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Interest
MATHematics MAGnification™

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INTEREST MAGNIFICATION

This is one of a series of very short books on math, statistics, and physics called "Math Magnifications." The "magnification" refers to focusing on a particular topic that is pivotal in or emblematic of mathematics.

OUTLINE

This Magnification will describe some essential financial literacy. Our outline will be intuitive, which is a polite way of saying that key ideas might be left undefined; the Introduction will give basic definitions and refer the reader to other definitions needed.

If you open a savings account, or otherwise allow someone else to do things with your money (this is called an *investment*), you should expect something in return. This return is often called *interest*. The more money you invest, the more you should get in return. The simplest way to do this is to make interest a percentage of the money invested.

Compound interest means you get interest on your interest.

We will illustrate interest with two jars, one that is fertile in the sense of generating interest and one that is barren in the sense of not generating interest.

We will also calculate the popular quantification of an account known as the *annual yield*, a way to compare the returns on different investments.

Compounding more often, all other things being equal, means more money. We will also discuss what is really a calculus concept, *compounding continuously*, meaning letting the number of compoundings per year go to infinity. It is not clear at first glance whether the return will also go to infinity.

For this Magnification, students should know how to calculate percentages, including switching from percentages to decimals and vice versa, and be familiar with exponential notation. Although we will also give verbal expressions of formulas, familiarity with algebraic notation ([2] is more than sufficient) would be helpful. Possession and knowledge of a scientific calculator sophisticated enough to do exponents, possibly with the famous irrational number e, will be necessary.

1. INTRODUCTION.

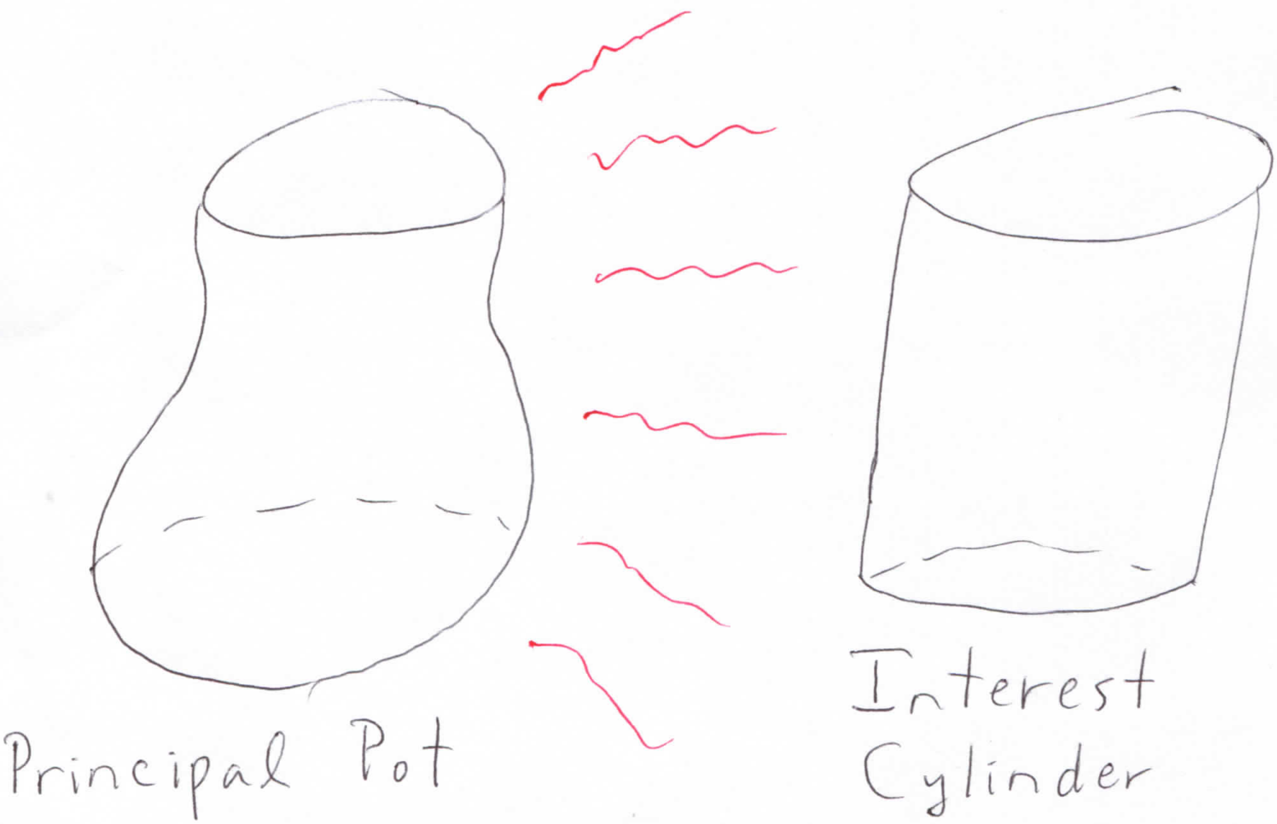
Definitions 1.1. In this Magnification, an **account** will be a place where you may deposit money. This deposit is a type of *investment.* The set up of such an account will include

(a) Specifying a fixed **APR** or **Annual Percentage Rate**; and

(b) Specifying a portion of the money in the account, denoted the **principal**. Said specification might cause the principal to change as time passes; and

(c) Specifying how the APR is applied to the principal, to put additional money, denoted **interest**, in your account.

