TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI

# Logarithms I
# MATHematics MAGnification™

Dr. Ralph deLaubenfels

TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI

Teacher-Scholar Institute

Columbus, Ohio

2020

## LOGARITHMS I MAGNIFICATION

This is one of a series of very short books on math, statistics, and physics called "Math Magnifications." The "magnification" refers to focusing on a particular topic that is pivotal in or emblematic of mathematics.

## OUTLINE

Every positive number has a *logarithm*, which we will shorten to *log*; said log is another number. See Section II for the definition of log.

As numbers increase, their corresponding logs also increase, but much more slowly. For example, log(1) (reads "log of 1") equals 0, log(10) (reads "log of 10") equals 1, log(100) (reads "log of 100") equals 2, log(1,000) (reads "log of 1,000") equals 3, etc.

Notice how "multiply by 10" changes to "add 1" when we apply log.

In general, the application of log often creates immense simplifications in many settings. This Magnification will focus on log changing multiplication to addition, as we just illustrated, division into subtraction, and exponentiation into multiplication; e.g., squaring will be changed into "multiply by two" and "taking square roots" will be changed into "multiply by one half."

Section I summarizes the information about exponents needed for this Magnification. Section II introduces logarithms. The arithmetic simplifications alluded to in the previous paragraph appear in Section III, where we give numerous examples. The Appendix gives rigorous definitions of exponents and logs and their properties, and shows how logs may greatly simplify differentiation.

For the many interesting applications of the logarithm *function*, including population growth and radioactive decay, see [2] or any reasonably complete precalculus book.

Except for the Appendix, this Magnification mathematically requires only knowledge of exponents and scientific notation, as may be found in [4]. The definitions and properties of exponents needed are reviewed in Section I of this Magnification. The Appendix requires calculus.

The primary prerequisite for this Magnification is the historical imagination needed to imagine a world without easily available calculators; this means time travel to fifty or more years in the past. Except for checking results (for the curious), no calculators should be used in this Magnification.

Throughout this Magnification, the symbol "$\equiv$" means "is defined to be" and the symbol "$\sim$" means "is approximately equal to."

# SECTION I:

# Exponents

This section is a quick summary of $b^x$, for $b$ a positive number and $x$ a real number; $b$ is the <u>base</u> and $x$ is the <u>exponent</u>. See [4] for much more on this subject.

The number $b^x$ is called <u>b raised to the x power</u>

or <u>the x power of b</u>

or <u>b to the x power.</u>

We begin with x a natural number 1, 2, 3, ..., more traditionally n.

## <u>Definition I.1</u>

An <u>exponent</u> $n = 1, 2, 3, \ldots$ counts the number of times a fixed number is multiplied.

More precisely, for b a positive number and $n = 1, 2, 3, \ldots$

$$b^n \equiv \underbrace{(b \times b \times b \times \cdots \times b)}_{n \text{ terms}}$$

# Discussion I.2

Suppose $b$ is a positive real number.

Note that

$$b^2 \times b^3 = (b \times b)(b \times b \times b) = b^5$$

$$= b^{2+3},$$ while

$$(b^2)^3 = (b \times b) \times (b \times b) \times (b \times b) = b^6$$

$$= b^{2 \times 3}$$

In general

**(Properties of Exponents)**

(1) $b^n b^m = b^{n+m}$ and

(2) $(b^n)^m = b^{nm}$

for $n$ & $m$ natural numbers.

For $x$ rational (meaning a ratio of integers), holding onto the properties of exponents above dictates the definitions of $b^x$:

$$b^0 b^m = b^{0+m} = b^m \quad (m = 1, 2, 3, \ldots)$$

tells us that

$$b^0 \equiv 1;$$

$$(b^{1/n})^n = b^{(\frac{1}{n} \times n)} = b^1 = b \quad (n = 1, 2, 3, \ldots)$$

implies that

$$b^{1/n} \equiv \sqrt[n]{b},$$

the positive $n^{th}$ root of $b$;

for $x$ a positive rational number, say $x = p/q$, $p, q$ natural numbers,

$$b^x \equiv (b^{1/q})^p;$$

Finally, for any rational $x$,

$$b^{-x} b^{x} = b^{(-x+x)} = b^{0} = 1$$

gives us

$$b^{-x} = \frac{1}{b^{x}} .$$

We have defined $b^x$ for $x$ rational, $b$ positive. Calculus is required to define $b^x$ for all real $x$, rational or not (see the Appendix). It can be shown that the properties of $b^x$ just discussed are true for all real $x$.

Here, for $b$ a positive real number, $x$, $x_1$, & $x_2$ real, we summarize the properties of $b^x$ needed for this Magnification.

# Exponentials Summarized

## I.3.

(1) For $n = 1, 2, 3, \ldots$, $b^n$ is $b$ multiplied by itself $n$ times, as in Definition I.1.

(2) $b^{x_1} b^{x_2} = b^{x_1 + x_2}$

(3) $(b^{x_1})^{x_2} = b^{x_1 x_2}$

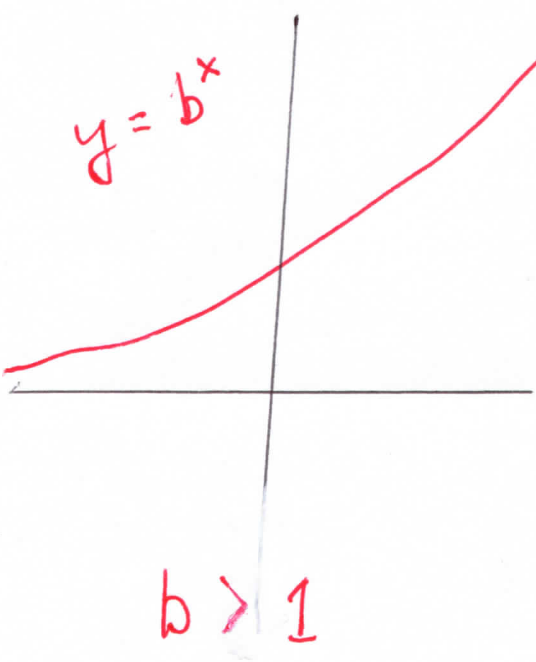
(4) $b^0 = 1$

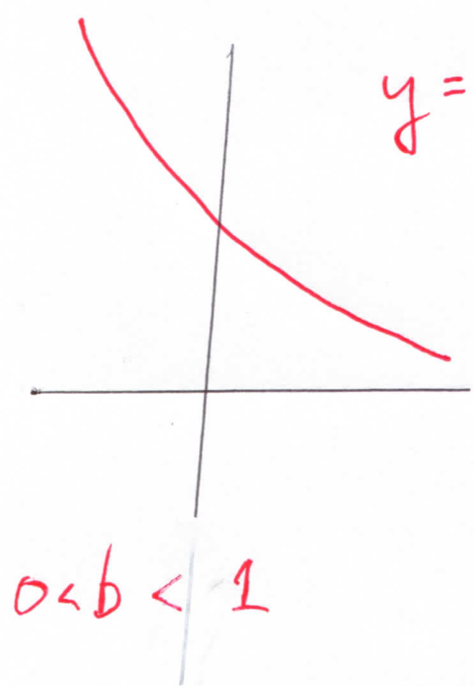
(5) $b^{-x} = 1/b^x$

(6) $\dfrac{b^{x_1}}{b^{x_2}} = b^{x_1 - x_2}$

(7) For $n = 1, 2, 3, \ldots,$ $b^{1/n}$ is a positive $n^{\text{th}}$ root of $b$.

(8) The graph of $y = b^x$ is monotone and has no holes. (see graphs below)



# Examples I.4

$(5^8)^3 = 5^{24}$ ; $5^8 \times 5^3 = 5^{11}$

$b^{1/2} = \sqrt{b}$, the square root of b;

e.g.,

$$49^{1/2} = 7.$$

$b^{1/3} = \sqrt[3]{b}$, the cube root of b;

e.g.,

$$8^{1/3} = 2.$$

$$27^{5/3} = (27^{1/3})^5 = 3^5 = 243.$$

$$100^{3/2} = (100^{1/2})^3 = 10^3 = 1{,}000.$$

$$100^{-3/2} = \frac{1}{100^{3/2}} = \frac{1}{1{,}000} = 0.001.$$

$$\left(\frac{1}{3\sqrt{3}}\right) \times 9 = \left(\frac{1}{3 \times 3^{1/2}}\right) \times 3^2 = 3^{-1-\frac{1}{2}+2} = 3^{1/2} = \sqrt{3}.$$

# SECTION II:

## Logarithms

In this section we will define <u>logarithms</u> (<u>logs</u>, for short).

**<u>Definition</u> <u>II.1</u>**

The <u>log</u>, short for <u>logarithm</u>, of a number to the base b, is the power to which b must be raised to equal the number.

