TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI

Population Growth
MATHematics MAGnification™

Dr. Ralph deLaubenfels

TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI

Teacher-Scholar Institute

Columbus, Ohio

2019

© 2019, Teacher-Scholar Institute, www.teacherscholarinstitute.com

POPULATION GROWTH MAGNIFICATION

This is one of a series of very short books on math, statistics, and physics called "Math Magnifications." The "magnification" refers to focusing on a particular topic that is pivotal in or emblematic of mathematics.

OUTLINE

Our goal is to predict the population in the future. This begins with modeling the present. In this Magnification, we will present and apply three popular models: linear, exponential, and logistic.

Prerequisites for this magnification are first-year high school algebra ([2] is more than sufficient).

1. INTRODUCTION

This section will mostly be terminology, in part to communicate the generality of our models.

Definition 1.0. The symbol "$\equiv$" means **is defined to be**.

Definitions 1.1. A **population** is the number of inhabitants of a specified kind in a specified place. "Inhabitants" could mean people, but does not have to. Other inhabitants of interest might be feral cats in the woods outside a big city, bugs on a farm, one-celled organisms in standing water, or money owed (this will be the subject of a future magnification).

A **generation** is any fixed length of time, at multiples of which we measure (that is, count) our population. A generation could be seconds, minutes, hours, days, weeks, months, years, decades, centuries, millenia.

For a reproducing organism, the most natural definition of generation is the average time that elapses between birth and giving birth.

Definitions 1.2. Population will be denoted P. P_0 will be the **initial population**, meaning the population now or whenever we start our clock.

For $N = 0, 1, 2, 3, \ldots,$

$$P_N \equiv \text{ population after } N \text{ generations.}$$

A list of all the populations in order, $P_0, P_1, P_2, P_3, \ldots,$ is a **sequence** of numbers.

Quick Summary 1.3. *Linear growth*, where the population is increased by the same number every generation, will be in Section 2; the general story is in 2.3. The corresponding sequence is then *arithmetic*. *Exponential growth*, where the population is multiplied by the same number every generation, will be in Section 3; the general story is in 3.3. The corresponding sequence is then *geometric*. *Logistic growth*, introduced in Section 4, is a modification of exponential that is more realistic, especially in the long term, and (usually, in practice) has the very desirable feature of converging to an equilibrium state in the long term; see 4.3 and 4.5.

Both linear and exponential growth will be given both a *recursive* and an *explicit* description. A recursive description defines the population in a generation entirely in terms of the population the previous generation; e.g., "the number of flat worms quadruples every day." An explicit description defines a population entirely in terms of the initial population and the number of generations that have elapsed since we started our clock; e.g., "$5(4^N)$ flat worms N days after January 1, $N = 0, 1, 2, 3, \ldots$"

Logistic growth is given only a recursive description.

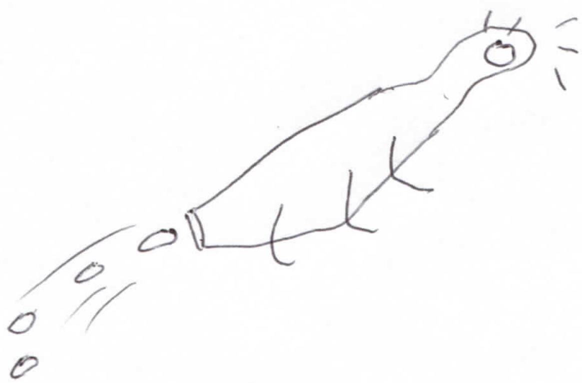
Remark 1.4. Another famous sequence, the *Fibonacci numbers*, also arises from a population model that may be considered a modification of exponential–more precisely, a modification of doubling where reproduction requires maturity–see [1].

We summarize our population formulas in Section 5. Section 6 has more examples, to prepare the hypothetical reader of this Magnification for the homework. Hints are given, on the page before the answers, for some of the homework.

2. LINEAR GROWTH

Definition 2.1. Linear growth means the same amount is added to the population every generation.

Example 2.2. In a beehive, all reproduction is done by the queen bee.



Suppose she produces 4 bees every minute.
If we have 10 bees at noon today, we'd like to know how many bees we'll have in the future.

Denote by

$$\begin{aligned} P_0 &\equiv \text{ the bee population at 12:00} \\ P_1 &\equiv \text{ the bee population at 12:01} \\ P_2 &\equiv \text{ the bee population at 12:02} \\ P_3 &\equiv \text{ the bee population at 12:03} \\ &\vdots \end{aligned}$$

A *recursive description* of the bee population is, in words, "add 4 every minute."

Starting with our P_0 of 10, we recursively obtain

$$\begin{aligned} P_1 &= P_0 + 4 = 10 + 4 = 14 \\ P_2 &= P_1 + 4 = 14 + 4 = 18 \\ P_3 &= P_2 + 4 = 18 + 4 = 22 \\ P_4 &= P_3 + 4 = 22 + 4 = 26 \\ &\vdots \end{aligned}$$

There is no theoretical limit to how many times we apply the recursive description, hence no theoretical limit to how far into the future we can predict the population of bees. But one could argue the *physical* limit of getting tired from repeated calculations. For example, if we wanted the number of bees at 1:00, we would add 4 60 times, to get from P_0 to our desired P_{60}.

Let's try for a shortcut to getting P_{60}. Our goal is to "add 4 60 times." This is the same as "adding 4×60." Let's illustrate this type of shortcut with P_0, P_1, P_2, P_3, P_4, calculated above, then jump ahead to getting P_{60} with this method.

$$
\begin{aligned}
P_0 &= 10 + 4 \times 0 = 10 \\
P_1 &= 10 + 4 \times 1 = 14 \\
P_2 &= 10 + 4 \times 2 = 18 \\
P_3 &= 10 + 4 \times 3 = 22 \\
P_4 &= 10 + 4 \times 4 = 26 \\
P_5 &= 10 + 4 \times 5 = 30 \\
&\cdot \\
&\cdot \\
&\cdot \\
&\cdot \\
P_{60} &= 10 + 4 \times 60 = 250.
\end{aligned}
$$

We feel emboldened to make a huge intellectual leap. For any $N = 0, 1, 2, \ldots$, let

$$P_N \equiv \text{ the population } N \text{ minutes after noon.}$$

Notice, in P_0, P_1, P_2, P_3, P_4 above, that the subscript of P equals the multiple of 4; e.g., rewriting P_{60} directly below, we have circled the two places where 60 appears.

$$P_{60} = 10 + 4 \times 60 = 250.$$

Follow this pattern, between subscript and what 4 is multiplied by, to get the very general formula

$$P_N = 10 + 4 \times N \equiv 10 + 4N, \quad N = 0, 1, 2, 3 \ldots.$$

The formula we just wrote down is an *explicit description* of the population. It means, for example, that P_{192}, the population 192 minutes after noon, is $10 + 4 \times 192 = 778$.