Definitions and Pictures 1.2. We will partition the money in an account by putting the principal into a **Principal Pot** and all other money into an **Interest Cylinder**. See the drawings immediately below.



This is an almost biological partitioning. The Principal Pot is fertile, like a field in the corn belt or yeast in water, in the sense that it generates money (the interest). The Interest Cylinder is barren; it generates nothing.

This Magnification will describe two ways of calculating principal and interest, hence the money in your account: *simple interest* (Section 2) and *compound interest* (Sections 3 and 4). Section 5 discusses *APY, Annual Percentage Yield,* a popular way of deciding which accounts are better. All formulas are summarized, both with words and with algebraic symbols, in Section 6.

2. SIMPLE INTEREST

Definition 2.1. Simple interest means an account such that

(a) The principal is always the money you initially deposited, call it P_0; and

(b) If i is the APR, written as a decimal, then the money in the account after N years is
$$P_0 + (iN)P_0 = P_0(1 + iN);$$
note that the interest is $(iN)P_0$, a linearly growing (see [1]) percentage of the principal.

Example 2.2. You invest 500 dollars in an account with simple interest and 10% APR.

After a year, the number of dollars in your account is
$$500 + (10\% \text{ of } 500) = 500 + (0.1)500 = 500 + 50 = 550.$$
Notice that the money in the account is in two pieces, a principal of 500 dollars and interest of 50 dollars.

After two years, the number of dollars in your account is
$$500 + (20\% \text{ of } 500) = 500 + (0.1)(2)500 = 500 + 100 = 600.$$
You still have a principal of 500 dollars, but your interest is now 100 dollars.

After three years, the number of dollars in your account is
$$500 + (30\% \text{ of } 500) = 500 + (0.1)(3)500 = 500 + 150 = 650.$$
The principal is still 500 dollars, but you now have 150 dollars of interest.

The expression from the last calculation can be written
$$500 + (0.1)(3)500 = 500\,(1 + (0.1)3)\,.$$
We will find it convenient to write our account money this way, to prepare for compound interest.

For example, if we want the money in your account after 10 years, we have $P_0 = 500, i = 0.1$, and $N = 10$, so you have
$$500\,(1 + (0.1)10) = 1,000 \text{ dollars}.$$

Examples 2.3. You deposit 200 dollars in a savings account that pays 24% annual interest, with no compounding (that is, simple interest). Find how much money you will have after

(a) A year

(b) 2 years

(c) 10 years

(d) 3 months

Answers.

(a) $200(1 + (0.24)(1)) = 248$ dollars

(b) $200(1 + (0.24)(2)) = 296$ dollars

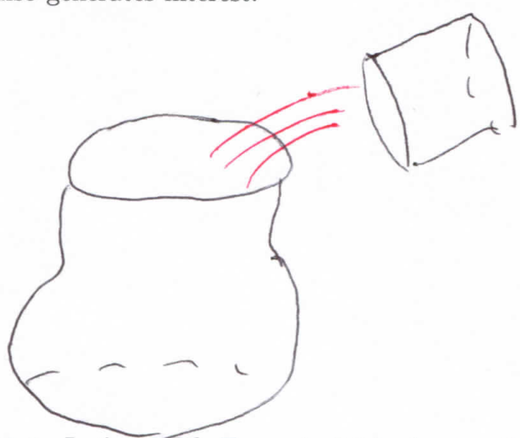
(c) $200(1 + (0.24)(10)) = 680$ dollars

(d) $200(1 + (0.24)(\frac{1}{4})) = 200(1 + \frac{0.24}{4}) = 212$ dollars

3. COMPOUND INTEREST

With simple interest, the money in your account grows, but said growth is consigned to the sterile Interest Cylinder (see the drawing in 1.2). The Principal Pot stays the same, generating money that piles up and stays in the Interest Cylinder.

With compound interest, the Interest Cylinder is emptied into the Principal Pot periodically, so that *it* also generates interest.