That is,

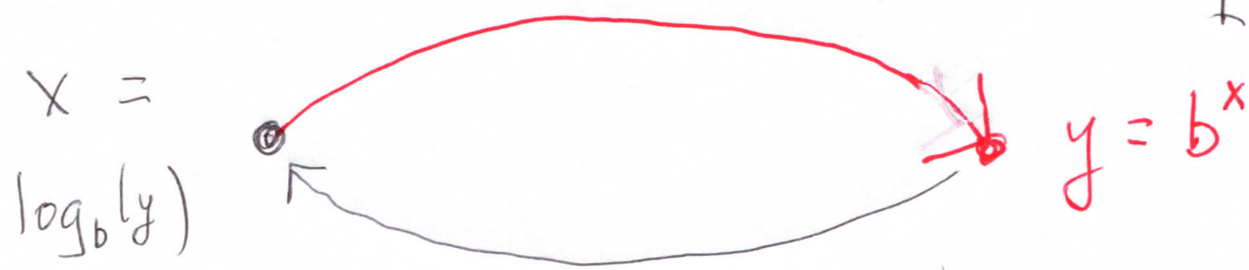
$$\log_b(y) = x$$

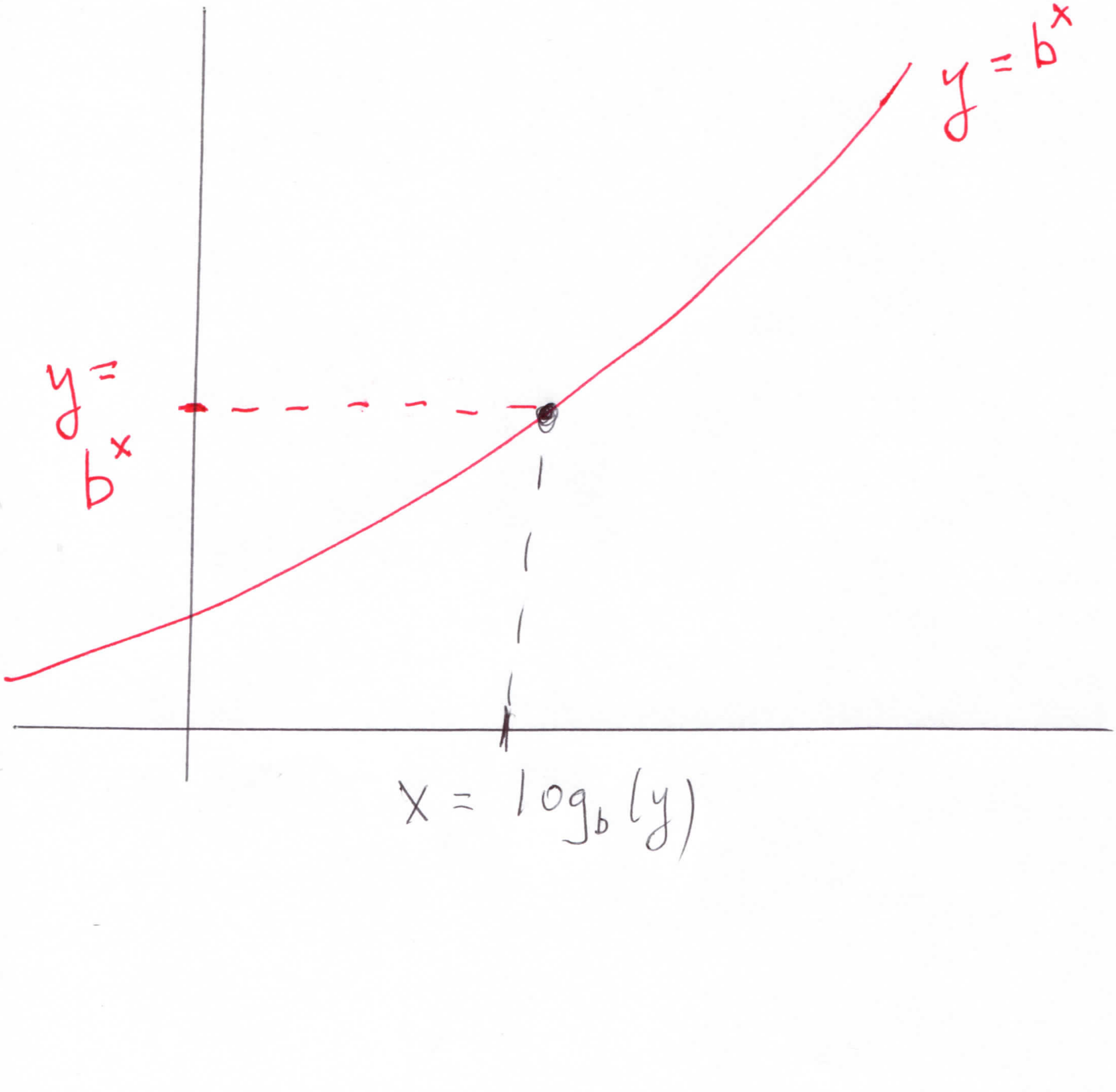
if and only if

$$y = b^x.$$

"$\log_b(y)$" reads the log, to the base b, of y.

In the language of precalculus (e.g., [2]), taking the logarithm is the inverse of exponentiation; see II.4(1) and the pictures on the next page.


$x = \log_b(y)$
$y = b^x$


$y = b^x$
$y = b^x$
$x = \log_b(y)$

# Examples II.2

(1) $\log_2(8) = \log_2(2^3) = \boxed{3.}$

In words, we are asking "What power of 2 equals 8?"

Since $2^3 = 8$, the answer is $\boxed{3.}$

(2) $\log_4(8) = \log_4(2^3) =$

$\log_4((4^{1/2})^3) = \log_4(4^{3/2}) = \boxed{3/2.}$

Note that $4^{3/2} = 8$; $\frac{3}{2}$ is the power of 4 that equals 8.

(3) $\log_{10}(10{,}000) = \log_{10}(10^4) = \boxed{4};$

since $10^4 = 10{,}000$, $\log_{10}(10{,}000) = 4$

(4) $\log_{1/3}(9) = \log_{1/3}(3^2)$

$= \log_{1/3}\left(\left(\left(\frac{1}{3}\right)^{-1}\right)^2\right) = \log_{1/3}\left(\left(\frac{1}{3}\right)^{-2}\right)$

$= \boxed{-2}$; note that $9 = \left(\frac{1}{3}\right)^{-2}$.

(5) $\log_5\left(\frac{1}{25}\right) = \log_5\left(\frac{1}{5^2}\right) =$

$\log_5(5^{-2}) = \boxed{-2}$ (since $5^{-2} = \frac{1}{25}$)

(6) $\log_3\left(\frac{1}{3\sqrt{3}}\right) = \log_3\left(\frac{1}{3^1 \times 3^{1/2}}\right) =$

$\log_3(3^{-3/2}) = \boxed{-3/2}$ (since $3^{-3/2} = \frac{1}{3\sqrt{3}}$)

## Discussion II.3.

Here is a sample of how logs simplify.

For any real $x_1$, $x_2$,

$$\log_b\left(b^{x_1} \times b^{x_2}\right) = \log_b\left(b^{(x_1+x_2)}\right)$$
$$= (x_1 + x_2) = \log_b(x_1) + \log_b(x_2).$$

In symbols, rewriting $y_1 \equiv b^{x_1}$, $y_2 \equiv b^{x_2}$, we have shown that, for positive $y_1$ and $y_2$,

$$\boxed{\log_b(y_1 y_2) = \log_b(y_1) + \log_b(y_2).}$$

In words,

the log of a product is the sum of the logs

Log changes multiplication (HARD) to addition (EASY).

See [3, Chapter 7, pp 183-184] for a quantification of multiplicative versus additive difficulty: for a pair of n-digit numbers, addition has on the order of n calculations, multiplication $n^2$.

Similarly, letting $y = b^x$,

$$\log_b(y^r) = \log_b((b^x)^r) = \log_b(b^{rx})$$
$$= rx = r\log_b(y)$$ ; that is,

Log changes exponentiation to multiplication:

$$\log_b(y^r) = r\log_b(y)$$

for r and y real, y positive. This is particularly useful when r is a fraction, so that roots are involved.

We summarize now the red-boxed assertions just made, along with other desired properties of log that follow from said assertions, where x is any real number, $y$, $y_1$, & $y_2$ are positive and real. Note that assertion (1) is stating precisely the inverse relationship between logarithms and exponentials, as in the drawings after the definition of $\log_b$.

The assertions in II.4 should be compared to the assertions in I.3.

# Logarithms Summarized

## II.4.

(1) $\log_b(b^x) = x$ and

$b^{\log_b(y)} = y.$

(2) $\log_b(y_1 y_2) = \log_b(y_1) + \log_b(y_2).$

(3) $\log_b(y^r) = r\log_b(y).$

(4) $\log_b(1) = 0.$

(5) $\log_b\left(\frac{1}{y}\right) = -\log_b(y).$

(6) $\log_b\left(\frac{y_1}{y_2}\right) = \log_b(y_1) - \log_b(y_2).$

# Examples II.5.

Let's redo an example from Examples II.2:

$$\log_3\left(\frac{1}{3\sqrt{3}}\right) = \log_3(1) - \log_3(3\sqrt{3})$$

$$= \log_3(1) - \left(\log_3(3) + \log_3(\sqrt{3})\right)$$

$$= \log_3(1) - \left(\log_3(3^1) + \log_3(3^{1/2})\right)$$

$$= 0 - \left(1 + \tfrac{1}{2}\right) = \boxed{-3/2.}$$

In general, $\log_b(\sqrt{y}) = \frac{1}{2}\log_b(y)$ & $\log_b(\sqrt[3]{y}) = \frac{1}{3}\log_b(y)$

$\log_{10}$ will be of particular interest in the next section. For example,

$$\log_{10}(2,950) = \log_{10}(2.95 \times 10^3)$$
$$= \log_{10}(2.95) + \log_{10}(10^3)$$
$$= \log_{10}(2.95) + 3.$$

This example demonstrates why tables of logarithms base 10 need only have logarithms of numbers between 1 & 10: before taking $\log_{10}$ of a number, write said number in scientific notation, as we just did:

$$2,950 = 2.95 \times 10^3.$$

# Examples II.6.

Two examples of logarithms are the Richter scale, measuring the intensity of earthquakes, and a pH number, measuring the strength of an acid or base. Being logarithms implies that, for example, a Richter scale measurement of 6 means an earthquake ten times as intense as a Richter scale measurement of 5, and a pH number of 4 means an acid ten times as strong as one with a pH number of 5. See II.4(2) with $b = 10 = y_2$.