A similarly compact *recursive description* of our bee population is

$$P_N = P_{N-1} + 4, \quad N = 1, 2, 3, \ldots;$$

this is shorthand for infinitely many statements

$$P_9 = P_8 + 4, P_{57} = P_{56} + 4, P_3 = P_2 + 4, \text{ etc.}$$

With the explicit description, we arrive at a population at any specified time immediately. For example, we could get

$$P_{2,000} = 10 + 4 \times 2,000 = 8,010,$$

without getting $P_1, P_2, P_3, \ldots P_{1,999}$ first, as we would have to do with our recursive description.

The recursive description focuses on the change in population, and often is our first descriptive representation from our initial information, e.g., "add 4 every minute."

In this example, $P_0 = 10$ is the *initial population* and 4 is the *common difference*, denoted d, between two consecutive populations.

The sequence of numbers $P_0, P_1, P_2, \cdots = 10, 14, 18, 22, \ldots$ is an *arithmetic sequence*, characterized by the difference between consecutive terms being constant.

LINEAR FORMULAS and TERMINOLOGY 2.3. Here's the general story for linear growth, where, for $N = 0, 1, 2, 3, \ldots,$

$$P_N \equiv \text{ the population after } N \text{ generations.}$$

$$\textbf{Recursive: } P_N = P_{(N-1)} + d, \;\; N = 1, 2, 3, \ldots;$$

$$\textbf{Explicit: } P_N = P_0 + dN, \;\; N = 0, 1, 2, 3, \ldots.$$

P_0 is the **initial population** and d is the **common difference**, the number that one adds to a population, to get the population the next generation.

The sequence $P_0, P_1, P_2, \ldots$ is then an **arithmetic sequence**, meaning a sequence of numbers where the difference between consecutive terms is constant.

Practice 2.4. On the next page, fill in missing populations $P_N, N = 3, 4, 5, 6$, both recursively and explicitly, and fill in P_{10} and P_{50} explicitly.

The page after the next page has answers.

LINEAR GROWTH
add 3 every day
(3 = d = common difference)
population of 8 today
($P_0 = 8$)

	RECURSIVE	EXPLICIT
N	$P_N = P_{N-1} + 3$: Add 3 to previous day	$P_N = 8 + 3N$: 8 plus 3x(number of days)
0	8	8
	(= P_0)	($P_0 = 8 + 3x0$)
1	11 = 8 + 3	11 = 8 + 3x1
	($P_1 = P_0 + 3$)	($P_1 = 8 + 3x1$)
2	14 = 11 + 3	14 = 8 + 3x2
	($P_2 = P_1 + 3$)	($P_2 = 8 + 3x2$)
3		
	($P_3 = P_2 + 3$)	($P_3 = 8 + 3x3$)
4		
	($P_4 = P_3 + 3$)	($P_4 = 8 + 3x4$)
5		
	($P_5 = P_4 + 3$)	($P_5 = 8 + 3x5$)
6		
	($P_6 = P_5 + 3$)	($P_6 = 8 + 3x6$)
.	……………………	………………………
.	……………………	……………………………
10	………………………	
		($P_{10} = 8 + 3x10$)
.	………………………	
.	………………………	
50	………………………	

LINEAR GROWTH
add 3 every day
(3 = d = common difference)
population of 8 today
($P_0 = 8$)

p. 7

	RECURSIVE	EXPLICIT
N	$P_N = P_{N-1} + 3$: Add 3 to previous day	$P_N = 8 + 3N$: 8 plus 3x(number of days)
0	8	8
	(= P_0)	($P_0 = 8 + 3x0$)
1	11 = 8 + 3	11 = 8 + 3x1
	($P_1 = P_0 + 3$)	($P_1 = 8 + 3x1$)
2	14 = 11 + 3	14 = 8 + 3x2
	($P_2 = P_1 + 3$)	($P_2 = 8 + 3x2$)
3	17 = 14 + 3	17 = 8 + 9
	($P_3 = P_2 + 3$)	($P_3 = 8 + 3x3$)
4	20 = 17 + 3	20 = 8 + 12
	($P_4 = P_3 + 3$)	($P_4 = 8 + 3x4$)
5	23 = 20 + 3	23 = 8 + 15
	($P_5 = P_4 + 3$)	($P_5 = 8 + 3x5$)
6	26 = 23 + 3	26 = 8 + 18
	($P_6 = P_5 + 3$)	($P_6 = 8 + 3x6$)
.	....	....
.	....	....
10	....	38 = 8 + 30
		($P_{10} = 8 + 3x10$)
.	....	
.	....	
50	....	158 = 8 + 3x50

Examples 2.5. Assume fractional values of P_N are possible.

(a) Suppose there are 9 scumslugs today and the population of scumslugs increases by 6 every day. How many scumslugs will there be 20 days after today?

(b) Write a recursive description of the scumslug population in (a).

(c) Write an explicit description of the scumslug population in (a).

(d) Which of the following sequences are arithmetic? For those that are, get the common difference.

(i) $0, 5, 10, 15, \ldots$

(ii) $2, 7, 12, 17, \ldots$

(iii) $2, 4, 8, 16, \ldots$

(iv) $0, 1, 4, 9, \ldots$

(v) $5, 8, 11, 14, \ldots.$

(e) Suppose $P_0 = 4, P_1 = 7, \ldots$ describes linear growth. Get P_{10}.

(f) Suppose P_N describes linear growth, $P_0 = 20$ and $P_8 = 60$. Get P_{16}.

(g) In (f), get an explicit description of P_N.

(h) Suppose P_N describes linear growth, $P_8 = 15$ and $P_{28} = 45$. Get the common difference d without getting P_0.

(i) In (h), get a recursive description of P_N without getting P_0.

(j) In (h), get P_0.

(k) Suppose $P_0, P_1, P_2, \ldots$ is an arithmetic sequence of populations. If $P_{10} = 97$ and $P_{30} = 257$, what is P_0?

(l) If $P_0 = 18$ and $P_N = P_{N-1} + 12$ is a recursive description of linear growth, find an explicit description.

(m) If $P_N = 123 + 13 \times N$ is an explicit description of linear growth, find a recursive description.

ANSWERS

(a) $9 + 6 \times 20 = 129$ scumslugs. (Add 6 20 times, to 9.)

