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Convergent Sequences
MATHematics MAGnification™

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CONVERGENT SEQUENCES MAGNIFICATION

This is one of a series of very short books on math, statistics, and physics called "Math Magnifications." The "magnification" refers to focusing on a particular topic that is pivotal in or emblematic of mathematics.

OUTLINE

The concept underlying all aspects of calculus is that of a *convergent sequence* or *limit of a sequence.*

We will informally define an infinite sequence of real numbers, denoted *sequence* for short, in Chapter I. We will informally define a *convergent* sequence and its limit in Chapter II, and give many examples, without proofs. The Appendix will give and illustrate the rigorous definitions of sequence and convergence.

The only prerequisites for this Magnification are, except for the Appendix, [9]; the Appendix might be more challenging.

Throughout this Magnification, the symbol "$\equiv$" means "is defined to be."

Speaking of definitions, we need to define the **factorial** of a nonnegative integer:

0! (reads "zero factorial") $\equiv 1$; 1! (reads "one factorial") $\equiv 1$; 2! (reads "two factorial") $\equiv 2\times 1 = 2$; 3! (reads "three factorial") $\equiv 3 \times 2 \times 1 = 6$; 4! (reads "four factorial") $\equiv 4 \times 3 \times 2 \times 1 = 24$;

for any nonnegative integer n,

$$n! \text{ (reads "n factorial")} \equiv n \times (n-1) \times (n-2) \times \cdots 5 \times 4 \times 3 \times 2 \times 1.$$

p. 2

CHAPTER I: SEQUENCES

We talked about popular sequences of populations in [6]. Here is a more general definition. See the Appendix for a formal definition of sequence.

Informal Definition 1.1. For this Magnification, a **sequence** will be a set of real numbers

$$\{x_n\}_{n=1}^{\infty} \equiv x_1, x_2, x_3, \ldots$$

indexed by the natural numbers $\mathbf{N} \equiv \{1, 2, 3, 4, \ldots\}$.

Repetition is possible; for example, the constant sequence

$$1, 1, 1, 1, \ldots \quad \text{or} \quad x_1 = 1, x_2 = 1, x_3 = 1, x_4 = 1, \ldots$$

defined by $x_n = 1$, for all natural numbers n, satisfies our definition of sequence.

The index n is often time, e.g., number of hours after noon, January 1, 2019. For any natural number n, x_n might then be a measurement made at time n, such as the temperature.

In conversational usage, "sequence" or "sequential" implies that order makes a difference; for example, the sequence

$$1.5, -1, 0, 0, 0, \ldots \quad \text{or} \quad x_1 = 1.5, x_2 = -1, x_3 = 0, x_4 = 0, x_5 = 0, \ldots, x_n = 0 \text{ for all } n \geq 3,$$

is a different sequence than

$$-1, 1.5, 0, 0, 0, \ldots \quad \text{or} \quad x_1 = -1, x_2 = 1.5, x_3 = 0, x_4 = 0, x_5 = 0, \ldots, x_n = 0 \text{ for all } n \geq e,$$

even though each sequence involves only the numbers $-1, 0, 1.5$.

Examples 1.2. Notation as compact as Definition 1.1 may be used to describe many sequences of interest. For example, the action of doubling every generation as in [6, Section 3], beginning with two organisms:

$$2, 4, 8, 16, \ldots,$$

may be described as

$$\{2^n\}_{n=1}^{\infty};$$

note that, when $n = 1, x_n \equiv x_1 = 2^1 = 2, n = 2$ makes $x_n \equiv x_2 = 2^2 = 4$, etc.; any member of the sequence may be calculated by replacing n with the particular value of n that is of interest.

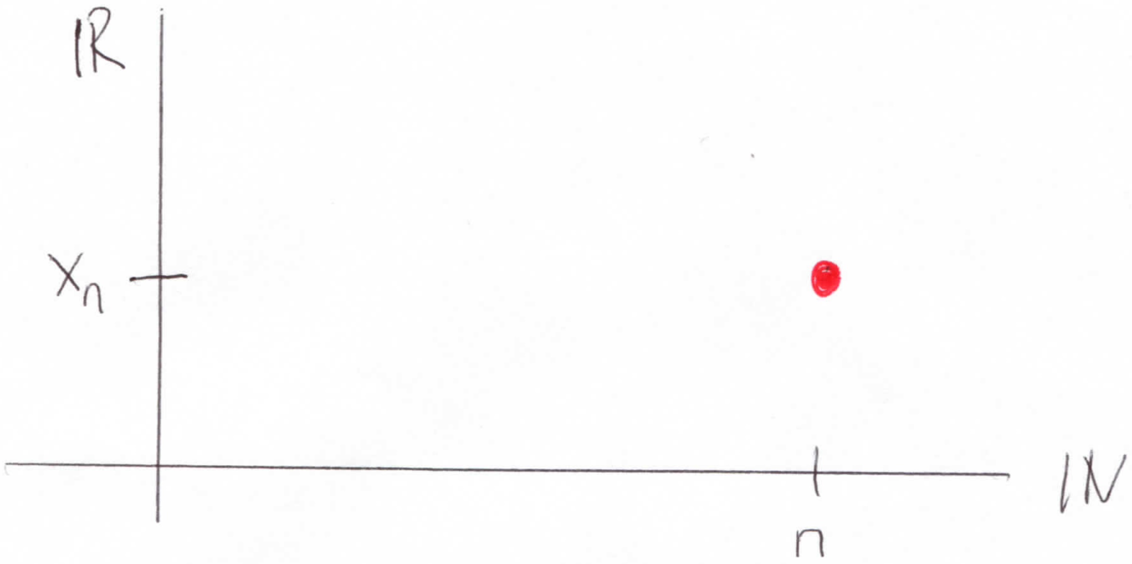
It is worth mentioning our favorite method of getting oscillation:

$$\{(-1)^n\}_{n=1}^{\infty} = (-1)^1, (-1)^2, (-1)^3, (-1)^4, (-1)^5, \cdots = -1, 1, -1, 1, -1, \ldots.$$

But the reader should be warned that sequences might not have the sort of explicit description we just illustrated. To construct an arbitrary sequence:

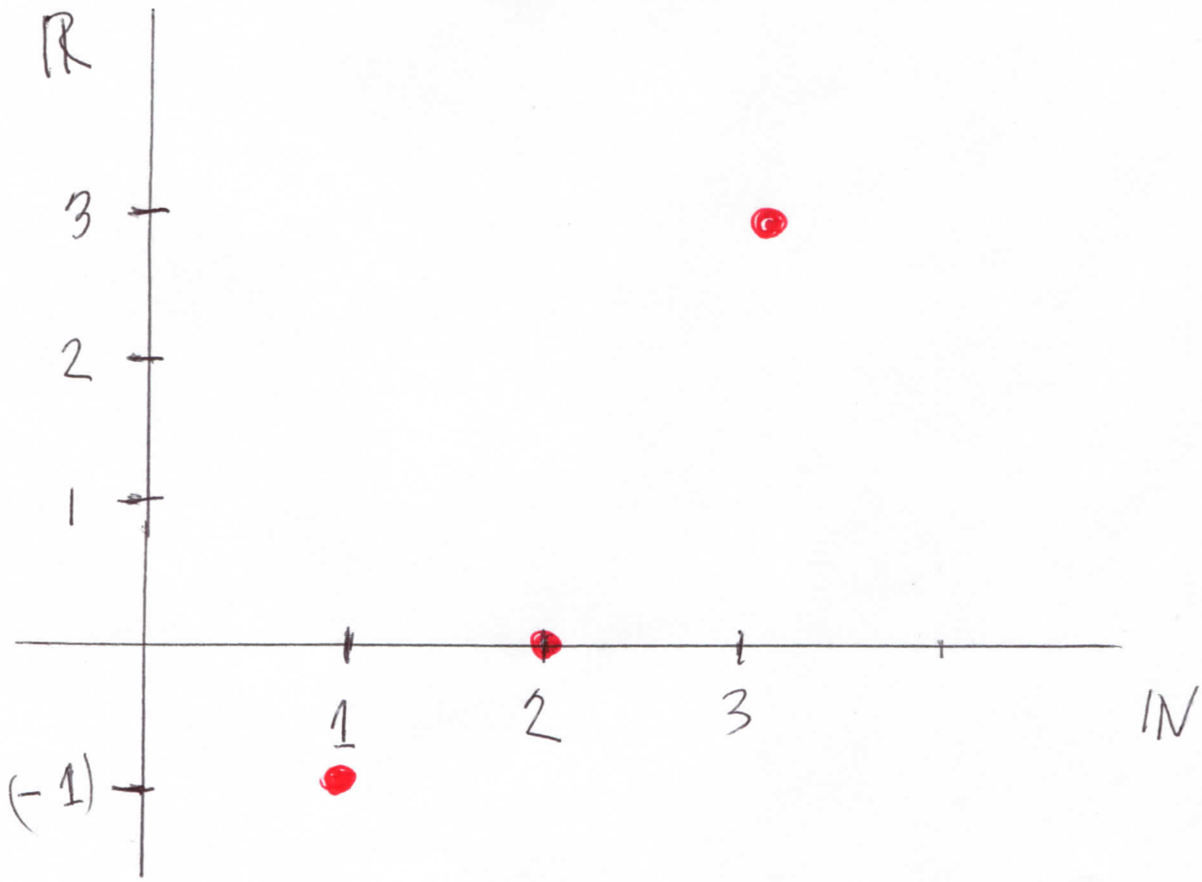
Imagine a fish bowl containing every real number. Get x_1 by reaching into the fish bowl and pulling out a randomly chosen real number, then return said number to the fish bowl. Get x_2 by again choosing a random real number from the fish bowl, then returning it. Continue sampling "with replacement" indefinitely. "With replacement" means that every number chosen is put back into the fish bowl before the next number is chosen; this makes repetition of numbers in the sequence possible.