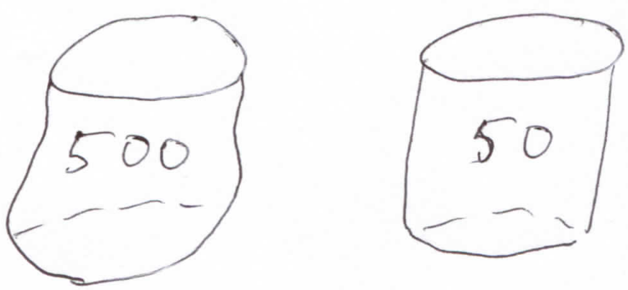


Example 3.1. Let's modify Example 2.2 by having the money be *compounded yearly.*

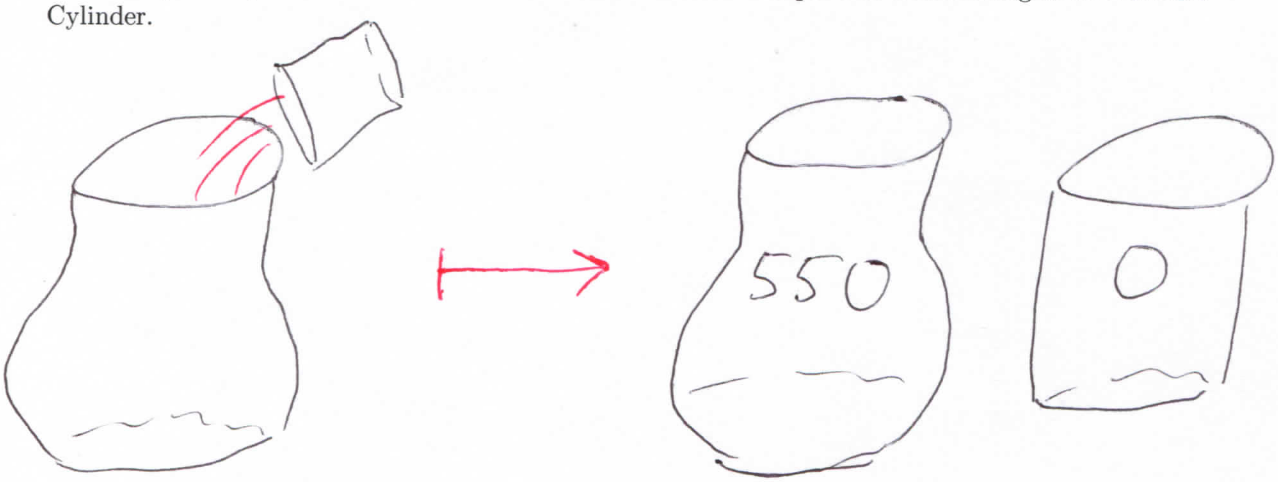
After one year, you still have, as with simple interest,

$$(500 + (10\% \text{ of } 500))$$

dollars in your account: 500 in the Principal Pot and 50 in the Interest Cylinder.



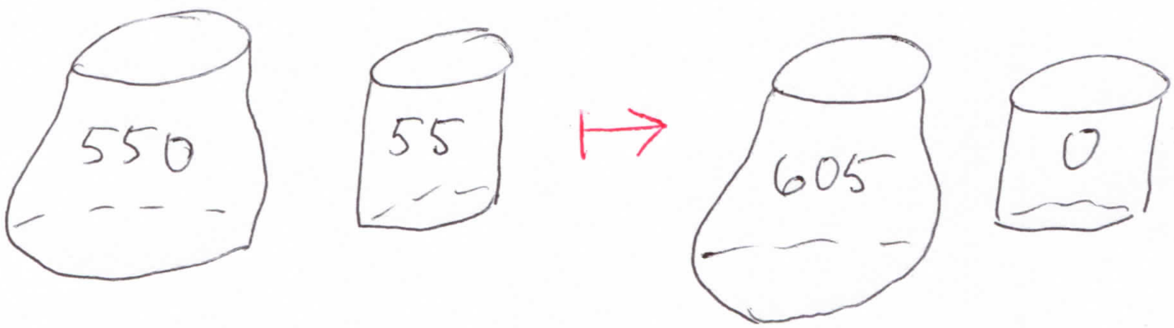
But now compounding enters the picture. The 50 dollars in the Interest Cylinder is poured into the Principal Pot, so that we now have 550 dollars in the Principal Pot and nothing in the Interest Cylinder.



After two years, the number of dollars in your account is

$$550 + (10\% \text{ of } 550) = 550 + (0.1)550 = 550 + 55 = 605 \text{ dollars};$$

at first, 550 dollars in the Principal Pot and 55 dollars in the Interest Cylinder, then, by the vigorous compounding action, 605 dollars in the Principal Pot and nothing in the Interest Cylinder.



After three years, by the same reasoning, your account will contain

$$605 + (0.1)605 = 665.50 \text{ dollars}.$$

Compare that 665.50, from annual compounding, to the 650 dollars we got after three years under simple interest; we see a slight improvement.

Our objection now is that our calculations are getting messy and possibly confusing; if we wanted to predict how much money would be in your account ten years from now, it would require ten calculations like the one we just made, with increasingly Byzantine fractions of a cent.

Easier Outlook 3.2. Here we will still refer to the account in Example 3.1. Each year, writing P for last years' account money, your account will contain

$$\text{P} + (10\% \text{ of P}) = (1 \times \text{P}) + ((0.1) \times \text{P}) = (1 + 0.1) \times \text{P};$$

in other words, *each year we multiply by* $(1 + 0.1)$.

Let's see if this gives us the same amounts of money as in Example 3.1.