(b) The common difference here is $d = 6$, so

$$P_N = P_{N-1} + 6, \quad N = 1, 2, 3, \ldots$$

(c) $P_0 = 9$, and $d = 6$, so

$$P_N = 9 + 6N, \quad N = 0, 1, 2, \ldots$$

(d) We need to look at differences between consecutive terms.
In (i), $(5 - 0) = (10 - 5) = (15 - 10) = 5$, so (i) is arithmetic, with common difference $d = 5$.
In (ii), $(7 - 2) = (12 - 7) = (17 - 12) = 5$, so same conclusion for (ii) as for (i).
In (iii), $(4 - 2) \neq (8 - 4)$, so (iii) is not arithmetic.
In (iv), $(1 - 0) \neq (4 - 1)$, so (iv) is not arithmetic.
In (v), $(8 - 5) = (11 - 8) = (14 - 11) = 3$, so (v) is arithmetic, with common difference $d = 3$.

(e) The common difference d is $(7 - 4) = 3$, so $P_{10} = P_0 + 3 \times 10 = 4 + 3 \times 10 = 34$.

(f) Denote by d the common difference. To get from P_0 to P_8 we must add d 8 times; that is, $60 = P_8 = P_0 + d \times 8 = 20 + 8d$ implies that d equals 5, so that $P_{16} = P_0 + 5 \times 16 = 20 + 5 \times 16 = 100$.

Alternatively, we might have noticed that getting from P_8 to P_{16} is the same number of generations (16 - 8 = 8) as getting from P_0 to P_8, thus we are making the same increase in population:

$$40 = (60 - 20) = (P_8 - P_0) = (P_{16} - P_8) = (P_{16} - 60),$$

and solving for P_{16} again gives us 100.

(g) In (f) we got $d = 5$, thus

$$P_N = P_0 + 5 \times N = 20 + 5N, \quad N = 0, 1, 2, 3, \ldots.$$

(h) Denote by d the common difference. Getting from P_8 to P_{28} requires that we add $d(28 - 8) = 20$ times; that is,

$$45 = P_{28} = P_8 + d \times 20 = 15 + 20d,$$

which implies that $d = 1.5$.

(i) $P_N = P_{N-1} + 1.5, \quad N = 1, 2, 3, \ldots.$

(j) We are going 8 generations *backwards* from P_8, so we *subtract* d×8 from P_8, to get P_0:

$$P_0 = P_8 - d \times 8 = 15 - (1.5) \times 8 = 3.$$

Or we could set up an equation:

$$15 = P_8 = P_0 + d \times 8 = P_0 + 12,$$

and we may solve for P_0 to again get 3.

(k) This is similar to (h)—(j):

$$257 = P_{30} = P_{10} + d \times (30 - 10) = 97 + 20d$$

implies that the common difference d equals 8, so that

$$P_0 = P_{10} - d \times 10 = 97 - 80 = 17.$$

Alternatively, it takes 20 generations backwards to get from P_{30} to P_{10} and 10 generations backwards to get from P_{10} to P_0, so we need half the difference between P_{30} and P_{10}:

$$80 = \frac{1}{2} \times 160 = \frac{1}{2} \times (257 - 97) = \frac{1}{2} \times (P_{30} - P_{10});$$

we subtract 80 from P_{10} to get to P_0:

$$P_0 = P_{10} - 80 = 97 - 80 = 17.$$

(l) The common difference d equals 12, so

$$P_N = P_0 + 12 \times N = 18 + 12N, \quad N = 0, 1, 2, 3, \ldots.$$

(m) The common difference d equals 13, so

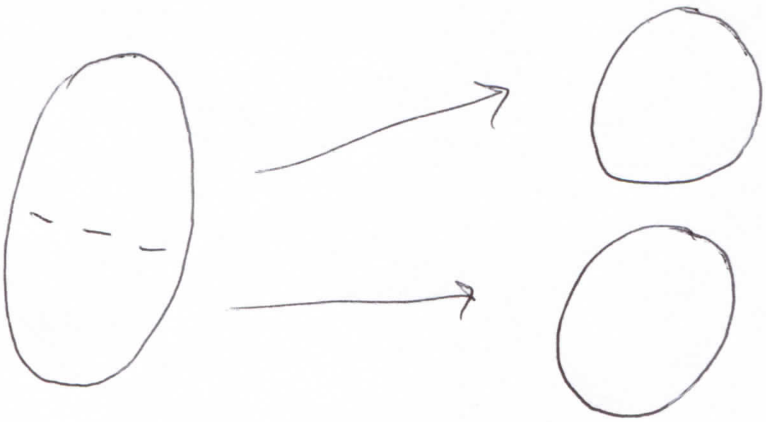
$$P_N = P_{N-1} + 13, \quad N = 1, 2, 3, \ldots.$$

3. EXPONENTIAL GROWTH

This section will be more cursory than Section 2, because the exposition and results are so analogous to Section 2. Here is a quick summary: addition in Section 2 is changed to multiplication in this section.

Definition 3.1. **Exponential growth** means you multiply the population by a fixed number every generation.

Example 3.2. An organism doubles every minute. If we have ten of said organism at noon today, how many can we expect in the future?



As in Section 2, denote by

$$\begin{aligned} P_0 &\equiv \text{ the population at 12:00} \\ P_1 &\equiv \text{ the population at 12:01} \\ P_2 &\equiv \text{ the population at 12:02} \\ P_3 &\equiv \text{ the population at 12:03} \\ &\ \vdots \\ P_N &\equiv \text{ the population at } N \text{ minutes after noon} \\ &\ \vdots \end{aligned}$$

A *recursive description* of the population is

$$P_N = 2(P_{N-1}), \quad N = 1, 2, 3, \ldots;$$

in words, "double every minute" or "multiply by 2 every minute." This is shorthand for

$$P_1 = 2 \times P_0 = 2 \times 10 = 20$$
$$P_2 = 2 \times P_1 = 2 \times 20 = 40$$
$$P_3 = 2 \times P_2 = 2 \times 40 = 80$$
$$P_4 = 2 \times P_3 = 2 \times 80 = 160$$
$$\vdots$$

As with recursive linear growth, the slowness of getting populations far in the future should concern us. For example, to get P_{20}, the population at 12:20, we would have to multiply by 2 20 times. We now cleverly observe, analogous to Section 2, that "multiply by 2 20 times" means "multiply by 2^{20}"; in the following, $(2 \times 2 \times 2 \times \cdots \times 2)$ contains 20 repetitions of 2:

$$P_{20} = 10 \times (2 \times 2 \times 2 \times \cdots \times 2) = 10 \times (2^{20}) = 10,485,760.$$

An *explicit description* of our population is the formula

$$P_N = 10 \times 2^N \equiv 10(2^N), \quad N = 0, 1, 2, 3, \ldots.$$

In this example, $r = 2$ is the *common ratio* for the *geometric sequence* of populations

$$P_0, P_1, P_2, \cdots = 10, 20, 40, \ldots$$

As in Section 2, we are ready to generalize.