Picture 1.3. We can get a good picture of a sequence, by making the natural numbers **N** the horizontal axis in the plane and the real numbers **R** the vertical axis: for each $n = 1, 2, 3, \ldots$, make a fat dot for x_n n units to the right of the origin and x_n units above the origin, as drawn directly below.



Example 1.4. Below we have drawn the first three elements of the sequence

$$\{(n^2 - 2n)\}_{n=1}^{\infty} = -1, 0, 3, 8, \ldots$$



CHAPTER II: CONVERGENCE

Everyone must deal with *error*, or, to put it more politely, *approximations.* For example, if I'm buying rutabaga seed for a disc, of radius one mile, of farmland, neither I nor my calculator can calculate with the actual area of π miles squared; whether we admit it or not, we will work with a decimal approximation of π. Error is inevitable.

The *accuracy* of an approximation, meaning the distance between the approximation and the real value

$$|\text{ approximation } - \text{ real value }|,$$

must be worried about.

Two decimal places of a decimal expansion seems to be popular; e.g., $\pi \sim 3.14$, where "$\sim$" means "is approximately equal to," a sort of morally compromised equality sign.

There might be circumstances where we are not satisfied with the accuracy of two decimal places. Retinal laser surgery springs to mind. In general, we don't know in advance how much accuracy we, or future conscious entities, might need.

Ideal would be to have *any* specified accuracy of approximation available; this leads to a *sequence* of approximations, from which anyone can select any desired accuracy.

Informal Definition 2.1. Suppose $\{x_n\}_{n=1}^{\infty}$ is a sequence and L is a real number. We say that the **limit, as n goes to ∞, of x_n equals** L, or x_n **converges to L as n goes to** ∞, denoted

$$\lim_{n\to\infty} x_n = L \qquad \text{or} \qquad x_n \to L \quad \text{as } n \to \infty,$$

if any specified accuracy, in approximating L with x_n, may be guaranteed by making n sufficiently large; that is,

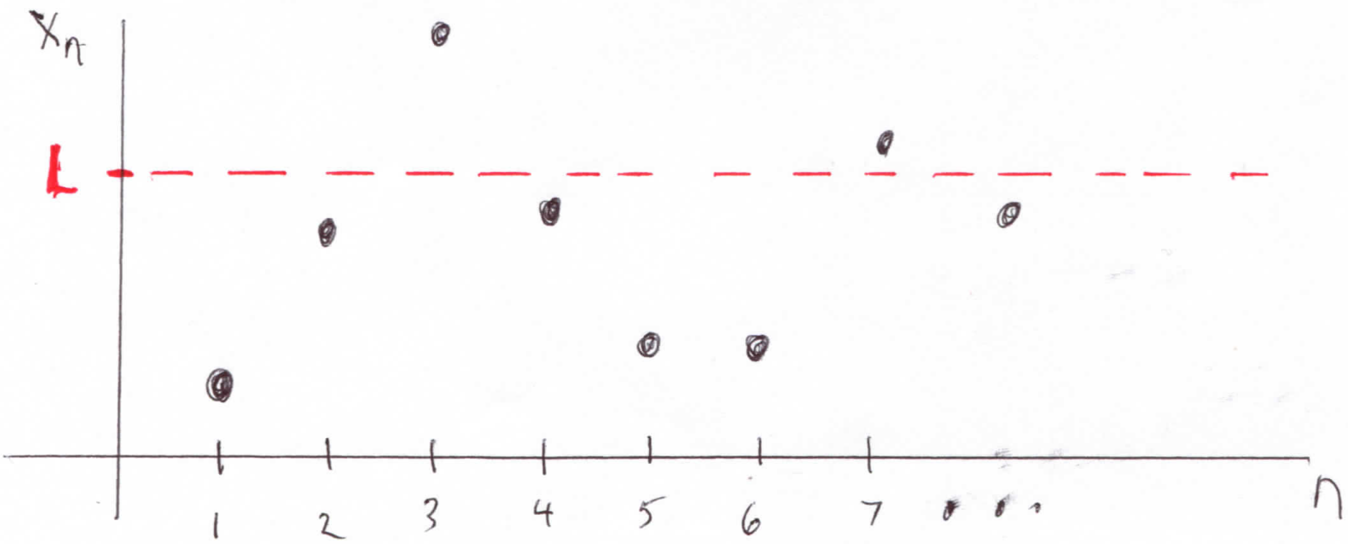
$$|x_n - L|$$

may be forced to be as small as we like by making n sufficiently large.

In the picture below, the red dotted line $y = L$, sometimes called a **horizontal asymptote**, is the limit that x_n, represented by black dots, is getting arbitrarily close to, as n gets arbitrarily large.

If, for any n, x_n is a physical state at time n, L is an *equilibrium state* that x_n aspires to, getting arbitrarily close to L but not necessarily equaling L at any time.

See the Appendix for a formal definition of limit.

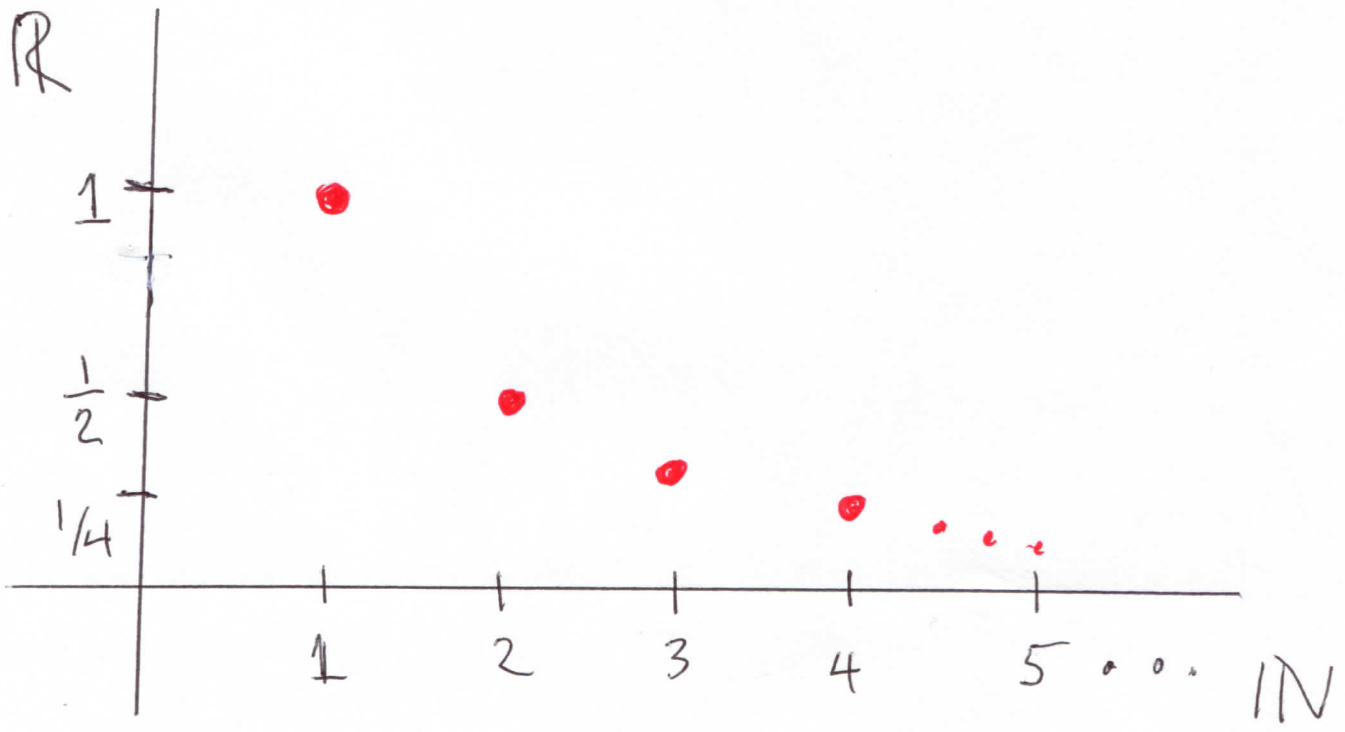


Examples 2.2

(1).

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

might seem believable: a fraction gets smaller when we enlarge the denominator.



Note that $\frac{1}{n}$ never <u>equals</u> 0, although it gets arbitrarily close to 0.