## Historical Notes II. 7.

By the sixteenth century, problems in navigation and astronomy involved multiplication and division of numbers with as many as 12 digits. John Napier introduced logarithms, taking 20 years to produce the first logarithm table, appearing in 1614. See [1] and [3, Chapter 7].

Slide rules (to appear in a future Magnification), precursors of calculators, are constructed with logarithms. The authors of this Magnification are old enough to appreciate, in the movie "Apollo 13" (the move is made in 1995 and describes events in 1970),

the pivotal appearance of
slide rules.

# SECTION III:

# Arithmetic with log tables

In this section, we will use logarithm properties, specifically II.4 with $b = 10$, and the log tables on the last & second-to-last pages of this Magnification, to greatly simplify arithmetic.

## Definition III.1.

Logarithm, or log for short, means $\log_{10}$, log to the base 10.

## Examples III.2.

Let's practice using the log tables. The table on the last page of this Magnification rounds logarithms to 4 decimal places, the table on the second-to-last page to 3 decimal places.

Each table has four pairs of columns, each pair labelled "number" on the left and "logarithm" on the right.

For example, if we wanted log(1.24), we'd look at the first pair of columns: (4 decimal places table)

| number | logarithm |
|---|---|
| ⋮ | ⋮ |
| 1.23 | 0.0899 |
| $\boxed{1.24}$ | $\boxed{0.0934}$ |
| 1.25 | 0.0969 |
| ⋮ | ⋮ |

and conclude that

$$\log(1.24) \simeq \boxed{0.0934}$$

If we had used the 3 decimal places column, we'd look at

| number | logarithm |
|---|---|
| ⋮ | ⋮ |
| 1.23 | 0.090 |
| 1.24 | 0.093 |
| 1.25 | 0.097 |
| ⋮ | |

and conclude that

$$\log(1.24) \simeq \boxed{0.093}$$

Since $\log \equiv \log_{10}$, we have equivalently discovered that

$$1.24 \simeq 10^{0.093}.$$

If we asked what number is the log of 1.74 (to 3 decimal places), we'd be at the second pair of columns of numbers & logarithms:

| number | logarithm |
|---|---|
| ⋮ | ⋮ |
| 1.73 | 0.238 |
| $\boxed{1.74}$ | $\boxed{0.241}$ |
| 1.75 | 0.243 |
| ⋮ | ⋮ |

and conclude that

$$\log(1.74) \simeq 0.241$$

(equivalently, $1.74 \simeq 10^{0.241}$)

If we wanted $\log(3)$ (to 4 decimal places), we'd look at the third pair of columns

| number | logarithm |
| --- | --- |
| 3.00 | 0.4771 |

and see that $\log(3) \simeq 0.4771$

(equivalently, $3 \simeq 10^{0.4771}$)

Now suppose we would like the log of the number 430?

We appear to be doomed to disappointment because 430 does not appear on our log tables; the numbers in the log tables are always between 1 and 10.

Scientific notation looks promising because it always produces a number between 1 and 10, e.g.,

$$430 = 4.3 \times 10^2.$$

From a log table,

$$\log(4.3) \simeq 0.6335,$$

thus

$$\log(430) = \log(4.3 \times 10^2) =$$

$$\log(4.3) + \log(10^2) \simeq 0.6335 + 2$$

$$= \boxed{2.6335} \quad (430 \simeq 10^{2.6335})$$

Those pieces of the answer 0.6335 and 2 are called <u>mantissa</u> and <u>chacteristic</u>, respectively, and will be discussed in Definitions <u>III</u>.3.

We have been discussing taking the log of a number. We will also go in the opposite direction: given a number, find another number whose log is the original number.

For example, let's ask "What number has a log of 0.908?" Look for the appearance of 0.908 under a "logarithm" column of a 3-decimal place table:

| number | logarithm |
|---|---|
| ⋮ | |
| 8.0 | 0.903 |
| 8.1 | 0.908 |
| 8.2 | 0.914 |
| ⋮ | ⋮ |

Our desired number is $\boxed{8.1}$
since $\log(8.1) \simeq 0.908$
(equivalently, $8.1 \simeq 10^{0.908}$)

Now we must face a new restriction: the logs in our tables are between 0 & 1.

Suppose, for example, we wanted a number whose log is 5.387?

NOT in table of logs

The simplest way to get a number between 0 & 1 is to peel off the integer 5:

$$5.387 = 5 + 0.387$$

characteristic     mantissa

(see Defs. III.3)

Focus on the mantissa: we'd like a number whose log is 0.387.

From the log table;

| number | logarithm |
|---|---|
| 2.40 | 0.380 |
| 2.45 | 0.389 ← closest to 0.387 |

We use the information

$$\log(2.45) \simeq 0.389 \simeq 0.387$$
$$\log(10^5) = 5$$

$$\rightarrow \quad \log(2.45 \times 10^5) \simeq 5.387$$

log of product

sum of logs

Our answer is

$$\boxed{2.45 \times 10^5 = 245{,}000}$$

since $\log(245{,}000) \simeq 5.387$ (equivalently, $245{,}000 \simeq 10^{5.387}$)

## Definitions III.3

Any real number may be written as the sum of an integer and a nonnegative number less than one; when this is done with a logarithm, or an approximation of a logarithm, the integral part is called the <u>chacteristic</u> and the nonnegative part less than one is called the <u>mantissa</u>.

We saw a few examples of characteristic and mantissa in Examples III.2; here is another example:

the log of 3,850,000 is

$$\log(3,850,000) = \log(3.85 \times 10^6) = \log(3.85) + \log(10^6) \cong \underbrace{0.5855}_{\text{mantissa}} + \underbrace{6}_{\text{characteristic}}$$

$$= 6.5855,$$

giving us a characteristic of 6 and a mantissa of 0.5855.

Notice that the mantissa is the part of the logarithm 6.5855 to the right of the decimal point and the characteristic is the part to the left.

$$\underbrace{6}_{\text{characteristic}}.\underbrace{5855}_{\text{mantissa}}$$

The mantissa is the part that appears in log tables.

Notice also the appearance of scientific notation (see [4]) in our example. For any positive real number there is a natural relationship between its representation in scientific notation and the decomposition of its logarithm into mantissa and chacteristic! scientific notation for a positive real number $y$ writes it as

$$y = a \times 10^{x}$$

for some integer $x$ and nonnegative number $a$ that is less than ten. Then the logarithm of $y$ is

$$\log y = (\log a) + x;$$

$(\log a)$ is the mantissa and $x$ is the characteristic. This decomposition of $y$ and $(\log y)$ explains why log tables need include only nonnegative numbers $a$ less than 1, as we mentioned in Examples II. 5.

# Examples III.4

Let's identify characterstic and mantissa in some logarithms.

(1) What number is the log of 4,500?

SOLUTION: Start with scientific notation:

$$4{,}500 = 4.5 \times 10^3.$$

Then

$\log(4{,}500) = \log(4.5 \times 10^3) =$

$\log(4.5) + \log(10^3)$ (log of product is sum of logs: II.4(2))

$\simeq 0.6532 + 3$

mantissa: find in log table, characteristic

ANSWER: $\boxed{3.6532}$

(2) What number has a log of 23.538?

$\underbrace{23}_{\text{characteristic}}.\underbrace{538}_{\text{mantissa}}$

SOLUTION:

$23.538 = 23 + 0.538$

characteristic

mantissa: find in log table,

$$\log(3.45) \simeq 0.538$$
$$\log(10^{23}) = 23$$
$$\log(3.45 \times 10^{23}) = 23.538$$

log of product sum of logs

(II. 4(2))

ANSWER: $\boxed{3.45 \times 10^{23}}$

(3) Find the log of 3/400.

SOLUTION:

We need to write 3/400 as a decimal.

Since $\frac{3}{4} = 0.75$, $\frac{3}{400}$ equals

$$\frac{0.75}{100} = 0.0075 = 7.5 \times 10^{-3},$$

thus

$$\log\left(\frac{3}{400}\right) = \log(7.5 \times 10^{-3}) =$$

$$\log(7.5) + \log(10^{-3})$$ (log of product is sum of logs)

$$\simeq 0.875 + (-3)$$

mantissa    characteristic

ANSWER: $\boxed{\begin{array}{l} 0.875 - 3 \\ = -2.125 \end{array}}$

(4) Find a number whose log is $-4.75$.