After one year,

$$500 \times (1.1)^1 = 550 \text{ dollars}.$$

After two years,

$$500 \times (1.1)^2 = 605 \text{ dollars}.$$

After three years,

$$500 \times (1.1)^3 = 665.5 \text{ dollars}.$$

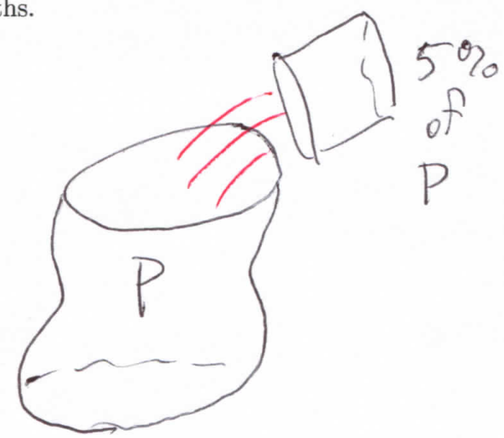
This matches our calculations in Example 3.1. More importantly, we can go immediately to the future. If we want the money in the account after ten years, we calculate

$$500 \times (1.1)^{10} = 1,296.87 \text{ dollars},$$

when rounded to two decimal places.

Examples 3.3. We may not want to wait a year before we compound, that is, before we empty the Interest Cylinder into the Principal Pot.

Let's have the same account as in Example 3.1, except that we will compound every six months. This means that, twice a year, the Interest Cylinder is emptied into the Principal Pot. Since only half a year has elapsed, only half as much interest has accumulated in the Interest Cylinder. Since half of 10% is 5%, we are increasing the principal by 5% of the previous principal, but we are doing it twice a year. As in 3.2, this is saying we multiply the principal by $(1 + 0.05) = 1.05$ every six months.



Twice a year:

$P \mapsto (1 + 0.05)P$

In other words, analogous to 3.2, *twice a year we multiply by* $(1 + \frac{0.1}{2}) = 1.05$.

For example, after ten years, we are multiplying by (1.05) twenty times; that is, we multiply our initial 500 dollars by $(1.05)^{20}$:

$$500 \times (1.05)^{20} \sim 1,326.65 \text{ dollars.}$$

An account may compound as often as it likes. Let's say we compound five times a year, but otherwise have the same account as in Example 3.1. Reasoning as we just did, the money in the account will be multiplied by

$$(1 + \frac{0.1}{5}) = 1.02$$

five times a year.

For the sake of comparison, let's look at the money in the account after ten years; since we are multiplying by 1.02 five times a year for ten years, the money in the account will now be

$$500 \times (1.02)^{50} \sim 1,345.79 \text{ dollars.}$$

Remarks 3.4. Notice that we get more money when we compound more often. Here are the number of dollars in the account after ten years, from an initial investment of 500 dollars, from 2.2, 3.2, and 3.3, under different numbers of compoundings per year:

No compounding (simple interest): 1,000

Compounding once a year for ten years: 1,296.87

Compounding twice a year for ten years: 1,326.65

Compounding five times a year for ten years: 1,345.79

This increase is believable, since compounding more often implies that money gets into the interest-generating Principal Pot sooner.

Examples 3.5. To motivate a general formula for compound interest, let's transcribe here the pre-calculated money in Remarks 3.4. Besides the ingredients from simple interest:

(a) P_0, the amount invested,

(b) N, the number of years in the account, and

(c) i, the annual percentage rate or APR, written as a decimal,

we also need

(d) k, the number of compoundings per year.

In each of the following amounts of money from Remarks 3.4, $P_0 = 500, N = 10$, and $i = 0.1$.

compounded once a year ($k = 1$): $500(1.1)^{10} = 500(1 + \frac{0.1}{1})^{10\times 1}$;

compounded twice a year ($k = 2$): $500(1.05)^{20} = 500(1 + \frac{0.1}{2})^{10\times 2}$;

compounded five times a year ($k = 5$): $500(1.02)^{50} = 500(1 + \frac{0.1}{5})^{10\times 5}$.

Here's the general formula. See Section 6 for a purely verbal description.

Compound interest 3.6. If P_0 dollars are invested in an account that compounds k times a year with APR i, written as a decimal, then the money in the account after N years is

$$P_0(1 + \frac{i}{k})^{Nk} \text{ dollars.}$$

Note that Nk, in the exponent of the formula above, is the number of compoundings that have occurred.

Examples 3.7. You deposit 200 dollars in a savings account that pays 24% annual interest. Find how much money you will have after fifty years, if the money is compounded

(a) Yearly

(b) Monthly

(c) Weekly

(d) Hourly (assume 365 days in a year).

(e) Every minute (assume 365 days in a year).

Answers.