(2). $$\lim_{n\to\infty} \frac{(2n^3 - 10n + 97)}{(6n^3 + n^2 + 19)} = \left(\begin{array}{l}\text{dividing} \\ \text{top + bottom} \\ \text{by } n^3\end{array}\right)$$

$$\lim_{n\to\infty} \frac{\left(2 - \frac{10}{n^2} + \frac{97}{n^3}\right)}{\left(6 + \frac{1}{n} + \frac{19}{n^3}\right)} = \frac{2}{6} = \frac{1}{3},$$

since $\frac{1}{n}$, hence any positive power of $\frac{1}{n}$, $\to 0$, as $n \to \infty$.

More informally,
in both the numerator &
denominator, the highest
power of n dominates, as
n gets large, so we could
reason that

$$\frac{(2n^3 - 10n + 97)}{(6n^3 + n^2 + 19)} \sim \frac{2n^3}{6n^3}$$

$$= \frac{1}{3}.$$

The techniques for this example work for any ratio of polynomials (called a <u>rational</u> <u>function</u>) so long as the highest power in the numerator does not exceed the highest power in the denominator.

(3). A decimal expansion for a number L contains a sequence converging to L.

For example, when we write

$$\pi = 3.14159\ldots$$

we mean that π is the limit of the sequence

$$3,\ 3.1,\ 3.14,\ 3.141,\ \ldots$$

(4). Infinite sums are another example of a sequence that we hope will converge.

For example, it can be shown that

$$\pi = \sum_{k=0}^{\infty} \frac{4(-1)^k}{(2k+1)} =$$

$$4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots$$

$k=0$ $\quad$ $k=1$ $\quad$ $k=2$ $\quad$ $k=3$ $\quad$ ...

Convergence of the infinite sum means the sequence of <u>partial sums</u>

$4, \left(4 - \frac{4}{3}\right) = \frac{8}{3}, \left(4 - \frac{4}{3} + \frac{4}{5}\right) = \frac{52}{15}, \ldots$

converges to π.

See Examples 2.2 (9). - (11). for more geometric examples of sequences converging to π.

(5). For $n = 1, 2, 3, \ldots$
the expression

$$\left(1 + \frac{1}{n}\right)^n$$

appears when compounding (money) n times a year.

The number "e," defined by

$$e \equiv \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

then appears when <u>compounding continuously</u>, meaning, informally, every instant.

See [5].

The number e is ubiquitous: It arises when the rate of growth of something is proportional to the amount present; e.g., population (more organisms implies more offspring). Radioactive decay is another example, which may be used to estimate how old an object is.

(6). The number e is also an infinite sum, meaning, as in Examples 2.2(4)., the limit of the sequence of partial sums

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} \equiv \lim_{n\to\infty}\left(\sum_{k=0}^{(n-1)} \frac{1}{k!}\right),$$

the limit of the sequence

$$1, \left(1+\frac{1}{1!}\right) \equiv \left(1+\frac{1}{1}\right) = 2,$$

$$\left(1+\frac{1}{1!}+\frac{1}{2!}\right) \equiv \left(1+\frac{1}{1}+\frac{1}{2\cdot 1}\right) = \frac{5}{2},$$

$$\left(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}\right) \equiv \left(1+\frac{1}{1}+\frac{1}{2\cdot 1}+\frac{1}{3\cdot 2\cdot 1}\right) = \frac{8}{3},$$

. . .

(7). Define $x_0 = 2$,
and, for $n = 0, 1, 2, \ldots$,

$$x_{n+1} \equiv \frac{1}{2}\left(x_n + \frac{3}{x_n}\right) \quad (*)$$

The statement $(*)$ is shorthand for infinitely many statements:

$$x_1 \equiv \frac{1}{2}\left(x_0 + \frac{3}{x_0}\right) = \frac{1}{2}\left(2 + \frac{3}{2}\right)$$
$$= 1.75 = \frac{7}{4};$$

$$x_2 \equiv \frac{1}{2}\left(x_1 + \frac{3}{x_1}\right) = \frac{1}{2}\left(\frac{7}{4} + \frac{3}{7/4}\right)$$
$$= \frac{97}{56} \simeq 1.732, \ldots$$

It can be shown that $x_n \to \sqrt{3}$, as $n \to \infty$.

Here is some intuition:

If $x_n \geq \sqrt{3}$, then $\frac{3}{x_n} \leq \frac{3}{\sqrt{3}} = \sqrt{3}$; similarly, if $x_n \leq \sqrt{3}$, then $\frac{3}{x_n} \geq \sqrt{3}$.

Thus x_{n+1}, from $(*)$, is the average of an <u>overestimate</u> of $\sqrt{3}$ with an <u>underestimate</u> of $\sqrt{3}$.

The statement (*) is an example of a <u>recursive</u> definition of a sequence.

In general, a recursive definition of a sequence has the form

$$(**) \quad x_{n+1} = g(x_n) \quad (n = 0, 1, 2, 3, \ldots)$$

for some fixed real-valued g on the real line.

Under the right conditions on g, $(**)$, combined with a specified x_0, defines a sequence that converges to a fixed point of g, meaning a number L such that $g(L) = L$.

Notice that, with

$$g(x) \equiv \frac{1}{2}\left(x + \frac{3}{x}\right),$$

$g(\sqrt{3}) = \sqrt{3}$; that is, $\sqrt{3}$ is a fixed point of g.

We discussed <u>logistic</u> population growth in [6] (see in particular [6, Definitions 4.3]) which is (* *) with

$$g(x) \equiv rx(1-x),$$

for some fixed number r.

For $1 < r < 3$, it can be shown that, for any x_0 between 0 & 1, the sequence defined by

$$x_{n+1} = g(x_n) \equiv r x_n (1 - x_n)$$

converges to $\left(1 - \frac{1}{r}\right)$;

note that, for this g,

$$g\left(1 - \frac{1}{r}\right) = \left(1 - \frac{1}{r}\right);$$

that is,

$\left(1 - \frac{1}{r}\right)$ is a fixed point of

$$g(x) \equiv r x (1 - x).$$

(8). The <u>geometric</u>
<u>series</u>

$$\sum_{k=0}^{\infty} r^k \equiv r^0 + r^1 + r^2 + r^3 + \cdots$$
$$= 1 + r + r^2 + r^3 + \cdots$$

is one of the few infinite series (sums) that converge to something familiar.

In [4, Remarks 11(a)] we show that, for any real r and natural number n,

$$S_n \equiv \sum_{k=0}^{(n-1)} r^k \equiv$$

$$(1 + r + r^2 + \dots + r^{n-1})$$

equals $\dfrac{(1-r^n)}{(1-r)}$.

For $|r| < 1$, it may be shown that

$$|r^n| \to 0 \quad \text{as } n \to \infty$$

thus

$$S_n \to \frac{1}{(1-r)} \quad \text{as } n \to \infty \, ;$$

this is what we mean when we say that

$$\sum_{k=0}^{\infty} r^k \equiv 1 + r + r^2 + \dots$$

$$= \frac{1}{(1-r)} \qquad (*) \qquad \left(\underline{\underline{\text{IF}}}\ |r|<1\right)$$

For example,

$$\sum_{k=0}^{\infty} \left(\tfrac{1}{2}\right)^k = \frac{1}{(1-\frac{1}{2})} = 2 ;$$

this is the resolution of one of Zeno's paradoxes (see [4]).

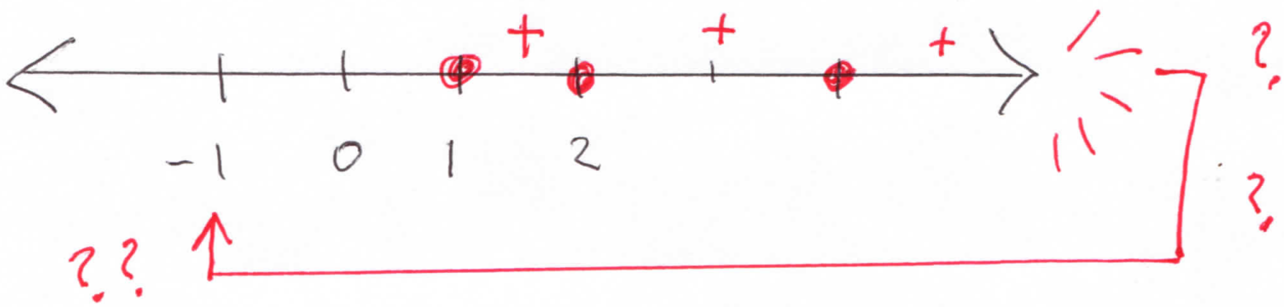
The formula $(*)$ can lead to peculiarities if we forget to insist that $|r|$ be less than 1. For example,