SOLUTION: The first integer greater than 4.75 is 5, thus we add 5, to get a positive number less than 1:

$-4.75 + 5 = 0.25$, thus

$$-4.75 = \underbrace{(-5)}_{\text{chacteristic}} + \underbrace{0.25}_{\text{mantissa}}$$

Using tables, we have

$$\log(10^{-5}) = (-5)$$
$$+ \quad \log(1.78) \simeq 0.25$$

---

$$\log(1.78 \times 10^{-5}) \simeq (-5) + 0.25$$
$$= (-4.75),$$

thus the number we want is

$$(1.78 \times 10^{-5}) = 0.0000178$$

# Examples III.5

Suppose we have forgotten $2 \times 3$, the product of 2 and 3. Watch what happens when we take logs (from the 3-decimal place table) and add:

$\log 2 \simeq 0.301$, $\log 3 \simeq 0.477$, $\log 2 + \log 3 \simeq 0.778$, which is the log of 6.

Perhaps our memory is jogged now: $2 \times 3 = 6$.

Notice the log relationships:

$$\log 2 + \log 3 = \log(2 \times 3);$$

this is an illustration of II.4(2)

"the log of the product is the sum of the logs."

Here is our preferred picture for discovering $2\times 3$, via log tables and addition:

$$
\begin{array}{l}
\log 2 \simeq 0.301 \\
+\log 3 \simeq 0.477 \\
\hline
\log(2\times 3) \simeq 0.778
\end{array}
$$

log of product

sum of logs

Now "unlog" 0.778; that is, find a number from a log table whose log is 6.

$$\log 6 \simeq 0.778 \simeq \log(2\times 3)$$

$\rightarrow \quad 2\times 3 = 6.$

NOTE that, on the 4-decimal place log table,

$$\left.\begin{aligned} \log 2 &\simeq 0.3010 \\ \log 3 &\simeq 0.4771 \end{aligned}\right\} \text{sum} = 0.7781$$

$\log 6 \simeq 0.7782$

NOT QUITE equal

This is a reminder that log tables contain approximations.

For simplicity, we will usually use our 3-decimal place log table.

Let's do more multiplication with logs & addition.

(1) What is $1.18 \times 3.75$?

SOLUTION:

$$
\begin{array}{rl}
 \log(1.18) & \simeq 0.072 \\
+ \underline{\log(3.75)} & \simeq \underline{0.574} \\
\log(1.18 \times 3.75) & \simeq 0.646
\end{array}
$$

log of product      sum of logs

To "unlog" 0.646, look at table:

| number | log |
|---|---|
| 4.4 | 0.643 |
| 4.5 | 0.653 |

closest to 0.646

$\log(4.4) \simeq 0.646 \simeq$

$\log(1.18 \times 3.75)$

$\to \boxed{1.18 \times 3.75 \simeq 4.4}$

(actual value is 4.425)

(2) Find $4.8 \times 7.7$.

SOLUTION: $\log(4.8) \simeq 0.681$

$+ \log(7.7) \simeq 0.886$

$\log(4.8 \times 7.7) \simeq 1.567$

log of product

sum of logs

To "unlog" 1.567, write as

$$\underset{\text{characteristic}}{1} + \underset{\text{mantissa}}{0.567}$$

$$\log(10) = 1$$

$$\log(3.70) \simeq 0.567 \quad (\text{from table})$$

$$\log(3.70 \times 10) \simeq 1.567$$

$$\rightarrow \log(37.0) \simeq 1.567 \simeq \log(4.8 \times 7.7)$$

$$\rightarrow \boxed{4.8 \times 7.7 \simeq 37.0}$$

(actual value is 36.96)

(3) 2,350 × 84 ?

SOLUTION: To get numbers we can look up in a log table, write each factor in scientific notation:

$$(2{,}350) \times 84 = (2.35 \times 10^3) \times (8.4 \times 10^1)$$

$$= \underbrace{(2.35 \times 8.4)}_{\text{focus on this}} \times 10^4$$

$$\log(2.35) \simeq 0.371$$

$$\log(8.4) \simeq 0.924$$

$$\log(2.35 \times 8.4) \simeq 1.295 = 1 + 0.295$$

$$\log(10) = 1$$

$$\log(1.97) \simeq 0.295$$

$$\log(1.97 \times 10) \simeq 1.295$$

$$\rightarrow \log(2.35 \times 8.4) \simeq 1.295$$

$$\simeq \log(1.97 \times 10)$$

$$\rightarrow (2.35 \times 8.4) \simeq 1.97 \times 10$$

$$\rightarrow (2{,}350) \times (84) = (2.35 \times 8.4) \times 10^4$$

$$\simeq (1.97 \times 10) \times 10^4 = \boxed{1.97 \times 10^5}$$

(actual answer is $1.974 \times 10^5$)

Let's illustrate what logs do to quotients, as described in II.4(6).

Suppose we've forgotten how to divide 8 by 4; that is, we want 8/4. Take logs and <u>subtract</u>:

$$\begin{array}{rcl} \log 8 & \simeq & 0.903 \\ -\ \log 4 & \simeq & \underline{0.602} \\ \hline \log\left(\frac{8}{4}\right) & \simeq & 0.301 \end{array}$$

log of quotient      difference of logs

From the 3-decimal log table,

$\log 2 \simeq 0.301 \simeq \log\left(\frac{8}{4}\right) \rightarrow \frac{8}{4} = 2.$

When we think of dividing as the opposite of multiplying, it is natural, since subtracting is the opposite of adding, to subtract logs rather than add, when dealing with a quotient.

Let's do some examples of of quotients, with the 3-decimal places log table.

(4) Approximate $4.4/1.29$.

$$
\begin{array}{rl}
\log(4.4) & \simeq 0.643 \\
-\ \underline{\log(1.29)} & \simeq \underline{0.111} \\
\log\left(\frac{4.4}{1.29}\right) & \simeq 0.532
\end{array}
$$

log of quotient — difference of logs

As with multiplication, we must "antilog": find a number whose log is as close as possible to 0.532.

| number | log |
|---|---|
| 3.40 | 0.531 ← closest to 0.532 |
| 3.45 | 0.538 |

$$\log\left(\frac{4.4}{1.29}\right) \simeq 0.532$$

$$\simeq \log(3.40) \longrightarrow$$

$$\boxed{\frac{4.4}{1.29} \simeq 3.40}$$

( from a calculator: $\frac{4.4}{1.29} \simeq 3.41$ )

(5) Find $\frac{11.9}{3{,}850}$.

We will do this in two ways.

SOLUTION 1: To be accessible to log tables, rewrite both the numerator and the denominator in scientific notation:

$$\frac{11.9}{3,850} = \frac{1.19 \times 10^1}{3.85 \times 10^3} =$$

$$\underbrace{\left(\frac{1.19}{3.85}\right)}_{\text{work on this}} \times 10^{-2}$$

$$\begin{array}{rcr} \log(1.19) & \simeq & 0.076 \\ -\ \log(3.85) & \simeq & 0.585 \\ \hline \log\left(\frac{1.19}{3.85}\right) & \simeq & -0.509 \end{array}$$

log of quotient

difference of logs

To get a number whose log is $-0.509$, rewrite said number as we did in Examples III, 4(4):

$-0.509 + 1 = 0.491 \rightarrow$

$-0.509 = (-1) + 0.491 \rightarrow$

$$
\begin{array}{rcl}
\log(10^{-1}) & = & (-1) \\
+\ \log(3.10) & \simeq & 0.491 \quad \text{(from table)} \\
\hline
\log(3.10 \times 10^{-1}) & \simeq & -0.509
\end{array}
$$

log of product　　　　sum of logs

so that

$\log\left(\frac{1.19}{3.85}\right) \simeq -0.509$

$\simeq \log(3.10 \times 10^{-1}) \rightarrow$

$\frac{1.19}{3.85} \simeq 3.10 \times 10^{-1} \rightarrow$

$$\frac{11.9}{3{,}850} = \frac{1.19}{3.85} \times 10^{-2}$$

$$\simeq (3.10 \times 10^{-1}) \times 10^{-2}$$

$$= \boxed{3.10 \times 10^{-3}}$$ $\left(\begin{array}{l}\text{actual value} \\ \sim 3.09 \times 10^{-3}\end{array}\right)$

SOLUTION 2:

To avoid negative logs, rewrite

$$\frac{11.9}{3{,}850} = \frac{11.9}{3.85 \times 10^{3}} = \underbrace{\left(\frac{11.9}{3.85}\right)}_{\text{work on this}} \times 10^{-3}$$

$\log(11.9) = \log(1.19 \times 10^1) =$
$\log(1.19) + \log(10^1) \simeq 0.076 + 1$;

now we may do our usual use of
II.4(6):

$$\begin{array}{rl} \log(11.9) & \simeq 1.076 \\ -\ \underline{\log(3.85)} & \simeq \underline{0.585} \\ \log\left(\frac{11.9}{3.85}\right) & \simeq 0.491 \end{array}$$

log of quotient

difference of logs

$$\rightarrow \log\left(\frac{11.9}{3.85}\right) \simeq 0.491 \simeq \text{(table)}$$

$$\log(3.10)$$

$$\rightarrow \frac{11.9}{3.85} \simeq 3.10 \rightarrow$$

$$\boxed{\frac{11.9}{3{,}850}} = \left(\frac{11.9}{3.85}\right) \times 10^{-3}$$

$$\boxed{\simeq 3.10 \times 10^{-3}}$$

(6) Approximate $\frac{1}{13}$ with log tables.