(a) $200(1 + \frac{0.24}{1})^{50\times 1} \sim 9,378,086.92$ dollars

(b) $200(1 + \frac{0.24}{12})^{50\times 12} \sim 28,915,656.22$ dollars

(c) $200(1 + \frac{0.24}{52})^{50\times 52} \sim 31,664,602.42$ dollars

(d) $200(1 + \frac{0.24}{(24\times 365)})^{50\times(24\times 365)} \sim 32,545,608.35$ dollars

(e) $200(1 + \frac{0.24}{(60\times 24\times 365)})^{50\times(60\times 24\times 365)} \sim 32,550,865.20$ dollars

Remark 3.8. The mathematical unifying theme for Sections 2 and 3 is the two simplest models for population growth. Simple interest is linear or arithmetic growth (adding a fixed number repeatedly), while compound interest is exponential or geometric growth (multiplying by a fixed number repeatedly); see [1].

For example, if you invest 100 dollars at 10% annual interest, your account after 1 year, 2 years, 3 years, ... is, in dollars,

$$110, 120, 130, \ldots,$$

an arithmetic sequence (add 10 every year).

If you invest 100 dollars at 10% annual interest compounded every year, your account after 1, 2, 3, ... years is, in dollars,

$$100(1.1), 100(1.1)^2, 100(1.1)^3, \ldots,$$

a geometric sequence (multiply by 1.1 every year).

4. COMPOUNDING CONTINUOUSLY

We have already observed, in Remarks 3.4, that compounding more times per year increases the money in your account. If we let the number of compoundings per year go to infinity (that is, get arbitrarily large), it is now natural to ask if the money in your account also goes to infinity.

Let's invest one dollar at 100% APR, for a year, and see how much we make in a year (rounded to 9 decimal places), for different numbers of compoundings per year.

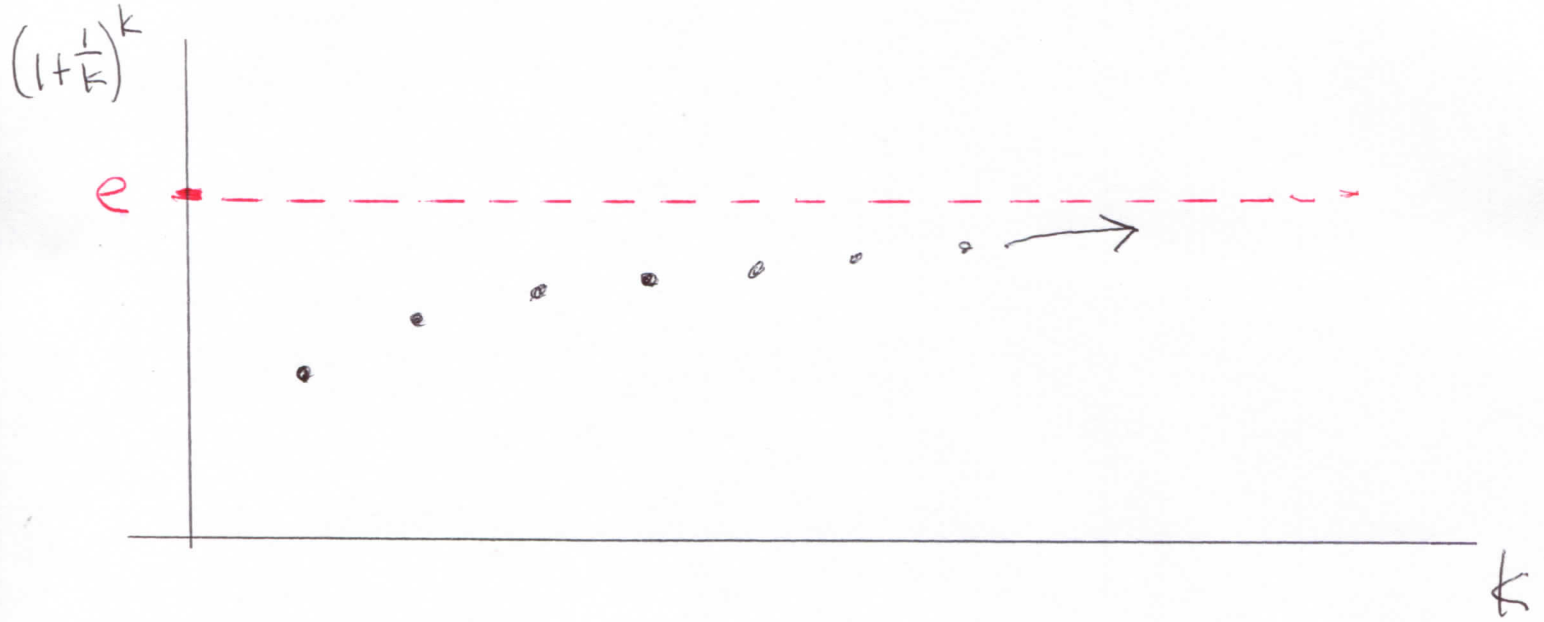
Compoundings Per Year	Money After a Year
1	$(1+\frac{1}{1})^1 = 2$
10	$(1+\frac{1}{10})^{10} \sim 2.59374246$
100	$(1+\frac{1}{100})^{100} \sim 2.704813829$
1,000	$(1+\frac{1}{1{,}000})^{1{,}000} \sim 2.716923932$
10,000	$(1+\frac{1}{10{,}000})^{10{,}000} \sim 2.718145927$
100,000	$(1+\frac{1}{100{,}000})^{100{,}000} \sim 2.718268237$
1,000,000	$(1+\frac{1}{1{,}000{,}000})^{1{,}000{,}000} \sim 2.718280469$