$$\sum_{k=0}^{\infty} 2^k \equiv 1 + 2 + 2^2 + \ldots$$

would then equal

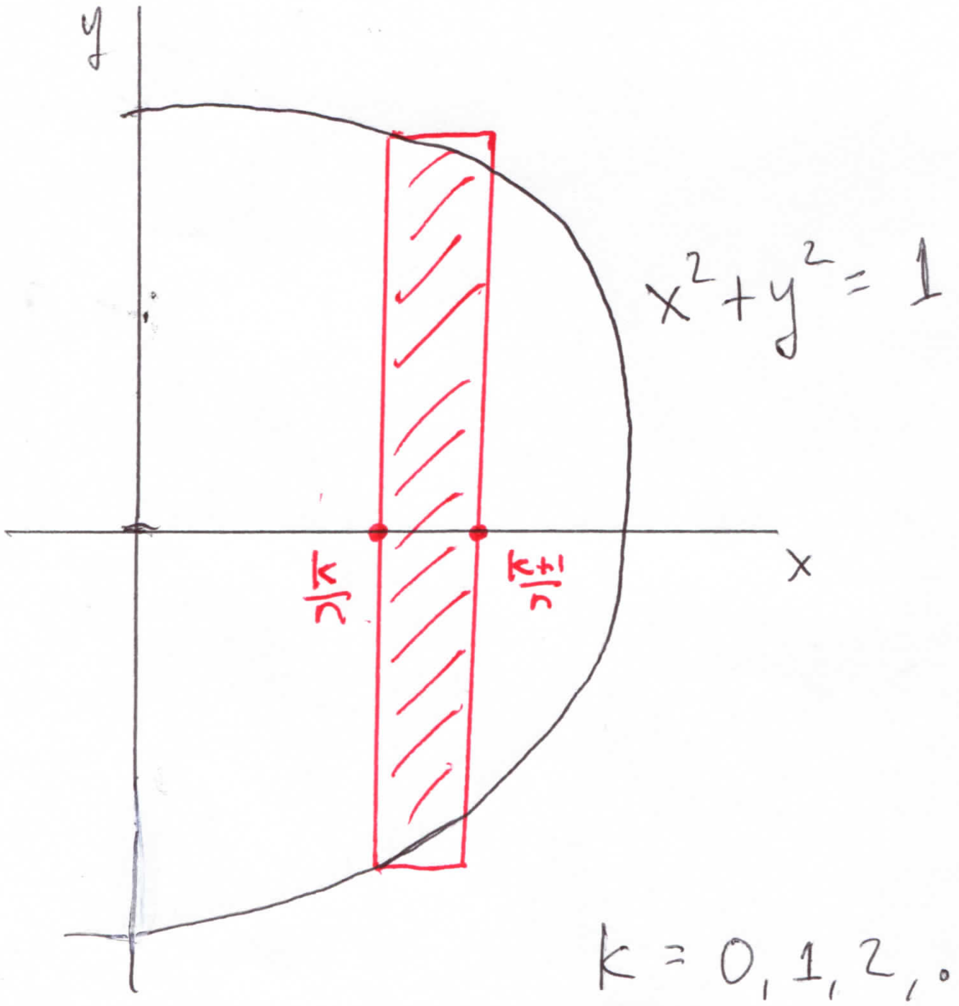
$$\frac{1}{(1-2)} = (-1);$$

somehow, adding increasingly large positive numbers gave us a negative number.



Letting $r = (-1)$ in $(*)$ also gives an interesting result; we will investigate the corresponding partial sums in Examples 2.2 (15).

(9). For arbitrary $n = 1, 2, 3, 4, \ldots$ approximate the right half of the unit disc $x^2 + y^2 \leq 1$ with rectangles as drawn below.



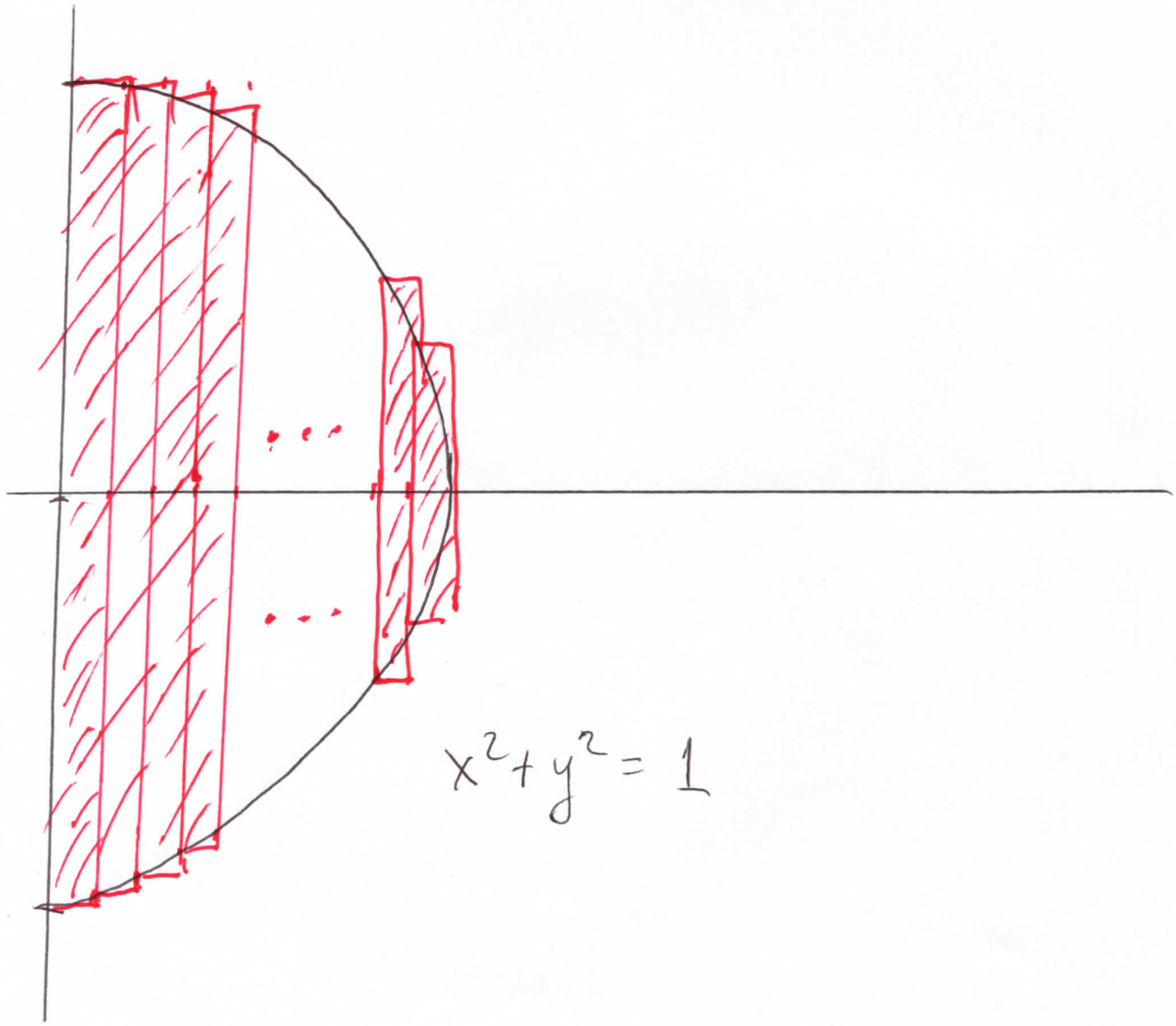
$k = 0, 1, 2, \ldots, (n-1)$

For $n = 1, 2, 3, \ldots,$

let P_n be the union of

the rectangles on the previous

page.



It can be shown that

$$\left(\text{area of } P_n\right) = \frac{2}{n}\sum_{k=1}^{n}\sqrt{\left(1-\left(\frac{k}{n}\right)^2\right)}$$

$$= \frac{2}{n}\left[\sqrt{\left(1-\left(\frac{1}{n}\right)^2\right)} + \sqrt{\left(1-\left(\frac{2}{n}\right)^2\right)} + \cdots + \sqrt{\left(1-\left(\frac{n-1}{n}\right)^2\right)} + \sqrt{\left(1-\left(\frac{n}{n}\right)^2\right)}\right]$$

(this is called a <u>Riemann</u> <u>sum</u> in calculus)

$$\rightarrow \frac{\pi}{2} = \left(\begin{array}{l}\text{area of right}\\ \text{half of unit disc}\end{array}\right)$$

as $n \to \infty$.