SOLUTION: To avoid negative logarithms, we will multiply by 100; that is, approximate $\frac{100}{13}$.

$$\log\left(\frac{100}{13}\right) = \log(100) - \log(13)$$

$$= 2 - \log(1.3 \times 10^1) = 2 - [\log(1.3) + \log(10^1)]$$

$$\simeq 2 - [0.114 + 1] = 0.886$$

$$\simeq \log(7.7) \to \frac{100}{13} \simeq 7.7 \to$$

$$\boxed{\frac{1}{13} \simeq 0.077}$$

(actual value $\sim 0.0769$)

# Examples III.6

We'd like to illustrate II.4(3). Compare, in the 3-decimal places log table,

$\log 2 \simeq 0.301$

$\log 8 \simeq 0.903$ &

$3 \log 2 \simeq 3 \times 0.301 = 0.903.$

If we wanted $2^3$, we would reason as follows, using II.4(3):

$\log(2^3) = 3 \log 2 \simeq 3 \times 0.301 =$

$0.903 \simeq \log 8$, thus $2^3 = 8$.

The exponent r in II.4(3) could be a fraction; for example, if we wanted $\sqrt{9}$, we would look up log 9 and multiply by $\frac{1}{2}$:

$$\log(\sqrt{9}) = \log(9^{1/2}) = \tfrac{1}{2}\log 9$$
$$\simeq \tfrac{1}{2} \times 0.954 = 0.477 \simeq \log 3,$$

thus $\sqrt{9} = 3$.

For $\sqrt[3]{8}$: $\log(\sqrt[3]{8}) = \log(8^{1/3})$

$$= \tfrac{1}{3}\log 8 \simeq \tfrac{1}{3} \times 0.903 = 0.301$$
$$\simeq \log 2 \rightarrow 2 = \sqrt[3]{8}.$$

Let's do some examples we don't know the answer to, using log tables, addition, & "easy" multiplication and division, such as multiplying or dividing by 2, 3, or 10.

(1) What is $1{,}330^5$?

SOLUTION: $\log(1{,}330^5) =$
$5 \times \log(1{,}330) = 5 \times \log(1.33 \times 10^3)$
$= 5 \times [\log(1.33) + \log(10^3)] \simeq$
$5 \times [0.124 + 3] = \frac{1}{2} \times 10 \times [0.124 + 3]$
$= \frac{1}{2} \times [1.24 + 30] = [0.62 + 15]$

For "antilogging" (that is, finding a number whose log is $[0.62 + 15]$):

$$\log(4.2) \simeq 0.62$$

$$+ \log(10^{15}) = 15$$

$$\log(4.2 \times 10^{15}) \simeq [0.62 + 15]$$

$\rightarrow \log(1{,}330^5) \simeq \log(4.2 \times 10^{15})$

$\rightarrow$ ANSWER: $\boxed{(1{,}330)^5 \simeq 4.2 \times 10^{15}}$

(actual value ~ $4.16 \times 10^{15}$)

(2) Use the 4-decimal place log table to approximate $(1.1)^{10}$.

SOLUTION: $\log(1.1^{10}) = 10\log(1.1) \simeq 10 \times 0.0414 = 0.414$:

| number | log |
|---|---|
| 2.55 | 0.4065 |
| 2.60 | 0.4150 ← closest to 0.414 |

$\rightarrow \log(1.1^{10}) \simeq 0.415 \simeq \log(2.60)$

$\rightarrow$ ANSWER: $\boxed{(1.1)^{10} \simeq 2.60}$

(actual value is ~ 2.59)

(3) SAME as (2), except $(1.01)^{100}$.

SOLUTION: $\log(1.01^{100}) =$
$100 \log(1.01) \simeq 100 \times 0.0043 =$
$0.43 \simeq$ (as in (2)) $\log(2.70)$

→ ANSWER: $\boxed{1.01^{100} \simeq 2.70}$

(actual value is $\simeq 2.7048$)

$\left(1 + \frac{1}{n}\right)^n$ converges to the "natural exponential" $e$, as $n$ goes to $\infty$.

(4) Use the 4-decimal place log table to approximate $\sqrt{2}$.

SOLUTION: $\log(\sqrt{2}) = \log(2^{1/2})$
$= \frac{1}{2}\log 2 \simeq \frac{1}{2} \times 0.301 =$
$0.1505$ :

| number | log |
|---|---|
| 1.41 | 0.1492 ← closest to 0.1505 |
| 1.42 | 0.1523 |

$\rightarrow \log(\sqrt{2}) \simeq \log(1.41) \rightarrow$

ANSWER: $\boxed{\sqrt{2} \sim 1.41}$

(actual value ~ 1.414)

(5) Get an approximation of $\sqrt[3]{100}$.

SOLUTION: $\log(\sqrt[3]{100}) =$
$\log(100^{1/3}) = \frac{1}{3} \times \log(100) =$
$\frac{1}{3} \times 2 = \frac{2}{3} \simeq \log(4.6)$

→ ANSWER: $\boxed{\sqrt[3]{100} \simeq 4.6}$

(actual value ~ 4.64)

(6) How about $\sqrt[5]{17}$?

SOLUTION: $\log(\sqrt[5]{17}) = \log(17^{1/5})$
$= \frac{1}{5} \log(17) \approx \frac{1}{5} \log(1.7 \times 10)$

$\simeq \frac{1}{5} \times \left[ \log(1.7) + \log(10^1) \right]$

$\simeq \frac{1}{5} \times \left[ 0.230 + 1 \right] = \frac{1}{5} \times 1.230$

$= \frac{1}{10} \times 2 \times 1.230 = \frac{1}{10} \times 2.460 =$

$0.2460 \simeq \log(1.76)$

$\longrightarrow$ ANSWER: $\boxed{\sqrt[5]{17} \simeq 1.76}$ (actual value ~ 1.762)

(7) Approximate $\frac{1}{\sqrt{13}}$.

SOLUTION: To avoid negative logs, we'll work with $10/\sqrt{13}$:

$\log\left(10/\sqrt{13}\right) = \log\left(10 \times 13^{-1/2}\right) =$

$\log(10) + \log\left(13^{-1/2}\right) = \log(10^1) - \frac{1}{2}\log(13)$

$= 1 - \frac{1}{2}\log\left(1.3 \times 10^1\right) =$

$$1 - \frac{1}{2} \times \left[ \log(1.3) + \log(10^1) \right]$$

$$\simeq 1 - \frac{1}{2} \times \left[ 0.114 + 1 \right] =$$

$$1 - 0.057 - \frac{1}{2} = 0.5 - 0.057$$

$$= 0.443 \simeq \log(2.75) \text{ or } \log(2.80)$$

| number | log | |
|---|---|---|
| 2.75 | 0.439 | equidistant |
| 2.80 | 0.447 | from 0.443 |

$$\rightarrow \frac{10}{\sqrt{13}} \simeq 2.75 \text{ or } 2.80$$

$\rightarrow$ ANSWER: $\boxed{\frac{1}{\sqrt{13}} \simeq 0.275 \text{ or } 0.280}$

(actual value ~ 0.277)

p. 78

## HOMEWORK

**1.** Find the log of $630,000$.

**2.** Find the number whose log is 2.5119.

**3.** Use log tables and addition to approximate each of the following.

(a) $156 \times 215$

(b) $57 \times 186$

(c) $\frac{78}{285}$

(d) $\frac{23{,}500}{171}$

(e) $\frac{1}{7}$

**4.** Use log tables, addition, and simple multiplication to approximate each of the following.

(a) $\sqrt{7}$

(b) $\frac{1}{\sqrt{7}}$

(c) Cube root (that is, third root) of 17

(d) Tenth root of 25

(e) Fifth root of 640

(f) $(3.55)^{20}$

(g) $10^{0.1}$

(h) $100^{0.01}$

(i) $1,000^{0.001}$

**5.** Use II.4(2) and (3) to prove II.4(6).

ANSWERS are on pages 79–91.

Homework #1 ANSWER

$$\log(630{,}000) = \log(6.3 \times 10^5)$$
$$= \log(6.3) + \log(10^5)$$
$$\simeq 0.799 + 5 = \boxed{5.799}$$

Homework #2 ANSWER

$$\log(3.25) \simeq 0.5119$$
$$+ \log(10^2) = 2$$

$$\log(\underbrace{3.25 \times 10^2}_{\boxed{325}}) \simeq 2.5119$$

Homework #3(a) ANSWER

$$\log(156 \times 215) = \log(156) + \log(215)$$

$$= \log(1.56 \times 10^2) + \log(2.15 \times 10^2)$$

$$\simeq \log(1.56) + \log(10^2) + \log(2.15)$$

$$+ \log(10^2)$$

$$\simeq 0.193 + 2 + 0.332 + 2 = 4.525$$

$$\log(3.35) \simeq 0.525$$

$$+ \underline{\log(10^4)} = \underline{\quad 4 \quad}$$

$$\log(3.35 \times 10^4) \simeq 4.525$$

$$\rightarrow 156 \times 215 \simeq 3.35 \times 10^4 = \boxed{33{,}500}$$

(actual value: 33,540)

# Homework #3(b) ANSWER

$$\log(57 \times 186) = \log(57) + \log(186)$$

$$= \log(5.7 \times 10^1) + \log(1.86 \times 10^2)$$

$$= \log(5.7) + \log(10^1) + \log(1.86)$$

$$+ \log(10^2)$$

$$\cong 0.756 + 1 + 0.270 + 2$$

$$= 4.026$$

$$\log(1.06) \simeq 0.026$$

$$+ \underline{\log(10^4)} = \underline{\quad 4 \quad}$$

$$\log(1.06 \times 10^4) \simeq 4.026$$

$$\rightarrow 57 \times 186 \simeq 1.06 \times 10^4 = \boxed{10{,}600}$$

(actual value: 10,602)

