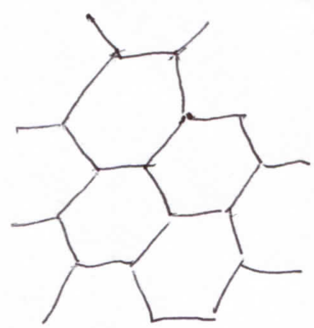
Compare Table 2.3, giving the interior angle measures that are *needed* for tiling, to Table 2.5, giving the interior angle measures that are *possible* for a regular n-gon. Said comparison will answer Big Question 2.1.

Answer 2.6, to Big Question 2.1. We can tessellate with copies of the same regular n-gon only when n = 3, 4, or 6.

That is, we can tessellate only with copies of an equilateral triangle, or with copies of a square, or with copies of a regular hexagon.







n=3

n=4

n=6

It is surprising that, out of the infinitely many possibilities (n-gons, with n equal to $3, 4, 5, \ldots$) there are only three regular polygons (n equal to three, four, or six) with which we can tessellate as in 2.1.

3. COMPARISON of TESSELLATIONS with TRIANGLES, TESSELLATIONS with SQUARES, and TESSELLATIONS with HEXAGONS

Assume throughout this section that a honeycomb stores food in containers whose cross sections are copies of a fixed regular polygon. Section 2 showed (Answer 2.6) that, to make the cross sections tessellations, the fixed regular polygon must be either a triangle, square, or hexagon; that is, we must do our tiling with only equilateral triangles, only squares, or only regular hexagons.

The question remains whether to choose equilateral triangles, squares, or regular hexagons.

We will show in this section (Theorem 3.1) that a tessellation with regular hexagons is superior to a tessellation with squares, which in turn is superior to a tesselation with equilateral triangles, in a way that we will now describe.

The containers described in the first sentence of this section are made of beeswax. On a cross section, this beeswax is the perimeter of the regular polygon, drawn in red below, and the stored food is in the interior of the regular polygon, shaded in black below.

Since beeswax is a big deal to produce, we'd like to store as much food as possible, for a fixed amount of beeswax. In terms of the polygonal cross section, we want, for a fixed perimeter, as much area as possible.

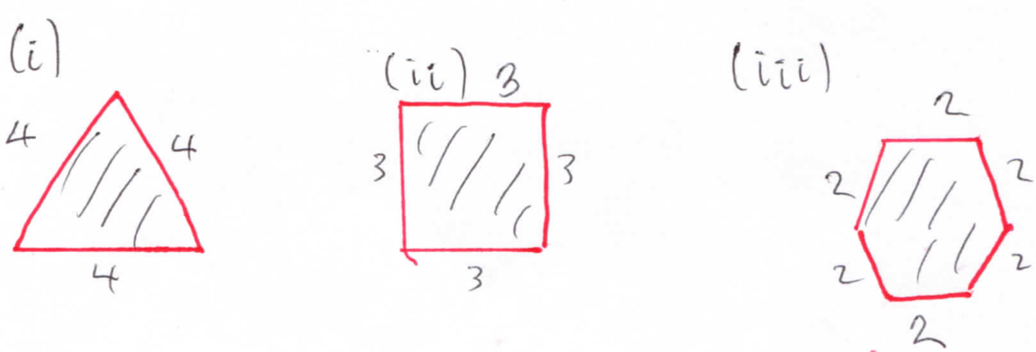
Theorem 3.1. (a) A regular hexagon has a greater area than a square with the same perimeter.

(b) A square has a greater area than an equilateral triangle with the same perimeter.

In particular, hexagons maximize $\frac{\text{area}}{\text{perimeter}}$, the ratio of area to perimeter, among the possible tessellations.

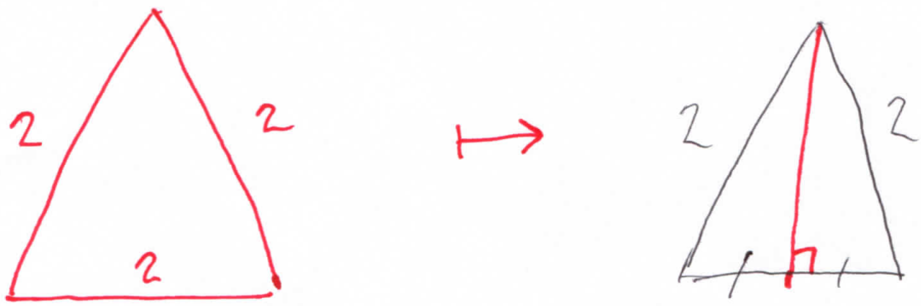
Proof. Just to keep our numbers simple, we will assume we have beeswax for a perimeter of 12 inches; the same argument, with more awkward numbers, would work for any perimeter.