However, we will leave it to the reader to observe that

$$(\text{perimeter of } P_n) = 6$$

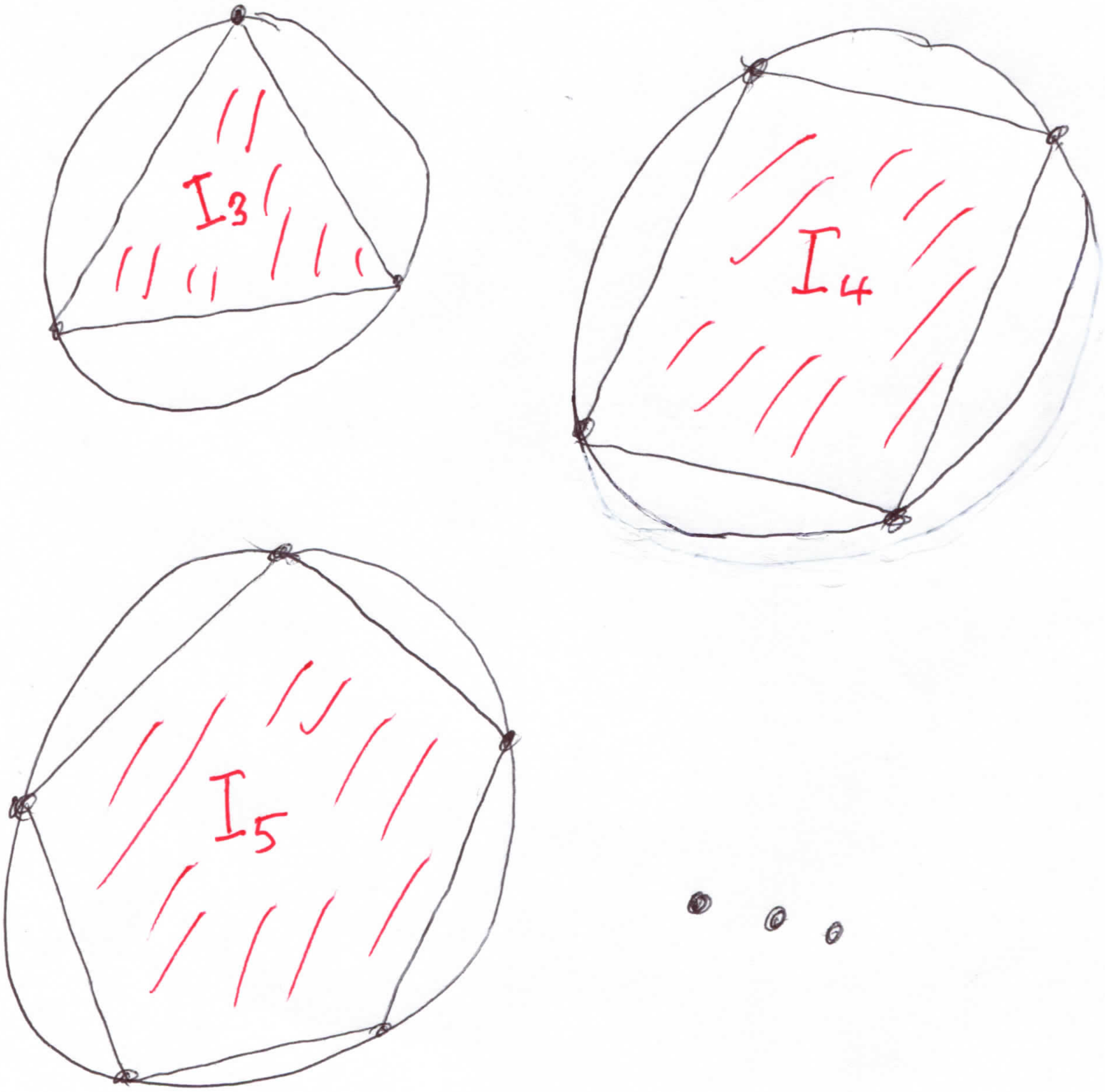
for all n, thus

$$(\text{perimeter of } P_n) \longrightarrow 6$$
$$\neq (2+\pi) = \left(\begin{array}{c}\text{perimeter of right} \\ \text{half of unit disc}\end{array}\right)$$

as $n \to \infty$.

(10). For $n = 3, 4, 5, \ldots,$

let I_n be a regular n-gon inscribed in the unit disc:



$\ldots$

Then it can be shown that, as $n \to \infty$,

$$\left(\text{area of } I_n\right) \longrightarrow \pi = \left(\begin{array}{c}\text{area of}\\ \text{unit disc}\end{array}\right)$$

and

$$\left(\text{perimeter of } I_n\right) \longrightarrow 2\pi = \left(\begin{array}{c}\text{perimeter}\\ \text{of unit disc}\end{array}\right)$$

For those readers who have seen trigonometry,

$$\left(\text{area of } I_n\right) = n\cos\left(\tfrac{\pi}{n}\right)\sin\left(\tfrac{\pi}{n}\right) \;\&$$

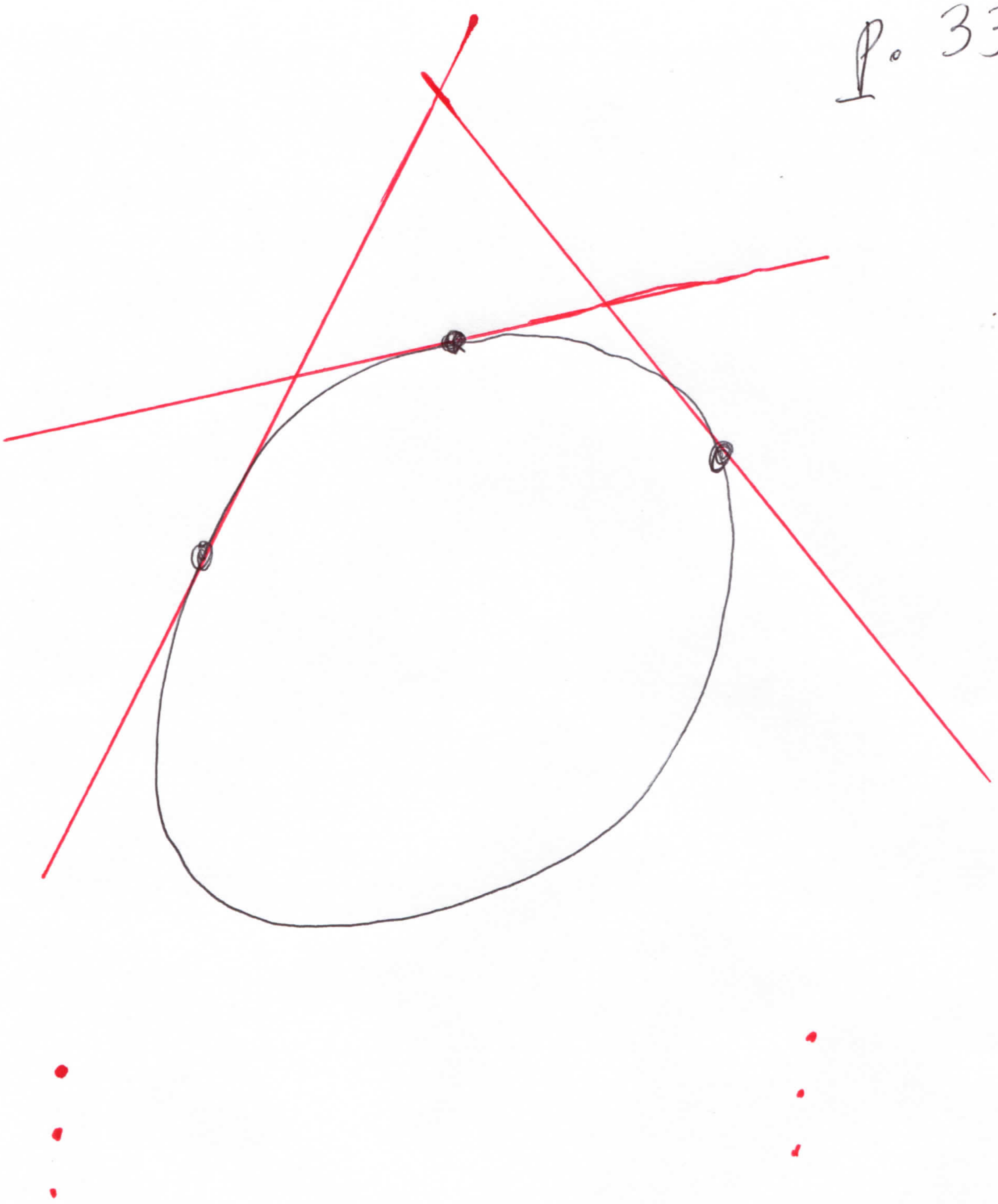
$$\left(\text{perimeter of } I_n\right) = 2n\sin\left(\tfrac{\pi}{n}\right),$$