# Homework #3(c) ANSWER

$\log\left(\frac{78}{285}\right)$ is negative, since $\frac{78}{285} < 1$, so let's look at

$\log\left(10 \times \frac{78}{285}\right) = \log\left(\frac{780}{285}\right) =$

$\log(780) - \log(285) =$

$\log(7.8 \times 10^2) - \log(2.85 \times 10^2)$

$= \log(7.8) + \log(10^2) - [\log(2.85) + \log(10^2)]$

$= \log(7.8) - \log(2.85) \simeq 0.892 - 0.455$

$= 0.437 \simeq \log(2.75)$

$\rightarrow 10 \times \frac{78}{285} \simeq 2.75 \rightarrow \frac{78}{285} \simeq \boxed{0.275}$

(actual value ~ 0.2737)

Homework # 3(d) ANSWER

$$\log\left(\frac{23{,}500}{171}\right) = \log(23{,}500) - \log(171)$$

$$= \log(2.35 \times 10^4) - \log(1.71 \times 10^2)$$

$$= \log(2.35) + \log(10^4) - [\log(1.71) + \log(10^2)]$$

$$\simeq 0.371 + 4 - [0.233 + 2]$$

$$= 2 + 0.138$$

$$\log(1.37) \simeq 0.138$$

$$+ \underline{\log(10^2)} = \underline{2}$$

$$\log(1.37 \times 10^2) \simeq 2 + 0.138$$

$$\rightarrow \frac{23{,}500}{171} \simeq 1.37 \times 10^2 = \boxed{137}$$

(actual value ~ 137.43)

# Homework # 3(e) ANSWER

$\log\left(\frac{1}{7}\right) < 0$, since $\frac{1}{7} < 1$,

so we'll look at $\frac{10}{7}$:

$$\log\left(\frac{10}{7}\right) = \log(10) - \log(7) \simeq$$
$$1 - 0.845 = 0.155 \simeq \log(1.43)$$

$$\rightarrow \frac{10}{7} \simeq 1.43 \rightarrow \boxed{\frac{1}{7} \simeq 0.143}$$

(actual value $\sim 0.1429$)

# Homework #4(a) ANSWER

$\log(\sqrt{7}) = \log(7^{1/2}) = \frac{1}{2}\log(7)$

$\simeq \frac{1}{2} \times 0.845 = 0.4225$

$\simeq \log(2.65)$

$\rightarrow \boxed{\sqrt{7} \simeq 2.65}$ (actual value: $\sim 2.646$)

# Homework #4(b) ANSWER

$\log\left(\frac{10}{\sqrt{7}}\right) = \log(10) - \frac{1}{2}\log(7) \simeq$

$1 - \frac{1}{2} \times 0.845 = 0.5775 \simeq \log(3.80)$

$\rightarrow \frac{10}{\sqrt{7}} \simeq 3.80 \rightarrow \boxed{\frac{1}{\sqrt{7}} \simeq 0.38}$

(actual value $\sim 0.378$)

# Homework #4(c) ANSWER

$$\log(\sqrt[3]{17}) = \log(17^{1/3}) =$$

$$\frac{1}{3} \times \log(17) = \frac{1}{3} \times \log(1.7 \times 10^1)$$

$$= \frac{1}{3} \times [\log(1.7) + \log(10^1)]$$

$$\simeq \frac{1}{3} \times [0.230 + 1] = 0.410$$

$$\simeq \log(2.55)$$

$$\rightarrow \boxed{\sqrt[3]{17} \simeq 2.55}$$

(actual value $\sim$ 2.57)

# Homework #4(d) ANSWER

$$\log\left(\sqrt[10]{25}\right) = \log\left(25^{1/10}\right)$$

$$= \frac{1}{10} \times \log(25) = \frac{1}{10} \times \log(2.5 \times 10^1)$$

$$= \frac{1}{10} \times \left[\log(2.5) + \log(10^1)\right]$$

$$\simeq \frac{1}{10} \times \left[0.398 + 1\right] = 0.1398$$

$$\simeq \log(1.38) \longrightarrow \boxed{\sqrt[10]{25} \simeq 1.38}$$

(actual value ~ 1.380)

ALTERNATIVELY: $\log\left(25^{1/10}\right)$

$$= \log\left(\left(5^2\right)^{1/10}\right) = \log\left(5^{2/10}\right) = \frac{2}{10}\log 5$$

$$\simeq \frac{2}{10} \times 0.699 = 0.1398 \simeq \log(1.38)$$

$\rightarrow \ldots$

# Homework #4(e) ANSWER

$$\log(\sqrt[5]{640}) = \log(640^{1/5})$$

$$= \frac{1}{5} \times \log(640) = \frac{1}{5} \times \log(6.4 \times 10^2)$$

$$= \frac{1}{5} \times \left[ \log(6.4) + \log(10^2) \right]$$

$$\simeq \frac{1}{5} \times [0.806 + 2] =$$

$$\frac{2}{10} \times [2.806] = \frac{1}{10} \times [5.612] = 0.5612$$

$$\simeq \log(3.65)$$

$$\longrightarrow \boxed{\sqrt[5]{640} \simeq 3.65}$$

(actual value ~ 3.64)

Homework # 4(f) ANSWER

$\log(3.55^{20}) = 20\log(3.55)$

$\simeq 20 \times 0.55 = 11 \rightarrow$

$\boxed{3.55^{20} \simeq 10^{11} = 100{,}000{,}000{,}000}$

("100 billion")

Homework # 4(g) ANSWER

$\log(10^{0.1}) = (0.1)\log(10) = 0.1$

$\simeq \log(1.26)$

$\rightarrow \boxed{10^{0.1} \simeq 1.26}$

# Homework # 4(h) ANSWER

$\log(100^{0.01}) = (0.01)\log(100)$

$= 0.01 \times 2 = 0.02 \simeq \log(1.05)$

$\rightarrow \boxed{100^{0.01} \simeq 1.05}$

# Homework # 4(i) ANSWER

$\log(1,000^{0.001}) = (0.001)\log(1,000)$

$= 0.001 \times 3 = 0.003 \simeq \log(1.01)$

$\rightarrow \boxed{1,000^{0.001} \simeq 1.01}$

p. 91

# Homework #5 ANSWER

$$\log_b\left(\frac{y_1}{y_2}\right) = \log_b\left(y_1 \times (y_2)^{-1}\right)$$

$$= \log_b(y_1) + \log_b\left((y_2)^{-1}\right) \quad (\text{by II.4(2)})$$

$$= \log_b(y_1) + (-1)\log_b(y_2) \quad (\text{by II.4(3)})$$

$$= \log_b(y_1) - \log_b(y_2).$$

p. 92

# APPENDIX

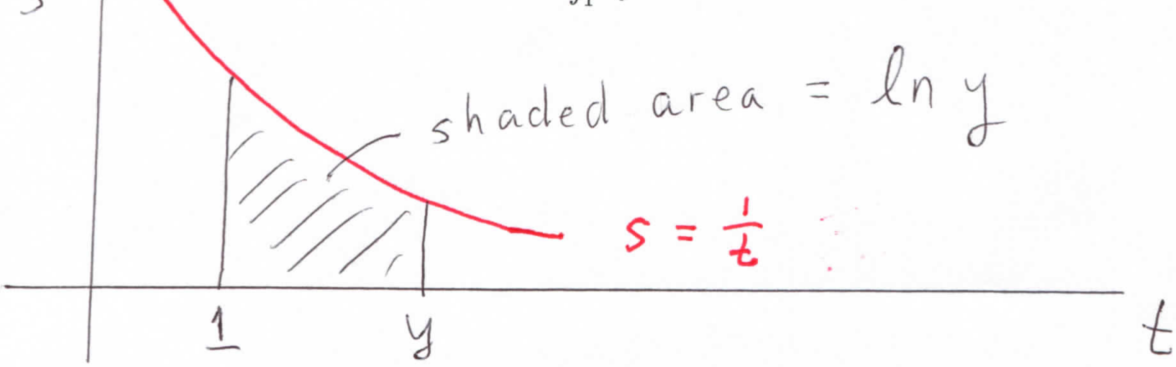
This section requires calculus.

In contrast with the precalculus introduction of exponentials, followed by logarithms as the inverse functions of exponentials, we use, in Examples APP.1, integration to define logarithms, for one particular base, then introduce a particular exponential as the inverse function of the particular logarithm. Along the way, we get exponentials $y = b^x$ defined for any real number $x$; as we mentioned in Section I, the precalculus approach defines $y = b^x$ only for rational $x$.

As another example of the simplifying properties of logarithm, we describe *logarithmic differentiation* in Examples APP.2.

**Examples APP.1.** Define a function ln ("natural logarithm") by

$$\ln y \equiv \int_1^y \frac{1}{t}\, dt \quad (y > 0).$$




Since $\frac{1}{t}$ is positive for $t > 0$, ln is an increasing function, so it has an inverse function

$$\exp \equiv (\ln)^{-1};$$

that is,

$$\exp(\ln y) = y \quad (y > 0); \quad \text{and} \quad \ln(\exp x) = x \quad (x \text{ real}).$$

By the Fundamental Theorem of Calculus, for $y > 0$,

$$\frac{d}{dy}(\ln y) = \frac{1}{y},$$

thus

$$\frac{d}{dx}(\exp x) = \frac{1}{\ln'(\exp x)} = \exp x \quad (x \text{ real}).$$

Let's justify the name "natural logarithm:"

**Theorem** ln is a logarithm; that is,

(1) $\ln(ab) = \ln a + \ln b \quad (a, b > 0)$; and

(2) $\ln(a^r) = r \ln a \quad (a > 0, r \text{ rational})$.