EXPONENTIAL FORMULAS and TERMINOLOGY 3.3. Here's the general story for exponential growth, where, for $N = 0, 1, 2, 3, \ldots,$

$$P_N \equiv \text{ the population after } N \text{ generations.}$$

$$\textbf{Recursive: } P_N = r(P_{(N-1)}), \quad N = 1, 2, 3, \ldots;$$

$$\textbf{Explicit: } P_N = P_0(r^N), \quad N = 0, 1, 2, 3, \ldots.$$

P_0 is the **initial population** and r is the **common ratio**, the number that one multiplies a population by, to get the population the next generation.

The sequence $P_0, P_1, P_2, \ldots$ is then a **geometric sequence**, meaning a sequence where the ratio between consecutive terms is constant.

Practice 3.4. On the next page, fill in missing populations $P_N, N = 3, 4, 5, 6$, both recursively and explicitly, and fill in P_{10} and P_{20} explicitly.

The page after the next page has answers.

EXPONENTIAL GROWTH
multiply by 3 every day
(3 = r = common ratio)
population of 2 today
(P_0 = 2)

	RECURSIVE	EXPLICIT
N	P_N = 3xP_{N-1}:	P_N = 2x(3^N):
	multiply previous day by 3	2 times (3 raised to Nth power)
0	2	2 = 2x1
	(= P_0)	(P_0 = 2 x (3^0))
1	6 = 3x2	6 = 2x3
	(P_1 = 3 x P_0)	(P_1 = 2 x (3^1))
2	18 = 3x6	18 = 2x9
	(P_2 = 3 x P_1)	(P_2 = 2 x (3^2))
3		
	(P_3 = 3 x P_2)	(P_3 = 2 x (3^3))
4		
	(P_4 = 3 x P_3)	(P_4 = 2 x (3^4))
5		
	(P_5 = 3 x P_4)	(P_5 = 2 x (3^5))
6		
	(P_6 = 3 x P_5)	(P_6 = 2 x (3^6))
.	....	....
.	....	....
10	....	
		(P_{10} = 2 x (3^{10}))
.	....	
.	....	
20	....	

EXPONENTIAL GROWTH
multiply by 3 every day
(3 = r = common ratio)
population of 2 today
(P_0 = 2)

	RECURSIVE	EXPLICIT
N	P_N = 3xP_{N-1}:	P_N = 2x(3^N):
	multiply previous day by 3	2 times (3 raised to Nth power)
0	2	2 = 2x1
	(= P_0)	(P_0 = 2 x (3^0))
1	6 = 3x2	6 = 2x3
	(P_1 = 3 x P_0)	(P_1 = 2 x (3^1))
2	18 = 3x6	18 = 2x9
	(P_2 = 3 x P_1)	(P_2 = 2 x (3^2))
3	54 = 3x18	54 = 2x27
	(P_3 = 3 x P_2)	(P_3 = 2 x (3^3))
4	162 = 3x54	162 = 2x81
	(P_4 = 3 x P_3)	(P_4 = 2 x (3^4))
5	486 = 3x162	486 = 2x243
	(P_5 = 3 x P_4)	(P_5 = 2 x (3^5))
6	1,458 = 3x486	1,458 = 2x729
	(P_6 = 3 x P_5)	(P_6 = 2 x (3^6))
.	....	....
.	....	....
10	....	118,098 = 2x59,049
		(P_{10} = 2 x (3^{10}))
.	....	
.	....	
20	....	6,973,568,802 = 2x(3^{20})

Examples 3.5. Allow fractional or irrational populations, where needed.

(a) Suppose there are 9 scumslugs today and the population of scumslugs is multiplied by 2.5 every day. How many scumslugs will there be 20 days from now?

(b) Write a recursive description of the scumslug population in (a).

(c) Write an explicit description of the scumslug population in (a).

(d) Identify each of the following sequences as arithmetic, geometric, or neither. If arithmetic, get the common difference; if geometric, get the common ratio.

(i) $1, 3, 9, 27, \ldots$

(ii) $2, 6, 18, 54, \ldots$

(iii) $2, 5, 8, 11, \ldots$

(iv) $1, 4, 9, \ldots$

(v) $0.5, 1, 2, 4 \ldots$

(vi) $0.5, 1, 1.5, 2, \ldots$

(e) Suppose $P_0 = 2, P_1 = 6, \ldots$ describes exponential growth. Get P_{10}.

(f) Suppose P_N describes exponential growth, $P_0 = 1$ and $P_2 = 9$. Get P_{20}.

(g) In (f), get an explicit description of P_N.

(h) Suppose P_N describes exponential growth, $P_3 = 250$ and $P_4 = 1,250$. Get the common ratio r without getting P_0.

(i) In (h), get a recursive description of P_N without getting P_0.

(j) In (h), get P_2 without getting P_0.

(k) In (h), get P_0.

(l) If $P_0 = 18$ and $P_N = 12 \times P_{N-1}$ is a recursive description of exponential growth, find an explicit description.

(m) If $P_N = 7 \times 3^N$ is an explicit description of exponential growth, find a recursive description.

ANSWERS

(a) $P_0 = 9$ and the common ratio r is 2.5:

$$P_{20} = 9 \times (2.5)^{20} \sim 818,545,232.$$

(b) $P_N = 2.5 \times P_{N-1}, \;\; N = 1, 2, 3 \ldots.$

(c) $P_N = 9 \times (2.5)^N, \;\; N = 0, 1, 2, \ldots.$

(d) For arithmetic, check the differences between consecutive terms; for geometric, check the ratios of consecutive terms.
(i) $\frac{3}{1} = \frac{9}{3} = \frac{27}{9} = 3$, so geometric with common ratio 3.
(ii) $\frac{6}{2} = \frac{18}{6} = \frac{54}{18} = 3$, so same conclusion as (i).
(iii) $(5-2) = (8-5) = (11-8) = 3$, so arithmetic with common difference 3.
(iv) $(4-1) \neq (9-4)$, so not arithmetic; $\frac{4}{1} \neq \frac{9}{4}$, so not geometric; neither.
(v) $\frac{1}{0.5} = \frac{2}{1} = \frac{4}{2} = 2$, so geometric with common ratio 2.
(vi) $(1-0.5) = (1.5-1) = (2-1.5) = 0.5$, so arithmetic with common difference 0.5.