We are comparing the shaded areas of a regular hexagon whose sides each measure 2, a square whose sides each measure 3, and a triangle whose sides each measure 4, as drawn below.

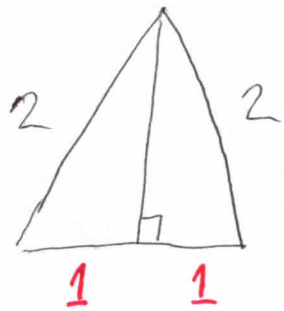


We will find it convenient to focus initially on a triangle whose sides each measure 2.

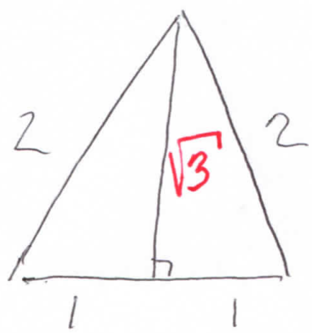
Draw a perpendicular line segment from a vertex to the opposite side, as drawn below.



By the Pythagorean theorem, the two line segments formed by the intersection of the aforementioned perpendicular line segment with the opposite side have equal length (see drawing above). Since their lengths add up to 2, each of the two line segments just mentioned have length 1, as drawn below.



By the Pythagorean theorem again, our perpendicular line segment has length $\sqrt{3}$.



Our triangle whose sides each measure 2 is now seen to have area

$$\frac{1}{2}(\text{base})(\text{height}) = (\frac{1}{2})(2)(\sqrt{3}) = \sqrt{3}.$$

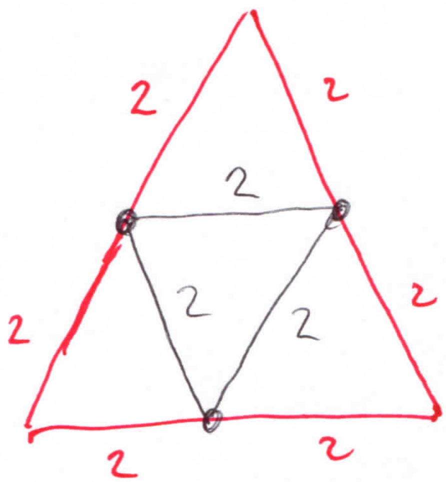
Now let's address the polygons (i), (ii), and (iii).

First, the easiest polygon: for a square whose sides each measure 3, the area is

$$3^2 = 9 \text{ for the area of (ii).}$$

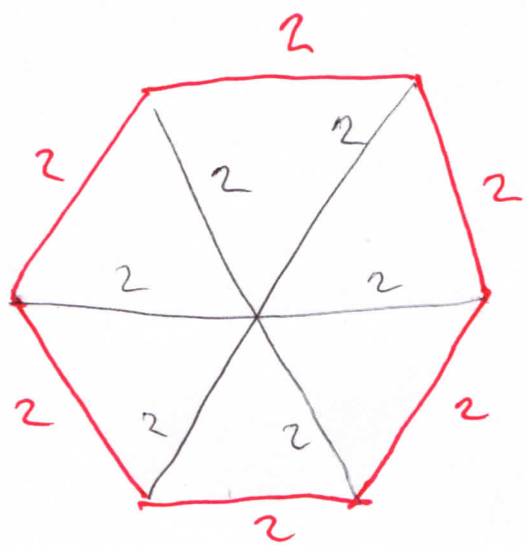
For the other polygons, we would like to argue that (i) is four triangles, each of whose sides measure 2, put together, while (iii) is six such triangles put together, as drawn below.

(i)



area = $4\sqrt{3} \sim 6.9$

(ii)

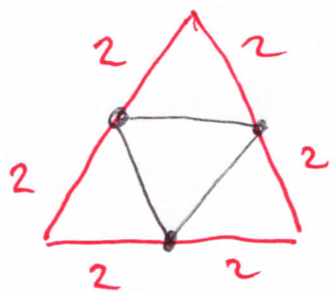


area = $6\sqrt{3} \sim 10.4$

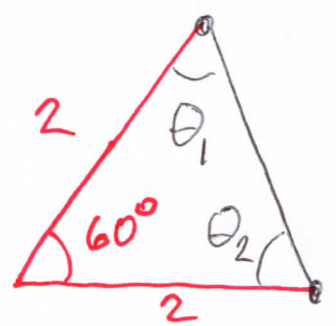
To justify the pictures we just drew (in particular, we need to believe that the black line segments each measure 2), we need the following, from [3, C.A page 15] (title not from [3]):

Sides versus Angles. In any triangle, two sides have equal length if and only if the angles opposite the sides have equal measure.