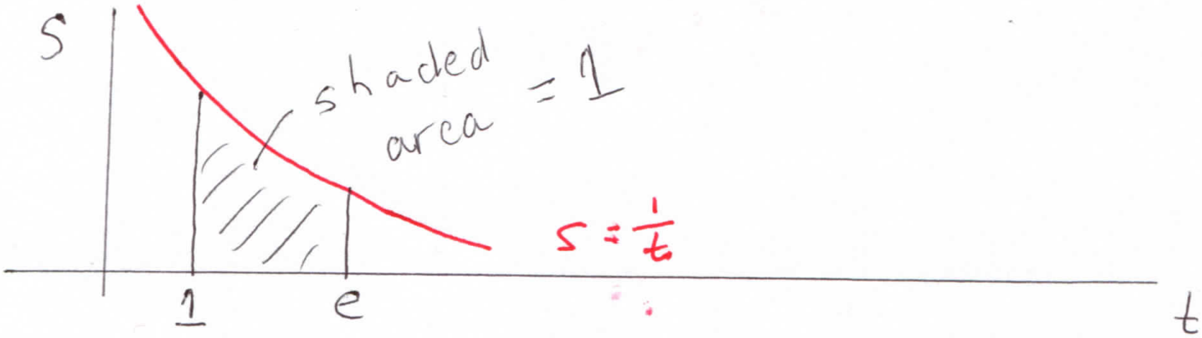
**Proof:** (1)

$$\ln(ab) \equiv \int_1^{ab} \frac{dt}{t} = \int_1^a \frac{dt}{t} + \int_a^{ab} \frac{dt}{t} = (\text{substitution } u \equiv \frac{t}{a} \text{ in second integral}) \int_1^a \frac{dt}{t} + \int_1^b \frac{du}{u} \equiv \ln a + \ln b.$$

(2) In the integral for $\ln(a^r)$, make the substitution $u \equiv t^{\frac{1}{r}}$, so that

$$du = \frac{1}{r} t^{(\frac{1}{r}-1)}\, dt = \frac{u\, dt}{rt}, \quad \text{hence } \frac{dt}{t} = \frac{r}{u}\, du: \ \ln(a^r) \equiv \int_1^{a^r} \frac{dt}{t} = \int_1^a \frac{r}{u}\, du = r \ln a.$$

**Definition.** $e \equiv \exp(1)$; that is, $e$ is a number such that $\ln e \equiv \int_1^e \frac{dt}{t} = 1$.




For $r$ rational,

$$\ln(e^r) = r \ln e = r = \ln(\exp(r)) \quad \text{(since ln and exp are inverse functions)},$$

thus

$$e^r = \exp(r).$$

For any real $x$, **DEFINE**

$$e^x \equiv \exp(x);$$

for $b > 0$,

$$b^x \equiv e^{x \ln b}.$$

NOTE that, for $x$ rational, properties of ln and exp guarantee that this definition of $b^x$ agrees with the algebraic definition.

**Definition.** For $b > 0$, $\log_b$ is the inverse function of $x \mapsto b^x$; that is,

$$\log_b y = x \iff y = b^x.$$

NOTE that $\log_b y = \frac{\ln y}{\ln b}$, since

$$b^{\frac{\ln y}{\ln b}} \equiv e^{\left(\frac{\ln y}{\ln b}\right) \ln b} = e^{\ln y} = y.$$

**Examples APP.2. Logarithmic Differentiation** is the following observation: if a function $f$ is differentiable at $c$ and $f(c) > 0$, then the composition $\ln(f)$ is differentiable at $c$, with

$$\frac{d}{dx}(\ln f)(c) = \frac{f'(c)}{f(c)}.$$

For example, if $f(x) \equiv x^{x^2}$, for $x > 0$, here is how we'd get $f'(x)$, for $x > 0$.

$$(\ln(f(x)) = x^2 \ln x,$$

thus

$$\frac{f'(x)}{f(x)} = \frac{d}{dx}(\ln(f(x)) = \frac{d}{dx}(x^2 \ln x) = 2x \ln x + x^2(\frac{1}{x}) = x(2 \ln x + 1),$$

so that

$$f'(x) = x(2 \ln x + 1) f(x) = x(2 \ln x + 1) x^{x^2}.$$

As another example, we can quickly derive the *power rule* with logarithmic differentiation: for $r$ real, $x > 0$,

$$\frac{(x^r)'}{x^r} = \frac{d}{dx}(\ln(x^r)) = \frac{d}{dx}(r \ln x) = \frac{r}{x},$$

thus

$$(x^r)' = x^r(\frac{r}{x}) = r x^{r-1}.$$

## REFERENCES

**1.** N. Bowditch, "The New American Practical Navigator."

**2.** R. deLaubenfels, Fun with Functions, Concise Textbooks Series https://teacherscholarinstitute.com/Books/PreCalculus.pdf, 1997.

**3.** H. L. Resnikoff and R. O. Wells, Jr., "Mathematics in Civilization," Dover, 1984.

**4.** J. Saxon, "Algebra 1. An Incremental Development," Second Edition, Saxon Publishers, Inc., 1990.

p. 95

**LOG TABLE (approximate to 3 decimal places)**

| number | logarithm | number | logarithm | number | logarithm | number | logarithm |
|---|---|---|---|---|---|---|---|
| 1.00 | 0.000 | 1.50 | 0.176 | 2.00 | 0.301 | 5.0 | 0.699 |
| 1.01 | 0.004 | 1.51 | 0.179 | 2.05 | 0.312 | 5.1 | 0.708 |
| 1.02 | 0.009 | 1.52 | 0.182 | 2.10 | 0.322 | 5.2 | 0.716 |
| 1.03 | 0.013 | 1.53 | 0.185 | 2.15 | 0.332 | 5.3 | 0.724 |
| 1.04 | 0.017 | 1.54 | 0.188 | 2.20 | 0.342 | 5.4 | 0.732 |
| 1.05 | 0.021 | 1.55 | 0.190 | 2.25 | 0.352 | 5.5 | 0.740 |
| 1.06 | 0.025 | 1.56 | 0.193 | 2.30 | 0.362 | 5.6 | 0.748 |
| 1.07 | 0.029 | 1.57 | 0.196 | 2.35 | 0.371 | 5.7 | 0.756 |
| 1.08 | 0.033 | 1.58 | 0.199 | 2.40 | 0.380 | 5.8 | 0.763 |
| 1.09 | 0.037 | 1.59 | 0.201 | 2.45 | 0.389 | 5.9 | 0.771 |
| 1.10 | 0.041 | 1.60 | 0.204 | 2.50 | 0.398 | 6.0 | 0.778 |
| 1.11 | 0.045 | 1.61 | 0.207 | 2.55 | 0.407 | 6.1 | 0.785 |
| 1.12 | 0.049 | 1.62 | 0.210 | 2.60 | 0.415 | 6.2 | 0.792 |
| 1.13 | 0.053 | 1.63 | 0.212 | 2.65 | 0.423 | 6.3 | 0.799 |
| 1.14 | 0.057 | 1.64 | 0.215 | 2.70 | 0.431 | 6.4 | 0.806 |
| 1.15 | 0.061 | 1.65 | 0.217 | 2.75 | 0.439 | 6.5 | 0.813 |
| 1.16 | 0.064 | 1.66 | 0.220 | 2.80 | 0.447 | 6.6 | 0.820 |
| 1.17 | 0.068 | 1.67 | 0.223 | 2.85 | 0.455 | 6.7 | 0.826 |
| 1.18 | 0.072 | 1.68 | 0.225 | 2.90 | 0.462 | 6.8 | 0.833 |
| 1.19 | 0.076 | 1.69 | 0.228 | 2.95 | 0.470 | 6.9 | 0.839 |
| 1.20 | 0.079 | 1.70 | 0.230 | 3.00 | 0.477 | 7.0 | 0.845 |
| 1.21 | 0.083 | 1.71 | 0.233 | 3.05 | 0.484 | 7.1 | 0.851 |
| 1.22 | 0.086 | 1.72 | 0.236 | 3.10 | 0.491 | 7.2 | 0.857 |
| 1.23 | 0.090 | 1.73 | 0.238 | 3.15 | 0.498 | 7.3 | 0.863 |
| 1.24 | 0.093 | 1.74 | 0.241 | 3.20 | 0.505 | 7.4 | 0.869 |
| 1.25 | 0.097 | 1.75 | 0.243 | 3.25 | 0.512 | 7.5 | 0.875 |
| 1.26 | 0.100 | 1.76 | 0.246 | 3.30 | 0.519 | 7.6 | 0.881 |
| 1.27 | 0.104 | 1.77 | 0.248 | 3.35 | 0.525 | 7.7 | 0.886 |
| 1.28 | 0.107 | 1.78 | 0.250 | 3.40 | 0.531 | 7.8 | 0.892 |
| 1.29 | 0.111 | 1.79 | 0.253 | 3.45 | 0.538 | 7.9 | 0.898 |
| 1.30 | 0.114 | 1.80 | 0.255 | 3.50 | 0.544 | 8.0 | 0.903 |
| 1.31 | 0.117 | 1.81 | 0.258 | 3.55 | 0.550 | 8.1 | 0.908 |
| 1.32 | 0.121 | 1.82 | 0.260 | 3.60 | 0.556 | 8.2 | 0.914 |
| 1.33 | 0.124 | 1.83 | 0.262 | 3.65 | 0.562 | 8.3 | 0.919 |
| 1.34 | 0.127 | 1.84 | 0.265 | 3.70 | 0.568 | 8.4 | 0.924 |
| 1.35 | 0.130 | 1.85 | 0.267 | 3.75 | 0.574 | 8.5 | 0.929 |
| 1.36 | 0.134 | 1.86 | 0.270 | 3.80 | 0.580 | 8.6 | 0.934 |
| 1.37 | 0.137 | 1.87 | 0.272 | 3.85 | 0.585 | 8.7 | 0.940 |
| 1.38 | 0.140 | 1.88 | 0.274 | 3.90 | 0.591 | 8.8 | 0.944 |
| 1.39 | 0.143 | 1.89 | 0.276 | 3.95 | 0.597 | 8.9 | 0.949 |
| 1.40 | 0.146 | 1.90 | 0.279 | 4.0 | 0.602 | 9.0 | 0.954 |
| 1.41 | 0.149 | 1.91 | 0.281 | 4.1 | 0.613 | 9.1 | 0.959 |
| 1.42 | 0.152 | 1.92 | 0.283 | 4.2 | 0.623 | 9.2 | 0.964 |
| 1.43 | 0.155 | 1.93 | 0.286 | 4.3 | 0.633 | 9.3 | 0.968 |
| 1.44 | 0.158 | 1.94 | 0.288 | 4.4 | 0.643 | 9.4 | 0.973 |
| 1.45 | 0.161 | 1.95 | 0.290 | 4.5 | 0.653 | 9.5 | 0.978 |
| 1.46 | 0.164 | 1.96 | 0.292 | 4.6 | 0.663 | 9.6 | 0.982 |
| 1.47 | 0.167 | 1.97 | 0.294 | 4.7 | 0.672 | 9.7 | 0.987 |
| 1.48 | 0.170 | 1.98 | 0.297 | 4.8 | 0.681 | 9.8 | 0.991 |
| 1.49 | 0.173 | 1.99 | 0.299 | 4.9 | 0.690 | 9.9 | 0.996 |