(e) The common ratio r is $\frac{P_1}{P_0} = \frac{6}{2} = 3$, so
$$P_{10} = 2 \times 3^{10} = 118,098.$$

(f) Denote by r the common ratio. $9 = P_2 = P_0 \times r^2 = r^2$, so $r = 3$, thus
$$P_{20} = P_0 \times 3^{20} = 3^{20} = 3,486,784,401.$$

(g) $P_N = 3^N, \quad N = 0, 1, 2, \ldots.$

(h) r equals the ratio of consecutive terms $\frac{P_4}{P_3} = \frac{1,250}{250} = 5$.

(i) $P_N = 5P_{N-1}, \quad N = 1, 2, 3, \ldots.$

(j) Since we're going backwards in time, from P_3 to P_2, divide by 5 instead of multiplying by 5:
$$P_2 = \frac{1}{5} \times P_3 = \frac{250}{5} = 50.$$
Alternatively, we could use (i): $250 = P_3 = 5 \times P_2$, and solve for P_2.

(k) We're going two generations backward to get from P_2 to P_0, so divide by 5 twice:
$$P_0 = \left(\frac{1}{5}\right)^2 \times P_2 = \frac{50}{5^2} = 2.$$
Or, we could use (i) twice, as in (j), first to get $P_1 = 10$, then to get $P_0 = 2$.

(l) The common ratio is $r = 12$, so
$$P_N = 18 \times 12^N, \quad N = 0, 1, 2, \ldots.$$

(m) The common ratio is now 3, so
$$P_N = 3P_{N-1}, \quad N = 1, 2, 3, \ldots.$$

Remark 3.6. Exponential growth with a common ratio greater than one is always eventually larger than any linear growth. Below we've graphed exponential growth in red, linear growth in black. Linear growth has a constant *rate* of growth (also known as *slope*), while, with exponential growth, the rate is also increasing. For a quick numerical example, compare 2^N (exponential growth) to $2N$ (linear growth), for $N = 0, 1, 2, \ldots.$

4. LOGISTIC GROWTH

Exponential growth, introduced in the previous section, seems plausible, at least in the short term, for both asexual (see Example 3.2) and sexual reproduction; for example, if a population breaks into pairs, and each pair has 6 offspring, then goes away, we would have the population tripling every generation.

Something that should give us pause, with this model of population growth, is how large populations can become. If we started with ten organisms, then the population 30 generations from now, if subject to tripling every generation, would be

$$P_{30} = 10 \times 3^{30} \sim 2,000,000,000,000,000 \equiv 2 \text{ quadrillion}.$$

If we were speaking to a class, instead of typing symbols into a word processor, we would now ask the class "When might exponential growth be unrealistic?" The answer we usually get to this question is something like "when the population runs out of food." A modification of exponential growth is needed when it would imply too big a population.

Definitions 4.1. The **carrying capacity**, usually denoted C, is the maximum population possible. We use this to define the **relative population**

$$p_N \equiv \frac{P_N}{C} \quad (N = 0, 1, 2, 3, \ldots),$$

where P_N is the population after N generations.

Note that $0 \leq p_N \leq 1$, for any N and any population.

Example 4.2. If the population P_{10} after 10 generations is 40,000, and the carrying capacity is 200,000, then the relative population after 10 generations is

$$p_{10} = \frac{40,000}{200,000} = 0.2 \text{ or } 20\%.$$

Definitions 4.3. The recursive description of exponential growth with common ratio r (see 3.3) becomes, after dividing by $C, p_N = rp_{N-1}$.

The **logistic growth model** is

$$p_N = rp_{N-1}(1 - p_{N-1}), \quad N = 1, 2, 3, \ldots.$$

The number r is called the **growth parameter**.

As the population approaches the carrying capacity, so that p_{N-1} approaches 1, the extra $(1 - p_{N-1})$ term in the logistic model slows down the population growth, thus making logistic growth more realistic than exponential growth.

Notice that our logistic growth model is a recursive description; there is no simple explicit description.

Examples 4.4. In all logistic calculations, it is very convenient to use a calculator that saves all data, including numbers that aren't showing on the screen. Our printed numbers below are all rounded to ten decimal places, but our calculator saves information beyond ten decimal places.

All examples below are logistic p_N.

(a) For $r = 3, p_0 = 0.1$, we have

$$p_1 = 3p_0(1 - p_0) = 3(0.1)(1 - 0.1) = 0.27;$$
$$p_2 = 3p_1(1 - p_1) = 3(0.27)(1 - 0.27) = 0.5913;$$
$$p_3 = 3p_2(1 - p_2) = 3(0.5913)(1 - 0.5913) = 0.72499293.$$

If we only want p_3 rounded to two decimal places, our answer would be 0.72.

Watch what happens if we round to two decimal places at each step of the recursion:

$$p_2 \sim 0.59 \rightarrow p_3 \sim 3(0.59)(1 - 0.59) = 0.7257,$$

which, rounded to two decimal places, equals 0.73.

The moral of this is that any rounding should be put off until the last step of the recursions.

(b) For $r = 1.6, p_0 = 0.8$, rounded to ten decimal places by our calculator,

$$p_1 = 1.6p_0(1 - p_0) = 1.6(0.8)(1 - 0.8) = 0.256;$$
$$p_2 = 1.6p_1(1 - p_1) = 1.6(0.256)(1 - 0.256) = 0.3047424;$$
$$p_3 = 1.6p_2(1 - p_2) = 0.3389991514;$$
$$p_4 = 1.6p_3(1 - p_3) = 0.3585259628;$$
$$p_5 = 1.6p_4(1 - p_4) = 0.3679761549;$$
$$p_6 = 1.6p_5(1 - p_5) = 0.3721115269;$$
$$p_7 = 1.6p_6(1 - p_6) = 0.3738312615;$$
$$p_8 = 1.6p_7(1 - p_7) = 0.3745303191;$$
$$p_9 = 1.6p_8(1 - p_8) = 0.3748117747;$$
$$p_{10} = 1.6p_9(1 - p_9) = 0.3749246532.$$

Notice that the numbers p_N in (b) seem to be getting close to a particular number as N gets large; for example, if we rounded to three decimal places, we have

$$p_0 = 0.8, p_1 = 0.256, p_2 \sim 0.305, p_3 \sim 0.339, p_4 \sim 0.359, p_5 \sim 0.368,$$
$$p_6 \sim 0.372, p_7 \sim 0.374, p_8 \sim 0.375, p_9 \sim 0.375, p_{10} \sim 0.375, \ldots$$