for $n = 3, 4, 5, \ldots$

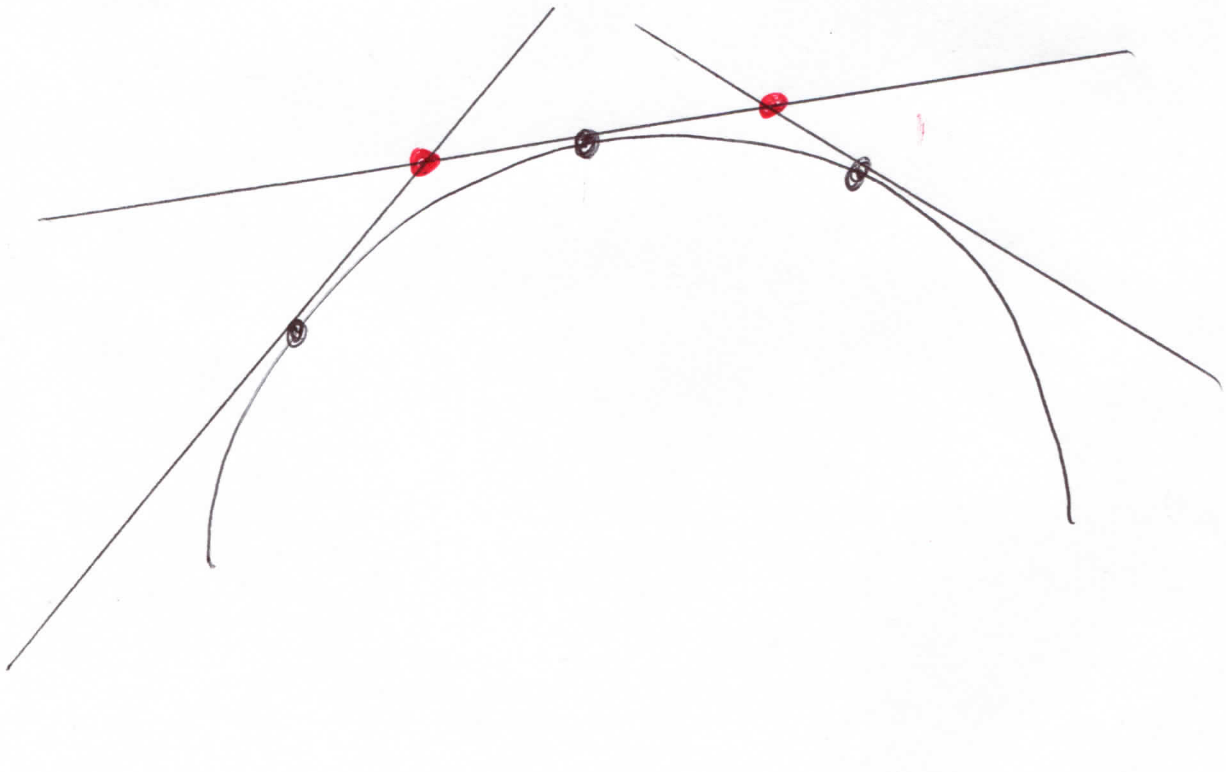
(11). For $n = 3, 4, 5, \ldots$

circumscribe the unit disc with an n-gon, denoted C_n, as follows.

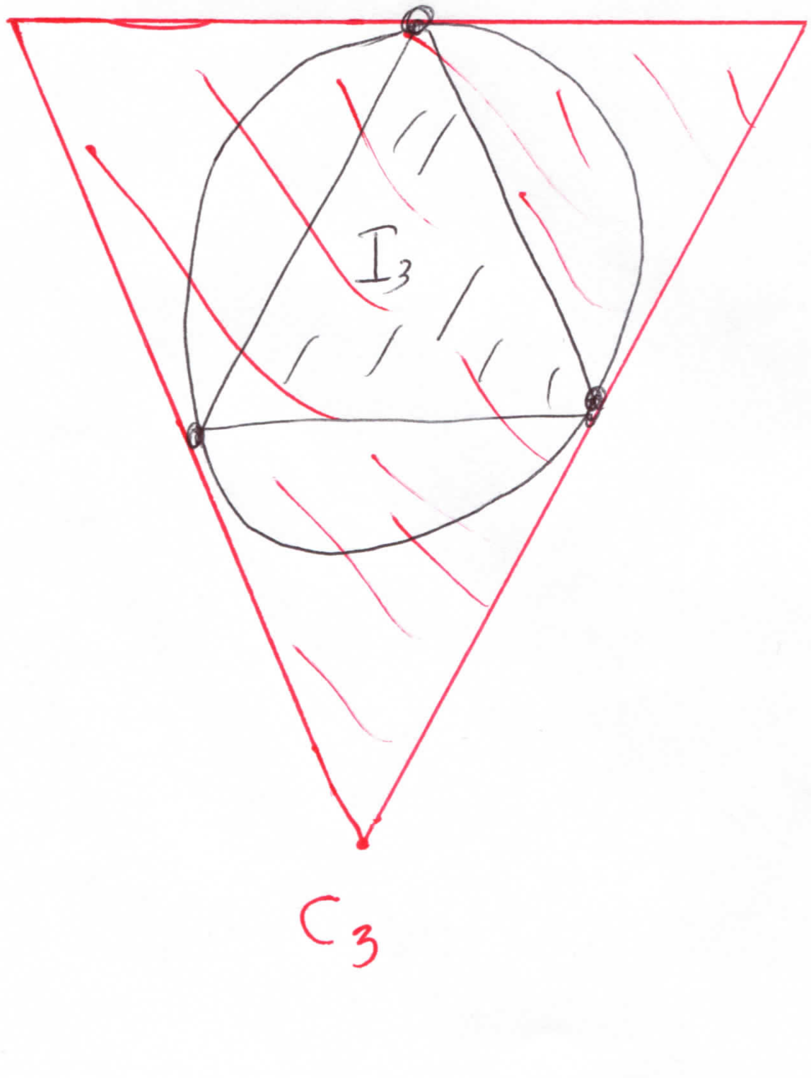
At each vertex of I_n from Examples 2.2(10), draw the tangent line to the unit disc, as on the next page, where tangent lines are red and vertices of I_n are black dots.

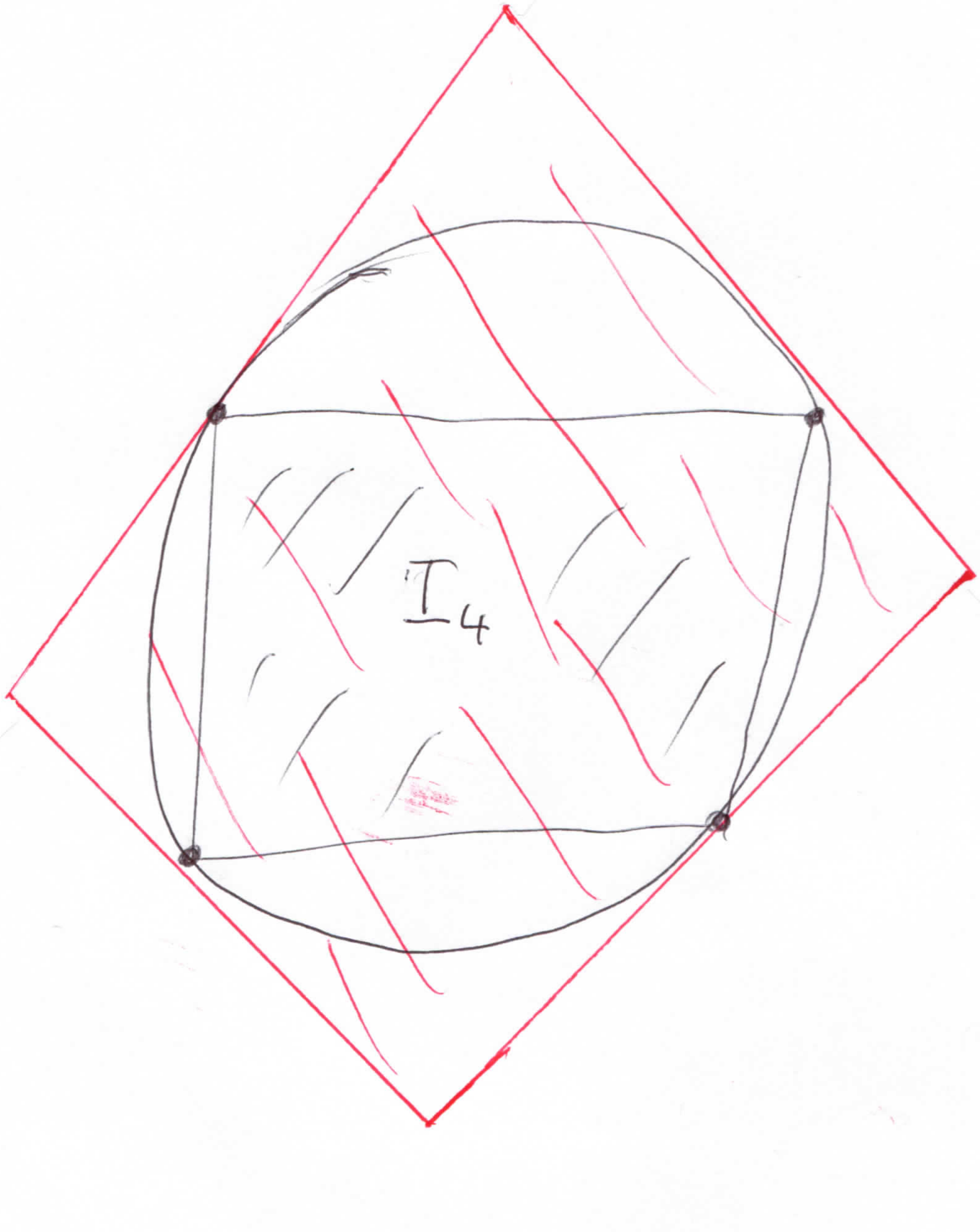


The vertices of C_n, drawn in red below, are the intersections of tangent lines from consecutive vertices of I_n.



Below and on the next page, we have drawn C_3 & C_4 in red and I_3 & I_4, from Examples 2.2(10)., in black.