**LOG TABLE (approximate to 4 decimal places)**

| number | logarithm | number | logarithm | number | logarithm | number | logarithm |
|---|---|---|---|---|---|---|---|
| 1.00 | 0.0000 | 1.50 | 0.1761 | 2.00 | 0.3010 | 5.0 | 0.6990 |
| 1.01 | 0.0043 | 1.51 | 0.1790 | 2.05 | 0.3118 | 5.1 | 0.7076 |
| 1.02 | 0.0086 | 1.52 | 0.1818 | 2.10 | 0.3222 | 5.2 | 0.7160 |
| 1.03 | 0.0128 | 1.53 | 0.1847 | 2.15 | 0.3324 | 5.3 | 0.7243 |
| 1.04 | 0.0170 | 1.54 | 0.1875 | 2.20 | 0.3424 | 5.4 | 0.7324 |
| 1.05 | 0.0212 | 1.55 | 0.1903 | 2.25 | 0.3522 | 5.5 | 0.7404 |
| 1.06 | 0.0253 | 1.56 | 0.1931 | 2.30 | 0.3617 | 5.6 | 0.7482 |
| 1.07 | 0.0294 | 1.57 | 0.1959 | 2.35 | 0.3711 | 5.7 | 0.7559 |
| 1.08 | 0.0334 | 1.58 | 0.1987 | 2.40 | 0.3802 | 5.8 | 0.7634 |
| 1.09 | 0.0374 | 1.59 | 0.2014 | 2.45 | 0.3892 | 5.9 | 0.7709 |
| 1.10 | 0.0414 | 1.60 | 0.2041 | 2.50 | 0.3979 | 6.0 | 0.7782 |
| 1.11 | 0.0453 | 1.61 | 0.2068 | 2.55 | 0.4065 | 6.1 | 0.7853 |
| 1.12 | 0.0492 | 1.62 | 0.2095 | 2.60 | 0.4150 | 6.2 | 0.7924 |
| 1.13 | 0.0531 | 1.63 | 0.2122 | 2.65 | 0.4232 | 6.3 | 0.7993 |
| 1.14 | 0.0569 | 1.64 | 0.2148 | 2.70 | 0.4314 | 6.4 | 0.8062 |
| 1.15 | 0.0607 | 1.65 | 0.2175 | 2.75 | 0.4393 | 6.5 | 0.8129 |
| 1.16 | 0.0645 | 1.66 | 0.2201 | 2.80 | 0.4472 | 6.6 | 0.8195 |
| 1.17 | 0.0682 | 1.67 | 0.2227 | 2.85 | 0.4548 | 6.7 | 0.8261 |
| 1.18 | 0.0719 | 1.68 | 0.2253 | 2.90 | 0.4624 | 6.8 | 0.8325 |
| 1.19 | 0.0755 | 1.69 | 0.2279 | 2.95 | 0.4698 | 6.9 | 0.8388 |
| 1.20 | 0.0792 | 1.70 | 0.2304 | 3.00 | 0.4771 | 7.0 | 0.8451 |
| 1.21 | 0.0828 | 1.71 | 0.2330 | 3.05 | 0.4843 | 7.1 | 0.8513 |
| 1.22 | 0.0864 | 1.72 | 0.2355 | 3.10 | 0.4914 | 7.2 | 0.8573 |
| 1.23 | 0.0899 | 1.73 | 0.2380 | 3.15 | 0.4983 | 7.3 | 0.8633 |
| 1.24 | 0.0934 | 1.74 | 0.2405 | 3.20 | 0.5051 | 7.4 | 0.8692 |
| 1.25 | 0.0969 | 1.75 | 0.2430 | 3.25 | 0.5119 | 7.5 | 0.8751 |
| 1.26 | 0.1004 | 1.76 | 0.2455 | 3.30 | 0.5185 | 7.6 | 0.8808 |
| 1.27 | 0.1038 | 1.77 | 0.2480 | 3.35 | 0.5250 | 7.7 | 0.8865 |
| 1.28 | 0.1072 | 1.78 | 0.2504 | 3.40 | 0.5315 | 7.8 | 0.8921 |
| 1.29 | 0.1106 | 1.79 | 0.2529 | 3.45 | 0.5378 | 7.9 | 0.8976 |
| 1.30 | 0.1139 | 1.80 | 0.2553 | 3.50 | 0.5441 | 8.0 | 0.9031 |
| 1.31 | 0.1173 | 1.81 | 0.2577 | 3.55 | 0.5502 | 8.1 | 0.9085 |
| 1.32 | 0.1206 | 1.82 | 0.2601 | 3.60 | 0.5563 | 8.2 | 0.9138 |
| 1.33 | 0.1239 | 1.83 | 0.2625 | 3.65 | 0.5623 | 8.3 | 0.9191 |
| 1.34 | 0.1271 | 1.84 | 0.2648 | 3.70 | 0.5682 | 8.4 | 0.9243 |
| 1.35 | 0.1303 | 1.85 | 0.2672 | 3.75 | 0.5740 | 8.5 | 0.9294 |
| 1.36 | 0.1335 | 1.86 | 0.2695 | 3.80 | 0.5798 | 8.6 | 0.9345 |
| 1.37 | 0.1367 | 1.87 | 0.2718 | 3.85 | 0.5855 | 8.7 | 0.9395 |
| 1.38 | 0.1399 | 1.88 | 0.2742 | 3.90 | 0.5911 | 8.8 | 0.9445 |
| 1.39 | 0.1430 | 1.89 | 0.2765 | 3.95 | 0.5966 | 8.9 | 0.9494 |
| 1.40 | 0.1461 | 1.90 | 0.2788 | 4.0 | 0.6021 | 9.0 | 0.9542 |
| 1.41 | 0.1492 | 1.91 | 0.2810 | 4.1 | 0.6128 | 9.1 | 0.9590 |
| 1.42 | 0.1523 | 1.92 | 0.2833 | 4.2 | 0.6232 | 9.2 | 0.9638 |
| 1.43 | 0.1553 | 1.93 | 0.2856 | 4.3 | 0.6335 | 9.3 | 0.9685 |
| 1.44 | 0.1584 | 1.94 | 0.2878 | 4.4 | 0.6435 | 9.4 | 0.9731 |
| 1.45 | 0.1614 | 1.95 | 0.2900 | 4.5 | 0.6532 | 9.5 | 0.9777 |
| 1.46 | 0.1644 | 1.96 | 0.2923 | 4.6 | 0.6628 | 9.6 | 0.9823 |
| 1.47 | 0.1673 | 1.97 | 0.2945 | 4.7 | 0.6721 | 9.7 | 0.9868 |
| 1.48 | 0.1703 | 1.98 | 0.2967 | 4.8 | 0.6812 | 9.8 | 0.9912 |
| 1.49 | 0.1732 | 1.99 | 0.2989 | 4.9 | 0.6902 | 9.9 | 0.9956 |