(c) The choice of initial relative population p_0 turns out to not make a difference, in what number we get close to as N gets large. For example, if r is still 1.6, but $p_0 = 0.1$, we leave it to the reader to calculate, then round to three decimal places,

$$p_0 = 0.1, p_1 = 0.144, p_2 \sim 0.197, p_3 \sim 0.253, p_4 \sim 0.303, p_5 \sim 0.338,$$
$$p_6 \sim 0.358, p_7 \sim 0.368, p_8 \sim 0.372, p_9 \sim 0.374, p_{10} \sim 0.375, p_{11} \sim 0.375 \ldots$$

(d) The choice of growth parameter r does make a difference, in what number we get close to as N gets large. For r equal to $\frac{4}{3}$, $p_0 = 0.3$, we get, rounded to ten places by our calculator,

$$p_1 = \frac{4}{3}(0.3)(1 - 0.3) = 0.28,$$
$$p_2 = \frac{4}{3}(0.28)(1 - 0.28) = 0.2688,$$
$$p_3 = \frac{4}{3}(0.2688)(1 - 0.2688) = 0.26206208,$$
$$p_4 = \frac{4}{3}p_3(1 - p_3) = 0.257847395,$$
$$p_5 = \frac{4}{3}p_4(1 - p_4) = 0.2551494878.$$

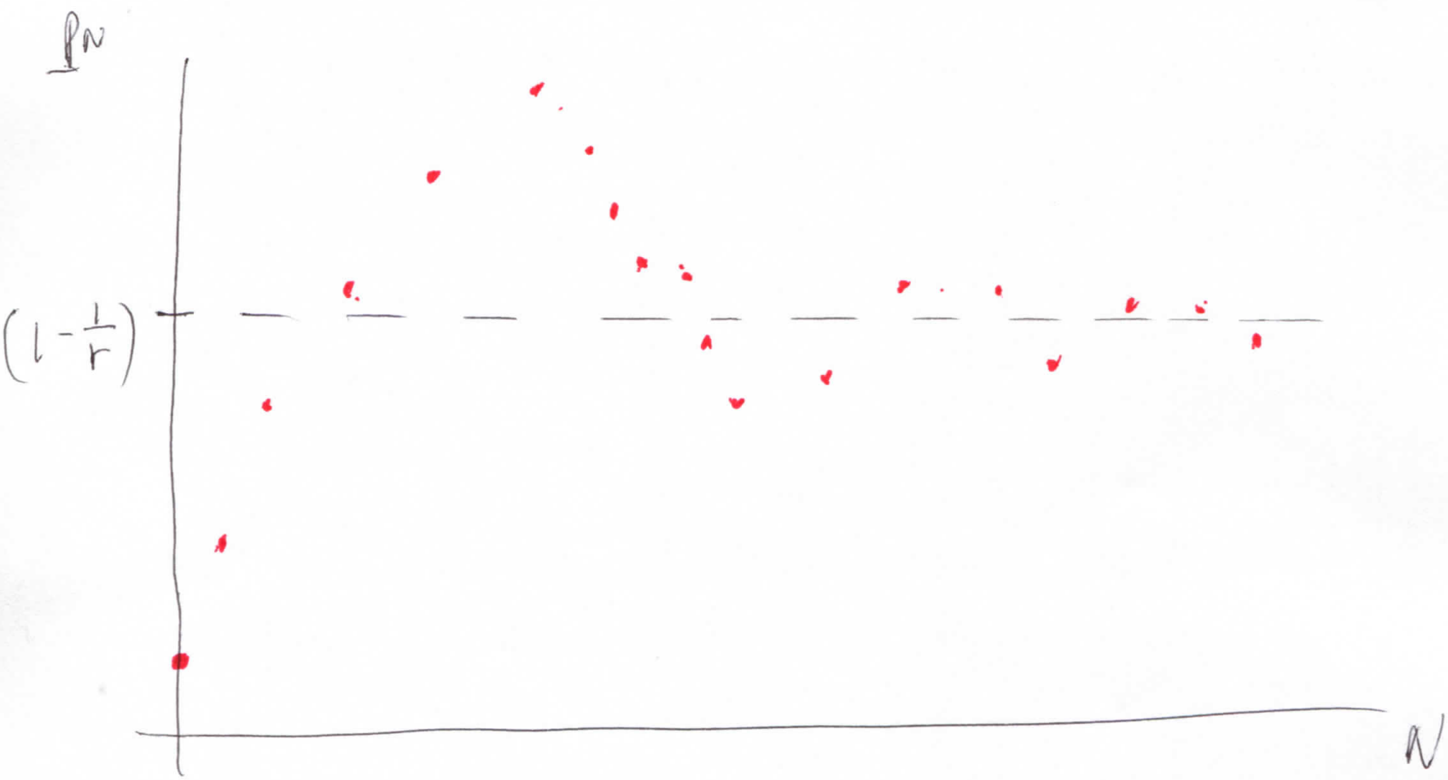
Here's the pattern.

Long-term behavior Theorem 4.5. For $1 < r < 3$, if p_N satisfies the logistic growth model as in Definitions 4.3, then p_N can be made arbitrarily close to $(1 - \frac{1}{r})$ by making N sufficiently large, *regardless* of p_0 between 0 and 1.

In the language of calculus, p_N *converges* to $(1 - \frac{1}{r})$, denoted $p_N \to (1 - \frac{1}{r})$ as $N \to \infty$, or

$$\lim_{N \to \infty} p_N = (1 - \frac{1}{r}),$$

shorthand for *the limit, as* N *goes to* ∞, *of* p_N, *equals* $(1 - \frac{1}{r})$.



For example, in Examples 4.4(b) and (c), $(1 - \frac{1}{r}) = (1 - \frac{1}{1.6}) = 0.375$; the long-term behavior of p_N in Examples 4.4(b) and (c) is convergence to 0.375.

In Examples 4.4(d), $(1 - \frac{1}{r}) = (1 - \frac{1}{\frac{4}{3}}) = 0.25$; the long-term behavior of p_N in Examples 4.4(d) is convergence to 0.25.

Practice 4.6. On the next page, fill in p_N (rounded to 4 decimal places) for $N = 1, 2, 3, \ldots, 15$. Also describe long-term behavior. The page after the next page has answers.