Definition and Factoid 4.1. It can be shown that there is a certain number, denoted **e**, that those numbers $(1+\frac{1}{k})^k$ (calculated directly above for $k = 1, 10, 100, 1{,}000, 10{,}000, 100{,}000$, and $1{,}000{,}000$) *converge* to as k goes to infinity. This means that we can force $(1+\frac{1}{k})^k$ to be as close as we like to e, by making k sufficiently large. The terminology is

$$\lim_{k\to\infty}(1+\frac{1}{k})^k = e,$$

shorthand for "limit, as k goes to infinity, of $(1+\frac{1}{k})^k$, equals e".

Here is the picture of this convergence.



We hope we are not revealing our ages when we assert our belief that most scientific calculators can give you decimal approximations of e: to nine decimal places

$$e \sim 2.718281828.$$

Compare that number to the numbers in the right-hand column of the table near the beginning of this section.

Definition 4.2. An account is **compounded continuously** if the money you get is the limit, as k goes to infinity, of the money you would get if you compounded k times per year.

Recall (see Section 3) that compounding k times a year means the Interest Cylinder is emptied into the Principal Pot (see Section 1) k times a year. With compounding continuously, the picture is that the Interest Cylinder is constantly being poured into the Principal Pot; in fact, since the Principal Pot is now generating interest that goes immediately back to the Principal Pot, one could remove the Interest Cylinder altogether. This is in sharp contrast to simple interest (see Section 2), where all the interest generated by the Principal Pot goes into the Interest Cylinder and stays there.

Refer now to our formula 3.6 for compounding k times a year.

Theorem 4.3. If P_0 dollars are invested in an account that compounds continuously with APR i, written as a decimal, then the money in the account after N years is

$$P_0 e^{iN} \text{ dollars.}$$

Proof: Let $m \equiv \frac{k}{i}$. Then the money in the account is

$$\lim_{k\to\infty} P_0(1+\frac{i}{k})^{Nk} = \lim_{m\to\infty} P_0(1+\frac{i}{im})^{N(im)} = P_0\left(\lim_{m\to\infty}(1+\frac{1}{m})^m\right)^{iN} = P_0 e^{iN}.$$

Example 4.4 (see Examples 3.7). You deposit 200 dollars in a savings account that pays 24% annual interest. Find how much money you will have after fifty years, if the money is compounded continuously.

Answer

$200e^{(0.24)(50)} = 200e^{12} \sim 32,550,958.28$ dollars. Compare to the Answers to Examples 3.7.

Remark 4.5. Denote, for $N = 0, 1, 2, 3, \ldots$, by P_N the amount of money in the account after N years. For those readers familiar with the language of calculus, the rate of growth of the money P_N in a continuously compounded account is proportional to P_N, while the rate of growth of the money in a simple interest account is proportional to the initial investment P_0. The constant of proportionality in both cases is i, the annual interest rate.

5. ANNUAL YIELD

With so many parameters in the calculation of compound interest, one might be forgiven for responding with great fear and confusion, when presented with many different accounts to choose among.

Definition 5.1. The value of an account may be quantified with its **Annual Yield**:

$$\frac{[(\text{money in account after a year}) - (\text{money deposited})]}{[\text{money deposited}]}.$$

When the annual yield is expressed as a percentage, it is called **Annual Percentage Yield** or **APY**.

Example 5.2. Suppose 800 dollars placed in an account grows to 1,000 dollars in a year. Find the APY of the account.

Answer. $\frac{[1{,}000-800]}{800} = 0.25$, or 25%.

Recommended Method to Calculate APY 5.3.

Invest one dollar, see what you've got in your account after one year, then subtract one.

Example 5.4. What is the APY of an account with 18% APR, compounded monthly?

Answer. See 3.6. A deposit of one dollar becomes, after one year,

$$\left(1 + \frac{0.18}{12}\right)^{12},$$

thus our APY equals

$$\left(\left[\left(1 + \frac{0.18}{12}\right)^{12}\right] - 1\right) \sim 0.1956,$$

or 19.56%.

What the APY Means 5.5.

If an account has a higher APY than another account, then the former account is superior to the latter account. An account with an APY of i_0 is equivalent, in terms of money made each year, to an account with APR i_0, compounded yearly.

For example, the account in Example 5.4 is equivalent to an account with APR of 19.56%, compounded yearly.

Example 5.6. Which of the following accounts is better?

(a) 22% APR compounded every year; or

(b) 20% APR compounded weekly.