Consider polygon (i), drawn in red below. Draw, in black, lines between midpoints of each side.



Focus on the leftmost triangle; the same argument will apply to the other triangles.



By "Sides versus Angles," $\theta_1 = \theta_2$, thus by the second paragraph of Discussion 2.4, measuring in degrees,

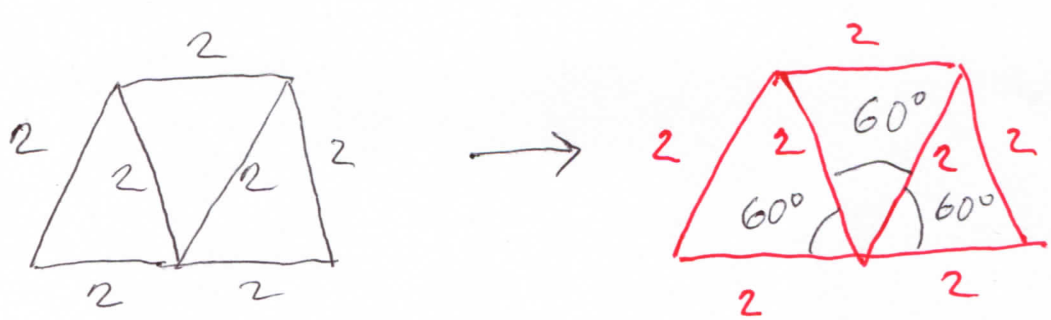
$$180 = 60 + \theta_1 + \theta_2 = 60 + 2\theta_1,$$

so that

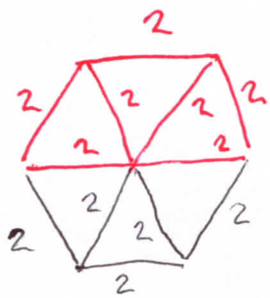
$$\theta_1 = \theta_2 = 60;$$

applying "Sides versus Angles" again tells us that the black line segment is of length 2, as desired.

Now consider polygon (iii). Three triangles, each of whose sides measure 2, may be placed side by side with no gaps as follows:



Since the sum of the indicated angle measures is 180 degrees, the base of the figure just drawn is a single line segment, thus said figure can be reflected through its base without creating any gaps, to get the desired picture of (iii).



Having drawn both (i) and (iii) as unions of triangles whose areas have been shown to be $\sqrt{3}$, we merely count the number of said triangles in each union, to get the area of (i) equal to $4\sqrt{3} \sim 6.9$ and the area of (iii) equal to $6\sqrt{3} \sim 10.4$.

Since $6\sqrt{3}$ is greater than 9, we get (a); since 9 is greater than $4\sqrt{3}$, we get (b). □

Other Hexagons 3.2. When basaltic lava cools, it sometimes breaks into columns whose top is a tessellation. Assuming the cooling is uniform, we might expect the tessellation to consist of copies of a fixed polygon.

As with honeycombs, Section 2 showed that the top of these columns should consist primarily of equilateral triangles, squares, or regular hexagons.

Theorem 3.1 implies that hexagons minimize $\frac{\text{perimeter}}{\text{area}}$, the ratio of perimeter to area, hence minimize the force required to break up a given area of lava. Thus, for a fixed mass of lava, as it cools the force trying to break it up increases continuously, and will reach the force needed to break up into hexagons before it reaches the force needed to break up into squares or triangles.

The analysis of the previous paragraph comes through approximately in fact: the top of the columns broken is predominantly a tessellation with fairly regular hexagons. See, for example, pictures of Devil's Postpile on the web, including Wikipedia. The qualifier "approximately" is due to the presence of other geologic factors.

See also pictures of Giant's Causeway in Northern Ireland.

REFERENCES

1. R. deLaubenfels, "Fibonacci Numbers and the Golden Ratio Magnification," https://teacherscholarinstitute.com/MathMagnificationsReadyToUse.html.

2. R. deLaubenfels, "Vectors Point to Geometry and Trigonometry," https://teacherscholarinstitute.com/Books/Vectors-Point-to-Geometry-and-Trigonomety.pdf (2019).

3. J. Saxon, "Algebra 1. An Incremental Development," Second Edition, Saxon Publishers, Inc., 1990.

4. Voltaire, "Candide."