LOGISTIC GROWTH:
p_N = rp_{N-1}(1 - p_{N-1})

N	p_N, r = 2	p_N, r = 2	p_N, r = 1.5
0	0.3000	0.9500	0.6000
	(= p_0)	(= p_0)	(= p_0)
1			0.3600
	(p_1 = 2p_0(1-p_0))		(p_1 = (1.5)p_0(1-p_0))
2			0.3456
	(p_2=2p_1(1-p_1))		(p_2=(1.5)p_1(1-p_1))
3			0.3392
	(p_3 = 2p_2(1-p_2))		(p_3=(1.5)p_2(1-p_2))
4			0.3362
5			0.3348
6			0.3340
7			0.3337
8			0.3335
9			0.3334
10			0.3334
11			0.3334
12			0.3333
13			
14			
15			

NOTE: (1 - (1/2)) = 0.5 (1 - (1/1.5)) = (1/3) ~ .3333

LOGISTIC GROWTH:

$p_N = rp_{N-1}(1 - p_{N-1})$

N	p_N, r = 2	p_N, r = 2	p_N, r = 1.5
0	0.3000	0.9500	0.6000
	($= p_0$)	($= p_0$)	($= p_0$)
1	0.4200	0.0950	0.3600
	($p_1 = 2p_0(1-p_0)$)		($p_1 = (1.5)p_0(1-p_0)$)
2	0.4872	0.1720	0.3456
	($p_2=2p_1(1-p_1)$)		($p_2=(1.5)p_1(1-p_1)$)
3	0.4997	0.2848	0.3392
	($p_3 = 2p_2(1-p_2)$)		($p_3=(1.5)p_2(1-p_2)$)
4	0.5000	0.4073	0.3362
5	0.5000	0.4828	0.3348
6	0.5000	0.4994	0.3340
7	0.5000	0.5000	0.3337
8	0.5000	0.5000	0.3335
9	0.5000	0.5000	0.3334
10	0.5000	0.5000	0.3334
11	0.5000	0.5000	0.3334
12	0.5000	0.5000	0.3333
13	0.5000	0.5000	0.3333
14	0.5000	0.5000	0.3333
15	0.5000	0.5000	0.3333

p_N converges to 0.5, as $N \to \infty$, since

$(1 - (1/2)) = 0.5$

p_N converges to $\frac{1}{3}$ as $N \to \infty$, since

$(1 - (1/1.5)) = (1/3) \sim .3333$

5. POPULATION GROWTH FORMULAS, summarized

P_0 means initial population, d is *common difference* (for linear growth), r is *common ratio* (for exponential growth). For any number N, we want to know

$P_N \equiv$ population after N days (or minutes, or years, or whatever the time unit is).

LINEAR GROWTH:

$$\text{Recursive: } P_N = P_{(N-1)} + d;$$
$$\text{Explicit: } P_N = P_0 + dN.$$

EXPONENTIAL GROWTH:

$$\text{Recursive: } P_N = r(P_{(N-1)});$$
$$\text{Explicit: } P_N = P_0(r^N).$$

LOGISTIC GROWTH: ($p_N \equiv \frac{P_N}{C}$, where C is *carrying capacity*)

$$\text{Recursive: } p_N = rp_{(N-1)}(1 - p_{(N-1)}).$$

LONG-TERM BEHAVIOR, LOGISTIC, $1 < r < 3$:

$$p_N \text{ converges to } (1 - \frac{1}{r}), \text{ as } N \text{ goes to infinity.}$$

EXAMPLES:

Linear, $d = 7, P_0 = 11$: Recursive: $P_N = P_{N-1} + 7$ MEANS "add 7 every day":

$$P_1 = P_0 + 7 = 11 + 7 = 18,$$
$$P_2 = P_1 + 7 = 18 + 7 = 25,$$
$$P_3 = P_2 + 7 = 25 + 7 = 32,$$
$$\ldots \text{ etc. } \ldots$$

Explicit: $P_N = 11 + 7N$ MEANS

$$P_1 = 11 + 7 \times 1 = 18,$$
$$P_2 = 11 + 7 \times 2 = 25,$$
$$P_3 = 11 + 7 \times 3 = 32,$$
$$\ldots \text{ etc. } \ldots$$
$$P_{127} = 11 + 7 \times 127 = 900,$$
$$\ldots \text{ etc. } \ldots$$

Exponential, $r = 2.5, P_0 = 11$: Recursive: $P_N = (2.5)P_{N-1}$ MEANS "multiply by 2.5 every day":

$$P_1 = (2.5)P_0 = (2.5) \times 11 = 27.5,$$
$$P_2 = (2.5)P_1 = (2.5) \times (27.5) = 68.75,$$
$$P_3 = (2.5)P_2 = (2.5) \times (68.75) = 171.875,$$
$$\ldots \text{ etc. } \ldots$$

Explicit: $P_N = 11(2.5^N)$ MEANS

$$P_1 = 11(2.5^1) = 27.5,$$
$$P_2 = 11(2.5^2) = 11 \times 6.25 = 68.75,$$
$$P_3 = 11(2.5^3) = 11 \times 15.625 = 171.875,$$
$$\ldots \text{ etc. } \ldots$$
$$P_{39} = 11(2.5^{39}) \sim 36,400,000,000,000,000,$$
$$\ldots \text{ etc. } \ldots$$

Logistic, $r = 2.5, p_0 = 0.7$:

$$p_1 = (2.5)(p_0)(1 - p_0) = (2.5)(0.7)(1 - 0.7) = 0.525,$$
$$p_2 = (2.5)(p_1)(1 - p_1) = (2.5)(0.525)(1 - 0.525) \sim 0.6234,$$
$$p_3 = (2.5)(p_2)(1 - p_2) \sim (2.5)(0.6234)(1 - 0.6234) \sim 0.5869, \ldots$$

Long-term behavior: p_N converges to $(1 - \frac{1}{2.5}) = 0.6$, as N goes to infinity.

6. MORE EXAMPLES

1. Suppose the initial relative population p_0 is 0.00001. If p_N, the relative population N days after January 1, grows according to the logistic growth model with growth parameter $r = 2.9$, what long-term behaviour of p_N should you expect?

2. Which of the following sequences are arithmetic, which are geometric, and which are neither? For those that are arithmetic, find the common difference d; for those that are geometric, find the common ratio r.

a. 1, 8, 27 ...
b. 1, 1, 2, 3, 5, 8, ...
c. 2, 6, 18, 54, ...
d. 2, 7, 22, 67, ...
e. 1, 6, 11, 16, 21, ...

3. Suppose a population increases by 6 every hour. If there are 13 at noon today, how many will there be at noon seven days from now?

4. Suppose a population is multiplied by (1.3) every day. If there are 10 on January 1, how many will there be on January 31?

5. A population of scumslugs grows according to the logistic growth model, with growth parameter $r = 2.5$,

$$p_N = (2.5)p_{N-1}(1 - p_{N-1}),$$

where p_N is the relative population N days after today. If p_0, the relative population today, equals 0.1, find the relative population p_3 three days from now.