C_4

Analogous to I_n
in Examples 2.2(10).:

As $n \to \infty$,

$$(\text{area of } C_n) \to \pi = \begin{pmatrix}\text{area of} \\ \text{unit disc}\end{pmatrix}$$

and

$$\begin{pmatrix}\text{perimeter of} \\ C_n\end{pmatrix} \to 2\pi = \begin{pmatrix}\text{perimeter} \\ \text{of unit disc}\end{pmatrix};$$

$$(\text{area of } C_n) = \frac{n \sin(\frac{\pi}{n})}{\cos(\frac{\pi}{n})}$$

$$\begin{pmatrix}\text{perimeter of} \\ C_n\end{pmatrix} = \frac{2n \sin(\frac{\pi}{n})}{\cos(\frac{\pi}{n})}$$

for $n = 3, 4, 5, \ldots$

We should mention here that the classical Greeks did not have the limit concept; among many other things, Zeno's paradoxes (see [10, Chapter 4], [4], and [1, Chapter 1], for example) may be resolved with the idea of limit. However, they had a technique that often led to the same conclusions that we now get from limits.

For example, consider the limits of areas in Examples (10). and (11). Since, for all n, I_n is contained in the unit disc and C_n contains the unit disc, a consequence of Examples (10). and (11). is that the area of the unit disc is π. Essentially this argument, with n equal to powers of 2, was used by the classical Greeks to get the area of the unit disc; see [2, pages 16-19], [10, Section 4.3], and [1, Chapter 19]. In lieu of limits, the classical Greeks used what is known as the <u>exhaustion method</u>; see the references above.

(12). We talked, in [3], about the <u>derivative</u> of a function f, at a point a. In terms of sequences, said derivative is

$$\lim_{n\to\infty} \frac{\left(f(a+\frac{1}{n}) - f(a)\right)}{(1/n)}$$

$$= \lim_{n\to\infty} n\left[f(a+\tfrac{1}{n}) - f(a)\right].$$

For example, if $f(x) \equiv x^2$, the derivative of f at $x=3$ is

$$\lim_{n\to\infty} n\left[(3+\tfrac{1}{n})^2 - 3^2\right],$$

which simplifies to

$$\lim_{n\to\infty} (6+\tfrac{1}{n}) = 6.$$

(13). We also, in [3], talked about the <u>integral</u> of a function f over an interval

$$[a, b] \equiv \{x \mid a \le x \le b\}.$$

For simplicity, we will state this for $[a, b] = [0, 1]$:

The integral of f over $[0, 1]$ is

$$\int_0^1 f(x)\, dx = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{(n-1)} f\left(\frac{k}{n}\right)$$

$$\equiv \lim_{n\to\infty} \frac{1}{n}\left[f(0) + f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n-1}{n}\right) \right].$$

For example, if $f(x) \equiv x^2$, then

$$\int_0^1 f(x)\,dx = \int_0^1 x^2\,dx =$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{(n-1)} \left(\frac{k}{n}\right)^2 = \lim_{n\to\infty} \frac{1}{n^3} \sum_{k=0}^{(n-1)} k^2$$

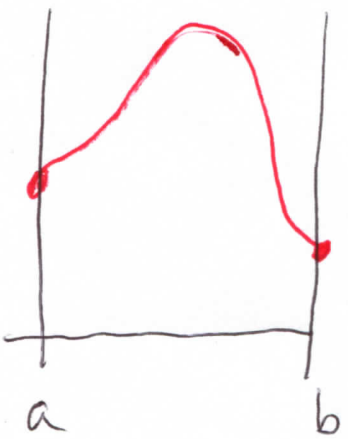
$$\equiv \lim_{n\to\infty} \frac{1}{n^3}\left[0^2 + 1^2 + 2^2 + \cdots + (n-1)^2 \right],$$

which can be shown to equal

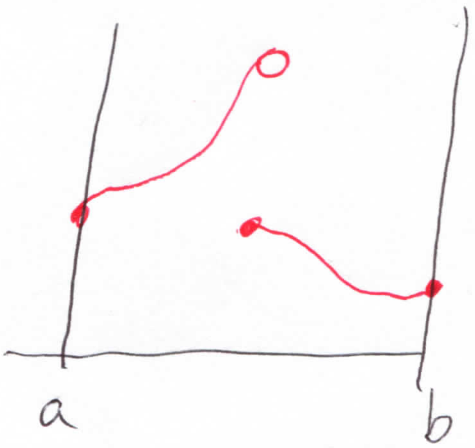
$$\lim_{n\to\infty} \frac{1}{6n^3}\left[(n-1)n(2n-1) \right]$$

$$= \lim_{n\to\infty} \frac{1}{6}\left[\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) \right] = \frac{1}{3}.$$

(14). At least on a closed interval $[a, b] \equiv \{x \mid a \le x \le b\}$, a real-valued function is (informally) <u>continuous</u> if its graph can be drawn without lifting pen from paper.



continuous



not continuous

It can be shown that a function f is continuous at a point c if & only if

$$f(c) = \lim_{n \to \infty} f(x_n) \quad \text{whenever}$$

$$c = \lim_{n \to \infty} x_n .$$

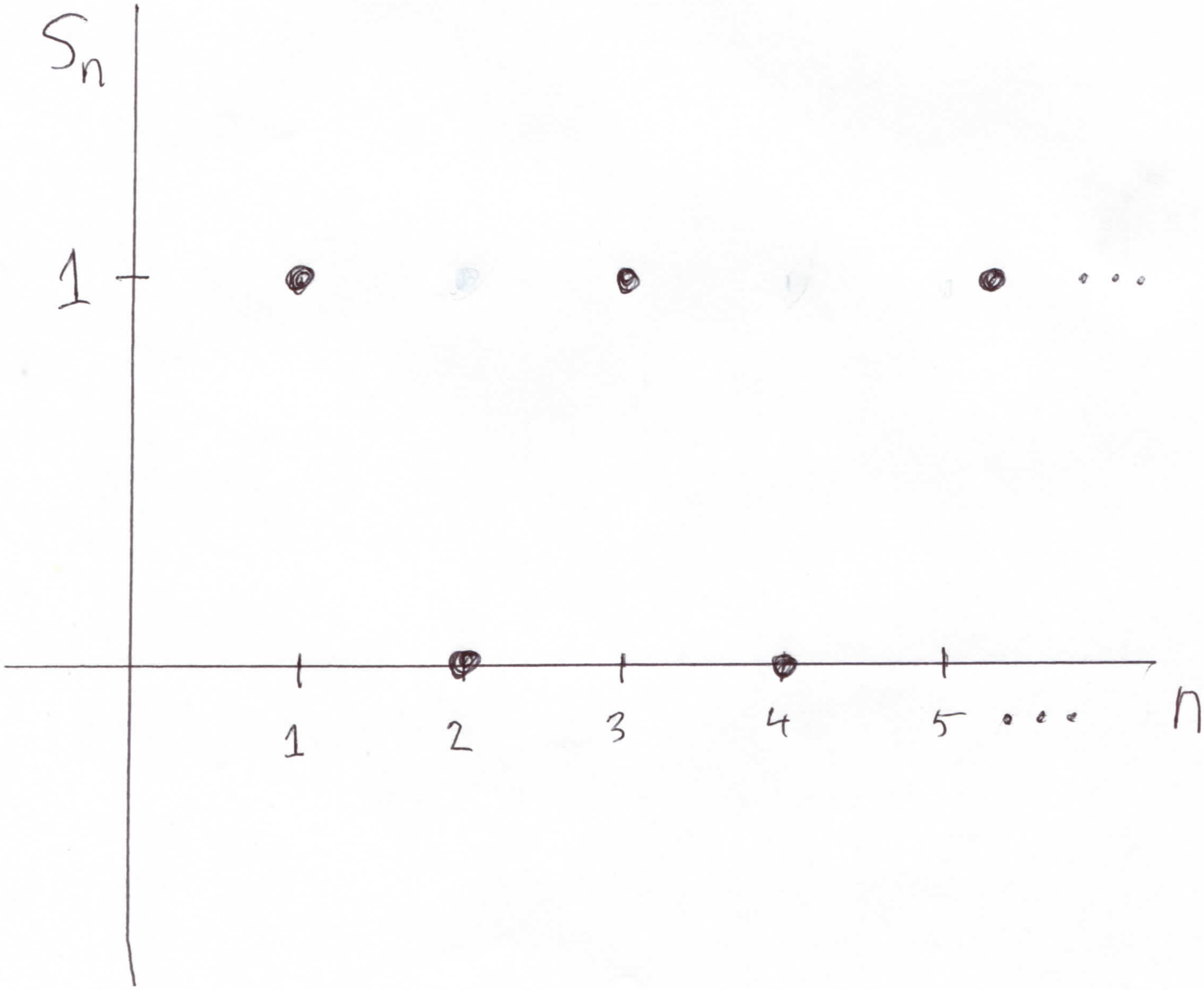
(15). In Examples 2.2(8), when $r = (-1)$, we get the sequence of partial sums

$$S_1 = 1, \; S_2 = 0, \; S_3 = 1, \; S_4 = 0,$$
$$S_5 = 1, \ldots;$$

that is,

$S_n = 0$ if n is even &

$S_n = 1$ if n is odd.