Answer. Compare APYs. For (a), the APY is $\left[(1 + \frac{0.22}{1})^1 - 1\right] = 0.22 = 22\%$ (see 5.5), while, for (b), the APY is

$$\left[(1 + \frac{0.2}{52})^{52} - 1\right] \sim 0.2209, \text{ or } 22.09\%.$$

Since (b) has a higher APY than (a), (b) is better. It also alliterates.

6. INTEREST FORMULAS summarized

INSTRUCTIONS for calculating savings with simple interest

a. WITH SYMBOLS: Money in account is

$$P_0(1 + iN),$$

where P_0 = amount invested, N = number of years in account, and i is the annual interest rate (as a decimal).

b. WITH WORDS:

1. Multiply annual interest (as a decimal) by the number of years.
2. Add 1 to the number from Step 1.
3. Multiply (number from Step 2) times (money invested).

INSTRUCTIONS for calculating savings with compound interest

a. WITH SYMBOLS: Money in account is

$$P_0(1 + \frac{i}{k})^{Nk},$$

where P_0 = amount invested, N = number of years in account, k = number of compoundings per year, and i is the annual interest rate (as a decimal).

b. WITH WORDS:

1. Change annual interest (as a decimal) to interest during compounding period: divide the annual interest by the number of compoundings in a year.
2. Add 1 to the number from Step 1.
3. Find the number of times the money is compounded: (number of compoundings in a year) times (number of years in account).
4. Raise (number from Step 2) to the (number from Step 3)th power.
5. Multiply (number from Step 4) times (money invested).

INSTRUCTIONS for calculating savings with interest continuously compounded

a. WITH SYMBOLS: Money in account is

$$P_0 e^{iN},$$

where P_0 = amount invested, N = number of years in account, and i is the annual interest rate (as a decimal).

b. WITH WORDS:

1. Multiply annual interest (as a decimal) by the number of years.
2. Raise e (approximately 2.718281828) to the number from Step 1.
3. Multiply (number from Step 2) times (money invested).

ANNUAL PERCENTAGE YIELD (APY)

$$\text{APY} \equiv \frac{[(\text{money in account after a year}) - (\text{money deposited})]}{[\text{money deposited}]}.$$

RECOMMENDATION: Invest one dollar in the formula for APY.

EXAMPLE (simple interest).

A savings account has 4% annual simple interest. Find the amount in the account if two thousand dollars is invested for 5 years.

WITH SYMBOLS:

$$2,000(1 + (.04)5) = 2,400 \text{ dollars.}$$

WITH WORDS:

1. Multiply: Interest times (number of years) equals (.04)5, which equals 0.2.
2. Add 1: 1 plus 0.2 equals 1.2.
3. Multiply: 2,000 times 1.2 equals 2,400.

EXAMPLE (compound interest).

A savings account has 4% annual interest, compounded quarterly; that is, every 3 months. Find the annual yield, and the amount in the account if two thousand dollars is invested for 5 years.

WITH SYMBOLS:

$$2,000(1 + \frac{.04}{4})^{5\times 4} \sim 2,440.38 \text{ dollars.}$$

WITH WORDS:

1. Interest per quarter: $\frac{0.04}{4}$ equals 0.01.
2. Add 1: 1 plus 0.01 equals 1.01.
3. Multiply: 5 times 4 equals 20.
4. $1.01^{20} \sim 1.22019004$ (save 1.01^{20} in calculator).
5. Multiply: 2,000 times (number from Step 4) $\sim 2,440.38$.

APY: Any investment amount will give the same APY for the same account, so let's make life simple with an investment of one dollar:

After one year, it grows to (calculating as above) $1(1 + \frac{.04}{4})^4$, so

$$\text{APY} = \frac{1(1 + \frac{.04}{4})^4 - 1}{1} = (1 + \frac{.04}{4})^4 - 1 = (1.01)^4 - 1 \sim 0.0406 = 4.06\%.$$

This means your account makes the same money as if you were compounding yearly with an APR of 4.06%.

EXAMPLE (continuously compounded interest).

A savings account has 4% annual interest, compounded continuously. Find the annual yield, and the amount in the account if two thousand dollars is invested for 5 years.

WITH SYMBOLS:

$$2,000e^{0.04\times 5} \sim 2,442.81 \text{ dollars.}$$

WITH WORDS:

1. Multiply: Interest times (number of years) equals (0.04)5, which equals 0.2.
2. Raise e to the power 0.2: $e^{0.2} \sim 1.221402758$ (save $e^{0.2}$ in calculator).
3. Multiply: 2,000 times $e^{0.2}$ equals 2,442.81.

APY: Invest one dollar, then subtract one from the resulting money after one year:

$$e^{0.04} - 1 \sim 0.0408 = 4.08\%.$$

This means your account makes the same money as if you were compounding yearly with an APR of 4.08%.

HOMEWORK

1. You deposit 1,000 dollars in a savings account that pays 6% annual interest, with no compounding. Find how much money you will have after

(a) A year

(b) 20 years

(c) 6 months

(d) One month

2. You deposit 1,000 dollars in a savings account that pays 6% annual interest. Find how much money you will have after twenty years, if the money is compounded

(a) Yearly

(b) Quarterly (4 times a year)

(c) Daily (assume 365 days in a year).