The sequence $\{S_n\}_{n=1}^{\infty}$ does <u>not</u> converge to 1, although we can get members of the sequence arbitrarily close to 1, because we are not <u>guaranteed</u> any specified accuracy in approximating 1 with S_n as n gets large; for example, $|S_n - 1| = 1$ will be greater than $\frac{1}{2}$ whenever n is even, regardless of how large n is.

It can be similarly shown that $\{S_n\}_{n=1}^{\infty}$ does not converge to anything; that is, $\{S_n\}_{n=1}^{\infty}$ has no limit.

Historical Remarks 2.3.

As seemingly disparate ideas evolve and are better understood, they often grow back on themselves. A good idea fuses together what appeared to be different ideas.

In physics, this was the case with electricity and magnetism, unified by Maxwell's equations. In mathematics, integration, differentiation, infinite series, and many other ideas that became part of calculus, were long worked on individually. Newton and Leibniz independently unified integration and differentiation in the late 17th century; this unification is called the Fundamental

Theorem of Calculus.
It took the limit concept to understand and unify all that is now considered a first-year calculus class. The idea of limit did not appear until the 19th century, introduced independently by Cauchy and Bolzano. See [2], [7], and [8].

APPENDIX

Here we briefly give rigorous definitions of sequence and limit of a sequence, then present illustrative applications.

Applying Definition APP. 2 as in Examples APP. 3 can be difficult for upper division and even graduate students. No particularly advanced mathematics is needed, but one must be nimble with fractions and inequalities.

Definition APP. 1.

An (infinite, real-valued) <u>sequence</u> is a function $f: \mathbb{N} \to \mathbb{R}$. It is popular to write x_n for $f(n)$, if x is the value of interest.

Definition APP. 2.

$\lim_{n\to\infty} x_n = L$ (See 2.1 for synonyms) if, for any $\epsilon > 0$, there exists a natural number $N(\epsilon)$ so that

$$|x_n - L| < \epsilon$$

whenever $n \geq N(\epsilon)$.

See the drawing on the next page. The Greek letter ϵ (epsilon) stands for <u>error</u> (in our approximation of L by x_n).

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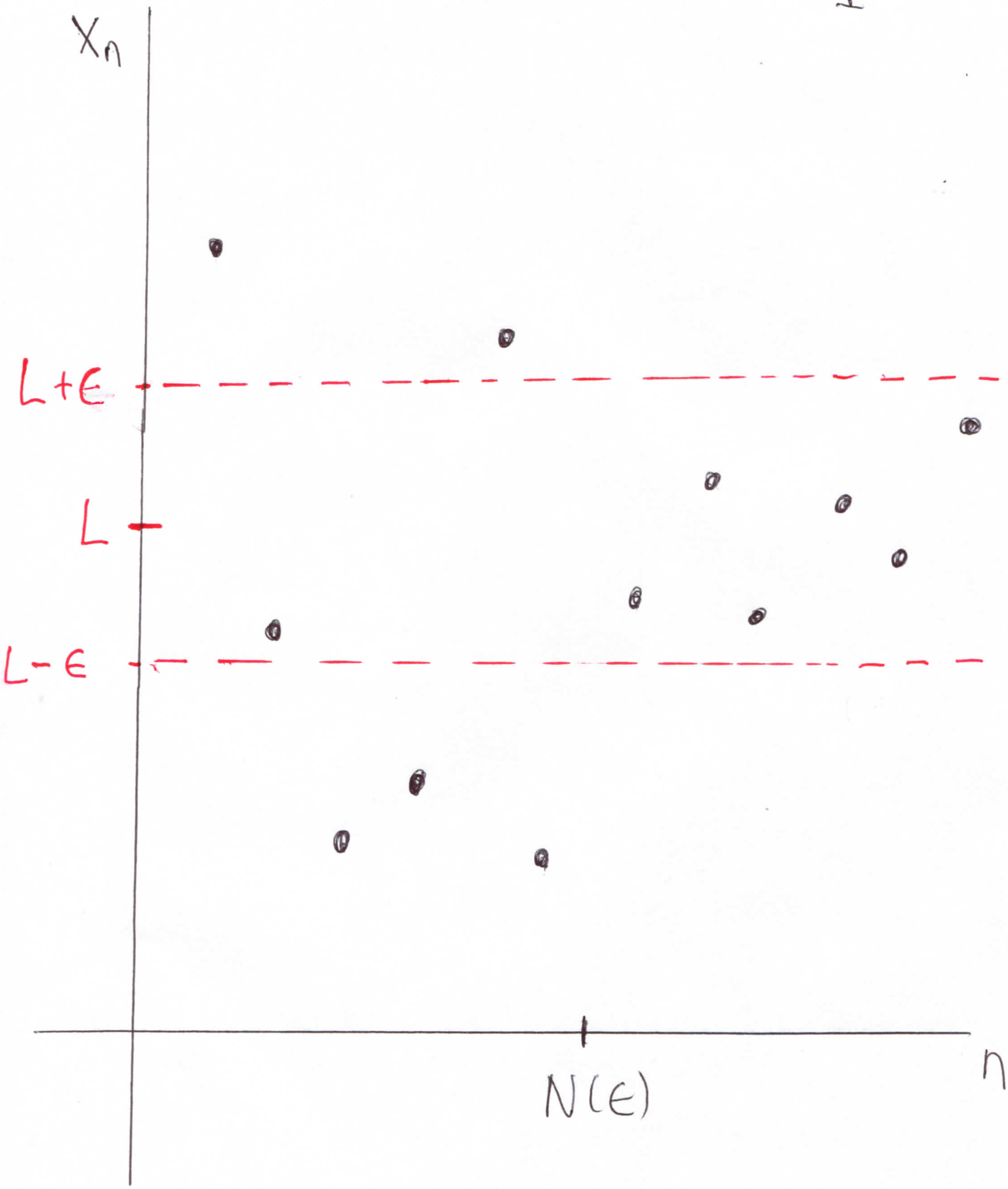

Examples APP.3

(1). Use Definition APP.2 to prove that

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

ANSWER: First do preliminary scratchwork:

$\frac{1}{n} < \epsilon$ if & only if $n > \frac{1}{\epsilon}$.

The scratchwork is not seen by the reader; what the reader sees begins NOW:

Proof:

Fix $\epsilon > 0$. Let $N(\epsilon)$ be an integer greater than $1/\epsilon$.

If $n \geq N(\epsilon)$, then

$$\left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| \leq \left|\frac{1}{N(\epsilon)}\right|$$

$$< \left|\frac{1}{(1/\epsilon)}\right| = \epsilon.$$

QED (Quod Erat Demonstrandum)

(2). Use Definition APP. 2 to prove that

$$\lim_{n\to\infty} \frac{(8n^3-n)}{(4n^3+\sqrt{n+1})} = 2.$$

ANSWER: For scratchwork, let

$$x_n \equiv \frac{(8n^3-n)}{(4n^3+\sqrt{n+1})}$$

$(n = 1, 2, 3, 4, \ldots)$

If we set $|x_n - 2| < \epsilon$, we will find it hard to solve for n, as in (1).

We need a simpler expression to set less than ϵ; said expression should be greater than $|x_n - 2|$, so that, when the simplified expression is less than ϵ, $|x_n - 2|$ will also be less than ϵ

To make a fraction larger, make the numerator larger & the denominator smaller:

$$|x_n - 2| =$$

$$\left| \frac{(8n^3 - n) - 2(4n^3 + \sqrt{n+1})}{(4n^3 + \sqrt{n+1})} \right| =$$

$$\left| \frac{(-n - 2\sqrt{n+1})}{(4n^3 + \sqrt{n+1})} \right| = \frac{(n + 2\sqrt{n+1})}{(4n^3 + \sqrt{n+1})}$$

$$< \frac{(n + 2n)}{4n^3} = \frac{3}{4n^2};$$

larger, if $n > 1$

smaller

set $\frac{3}{4n^2} < \epsilon$ & solve for n:

$$n > \sqrt{\frac{3}{4\epsilon}}.$$

Now we may begin the actual, formal proof, which is all the reader will see:

Proof that $\lim_{n \to \infty} x_n = 2$.

Let $\epsilon > 0$ be arbitrary.
Let $N(\epsilon)$ be an integer greater than $\left[\sqrt{\frac{3}{4\epsilon}} + 1\right]$.

If $n \geq N(\epsilon)$, then

$$|x_n - 2| = \left| \frac{(8n^3 - n) - 2(4n^3 + \sqrt{n+1})}{(4n^3 + \sqrt{n+1})} \right|$$

$$= \left| \frac{-n - 2\sqrt{n+1}}{4n^3 + \sqrt{n+1}} \right| =$$

$$\frac{(n + 2\sqrt{n+1})}{(4n^3 + \sqrt{n+1})} < \frac{(n+2n)}{4n^3} \quad \left(\begin{array}{l}\text{since} \\ n > 1\end{array}\right)$$

$$= \frac{3}{4n^2} \leq \frac{3}{4(N(\epsilon))^2}$$

$$< \frac{3}{4\left(\sqrt{\frac{3}{4\epsilon}}\right)^2} = \epsilon.$$

QED

Note that Examples APP.3(1). & the technique of Examples 2.2(2). would also give Examples APP.3(2). if we had general information about limits of sums, products, & square roots of sequences.

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