(d) Continuously.

3. What is the APY for an account that pays 36% APR and is compounded continuously?

4. You deposit ten thousand dollars in a savings account that pays 12% annual interest. Find how much money you will have after ten years, if the money is compounded

a. never (simple interest);

b. yearly;

c. semiannually (twice a year);

d. quarterly (4 times a year);

e. monthly;

f. daily (assume 365 days in a year);

g. hourly.

5. "e" is a famous irrational number, approximately equal to 2.7183. Get a decimal approximation of e raised to the $[0.12 \times 10]$ power, and compare it to your answers in HW no. 4.

6. Calculate the annual yield from a savings account that has 12% annual interest, compounded

a. Monthly;

b. Daily;

c. Hourly.

7. Calculate e raised to the 0.12 power and compare this to your answers in no. 6.

8. Decide, using annual yield calculations, which of the following savings accounts is a better deal.

a. 11% annual interest, compounded yearly; OR

b. 10% annual interest, compounded monthly.

9. Same question as no. 8, for

a. 47% annual interest, compounded semiannually (twice a year); OR

b. 45% annual interest, compounded quarterly (four times a year).

HOMEWORK ANSWERS

1. (a) $1,000(1+0.06) = 1,060$ dollars

(b) $1,000(1+(0.06)(20)) = 2,200$ dollars

(c) $1,000(1+\frac{0.06}{2}) = 1,030$ dollars

(d) $1,000(1+\frac{0.06}{12}) = 1,005$ dollars

2. (a) $1,000(1+0.06)^{20} \sim 3,207.14$ dollars

(b) $1,000(1+\frac{0.06}{4})^{20\times 4} \sim 3,290.66$ dollars

(c) $1,000(1+\frac{0.06}{365})^{20\times 365} \sim 3,319.79$ dollars

(d) $1,000e^{0.06\times 20} = 1,000e^{1.2} \sim 3,320.12$ dollars.

3. $\left[e^{0.36} - 1\right] \sim 0.4333 = 43.33\%$.

4. a. $10,000(1+(0.12)10) = 22,000$ dollars.

b. $10,000(1+0.12)^{10\times 1} \sim 31,058.48$ dollars.

c. $10,000(1+\frac{0.12}{2})^{10\times 2} \sim 32,071.35$ dollars.

d. $10,000(1+\frac{0.12}{4})^{10\times 4} \sim 32,620.38$ dollars.

e. $10,000(1+\frac{0.12}{12})^{10\times 12} \sim 33,003.87$ dollars.

f. $10,000(1+\frac{0.12}{365})^{10\times 365} \sim 33,194.62$ dollars.

g. $10,000(1+\frac{0.12}{24\times 365})^{10\times(24\times 365)} \sim 33,200.90$ dollars.

5. $e^{0.12\times 10} = e^{1.2} \sim 3.320116923$, so that $10,000e^{0.12\times 10} \sim 33,201.17$ (this would be your money from *compounding continuously*; see Section 4). The answers in HW no. 4 are getting closer and closer to $10,000e^{0.12\times 10}$ as compounding occurs more and more often.

6. a. $\left[(1+\frac{0.12}{12})^{12} - 1\right] \sim 0.1268 = 12.68\%$.

b. a. $\left[(1+\frac{0.12}{365})^{365} - 1\right] \sim 0.1275 = 12.75\%$.

c. a. $\left[(1+\frac{0.12}{24\times 365})^{24\times 365} - 1\right] \sim 0.1275 = 12.75\%$. (Note: it is only rounding that makes this answer look the same as the answer to b.; the answer to c. is actually slightly larger than the answer to b.)

7. $\left[e^{0.12} - 1\right] \sim 0.1275 = 12.75\%$, as with b. and c. in HW no. 6. Actually $\left[e^{0.12} - 1\right]$ is slightly larger than the answers to b. and c.; it is only rounding that made it look the same.

$\left[e^{0.12} - 1\right]$ is the annual yield for an account with 12% APR compounded continuously.

8. For a., the annual yield is $\left[(1+\frac{0.11}{1})^{1} - 1\right] = 0.11 = 11\%$; for b., the annual yield is $\left[(1+\frac{0.1}{12})^{12} - 1\right] \sim 0.1047 = 10.47\%$.

The account in a. is better because it has a higher annual yield, or APY.

9. Same strategy as with HW 8.:
For a., the annual yield is $\left[(1+\frac{0.47}{2})^{2} - 1\right] \sim 0.5252 = 52.52\%$; for b., the annual yield is $\left[(1+\frac{0.45}{4})^{4} - 1\right] \sim 0.5318 = 53.18\%$.

This time b. wins, because its annual yield is higher.

REFERENCES

1. R. deLaubenfels, "Population Growth Magnification," https://teacherscholarinstitute.com/MathMagnificationsReadyToUse.html.

2. J. Saxon, "Algebra 1. An Incremental Development," Second Edition, Saxon Publishers, Inc., 1990.