6. Suppose $P_N = P_{N-1} + 4$, for any number N; that is, the population increases by 4 every generation. If $P_{207} = 2,149$, find

(a) P_{210};
(b) P_{205}.
(NOTE: you do not need to get P_0.)

7. Suppose $P_N = (1.2)P_{N-1}$, for any number N; that is, the population is multiplied by (1.2) every generation. If $P_{17} = 21$, find

(a) P_{18};
(b) P_{19};
(c) P_{16}.
(NOTE: you do not need to get P_0.)

For 8–12, identify

(a) if the description is explicit or recursive;
(b) if the description is of linear, exponential, or logistic growth.
(c) the common difference, if linear; the common ratio, if exponential; the growth parameter, if logistic.

8. $p_N = 3p_{N-1}(1 - p_{N-1})$. $(p_1 = 3p_0(1 - p_0), p_2 = 3p_1(1 - p_1), ...)$

9. $P_N = 15 + 9N$. $(P_0 = 15, P_1 = 15 + 9, P_2 = 15 + 9 \times 2, ...)$

10. $P_N = 3P_{N-1}$. $(P_1 = 3P_0, P_2 = 3P_1, ...)$

11. $P_N = P_{N-1} + 15$. $(P_1 = P_0 + 15, P_2 = P_1 + 15, ...)$

12. $P_N = 10(3^N)$. $(P_0 = 10, P_1 = 10 \times 3, P_2 = 10 \times 3^2, ...)$

13. Suppose $P_0, P_1, P_2, ...$ is a sequence of populations, $P_{11} = 20$ and $P_{12} = 22$. Find P_{13}, if

(a) the population growth is linear;

(b) the population growth is exponential.

(NOTE: you do not need to get P_0.)

14. Suppose $p_0, p_1, p_2, ...$ is a sequence of relative populations following the logistic growth model with growth parameter $r = 2$. If $p_{100} = 0.7$, what is p_{101}?

15. Suppose $P_{50} = 100$ and $P_{52} = 900$.

(a) If P_N is linear, find the common difference d.

(b) If P_N is exponential, find the common ratio r.

ANSWERS

1. p_N gets arbitrarily close to $(1 - \frac{1}{2.9}) \sim 0.655$, as N gets large.

2. Checking consecutive ratios and consecutive differences, we deduce that e. is arithmetic with d $= 5$ $(5 = (6-1) = (11-6) = (16-11) = \ldots)$, c. is geometric with r = 3 $(3 = \frac{6}{2} = \frac{18}{6} = \frac{54}{18})$, the others are neither. For example, in part a., $(8-1) \neq (27-8)$, so the sequence is not arithmetic, while $\frac{8}{1} \neq \frac{27}{8}$ implies that the sequence is not geometric.

3. This is linear, with $P_0 = 13$, common difference 6; $P_{7\times 24} = 13 + 6 \times (7 \times 24) = 1021$.

4. This is exponential, with $P_0 = 10$, common ratio 1.3: $P_{30} = 10 \times (1.3)^{30} = 26,199.95644$ (rounding to 26,200 makes more physical sense).

5. $p_1 = 2.5(0.1)(1-0.1) = 0.225; p_2 = 2.5(0.225)(1-0.225) = 0.4359375; p_3 = 2.5(p_2)(1-p_2) = 0.6147399902 \sim 0.6147$ (to 4 decimal places).

6. (a) To move from P_{207} to P_{210}, we need to add 4 three times: $P_{210} = P_{207} + 4 \times 3 = 2,149 + 12 = 2,161$.

(b) We are moving 2 generations *backward* in time from P_{207}, so we *subtract* 4 two times: $P_{205} = P_{207} - 4 \times 2 = 2,149 - 8 = 2,141$.

Alternatively, we could've said $(4 \times 2) + P_{205} = P_{207}$, and solved for P_{205}.

7. This is an exponential analogue of no. 6.

(a) $P_{18} = (1.2) \times P_{17} = (1.2) \times 21 = 25.2$.

(b) $P_{19} = (1.2) \times P_{18} = (1.2) \times 25.2 = 30.24$.

(c) Moving backwards one generation, we divide by (1.2) instead of multiplying: $P_{16} = \frac{1}{1.2} \times P_{17} = \frac{21}{1.2} = 17.5$.

Alternatively, we could've observed that $(1.2) \times P_{16} = P_{17}$, and solved for P_{16}.

8. (a) recursive (b) logistic (c) growth parameter 3
9. (a) explicit (b) linear (c) common difference 9
10. (a) recursive (b) exponential (c) common ratio 3
11. (a) recursive (b) linear (c) common difference 15
12. (a) explicit (b) exponential (c) common ratio 3

13. (a) The common difference is $22 - 20 = 2$, so $P_{13} = P_{12} + 2 = 22 + 2 = 24$.

(b) The common ratio is $\frac{22}{20}$, so $P_{13} = \left(\frac{22}{20}\right) P_{12} = (1.1) \times 22 = 24.2$.

14. $p_{101} = 2(p_{100})(1 - p_{100}) = 2(0.7)(1 - 0.7) = 0.42$.

15. (a) $900 = P_{52} = P_{50} + d \times 2 = 100 + 2d$, so that d $= \frac{900-100}{2} = 400$.

(b) $900 = P_{52} = P_{50} \times r^2 = 100 \times r^2$, so that r $= \sqrt{\frac{900}{100}} = 3$.

HOMEWORK

0. Fill in the missing populations or relative populations in the spreadsheets at the end of this homework, analogously to 2.4, 3.4, and 4.6.

1. For each of the following sequences, identify it as arithmetic, geometric, or neither. If arithmetic, find the common difference d; if geometric, find the common ratio r.

a. 2, 5, 8, 11, ...
b. 2, 4, 18, 32, ...
c. 3, 6, 12, 24, ...
d. 0, 1, 3, 6, 10, ...
e. 1, 1, 2, 3, 5, 8, 13, ...

2. A population of scumslugs increases by 5 every day. If there are 12 today, and none die, how many will there be 30 days from now?

3. Another population of scumslugs decides to share the reproduction chores, so that the population is multiplied by four every day. Again starting with 12 today, how many will there be 10 days from now?

4. For each of the following, identify the description of the population growth as

(i) linear, exponential, or logistic; and

(ii) explicit or recursive.

If linear, get the common difference d. If exponential, get the common ratio r.

a. $p_N = 3p_{N-1}(1 - p_{N-1})$

b. $P_N = 10(3^N)$

c. $P_N = 3 + P_{N-1}$

d. $P_N = 10 + 3N$

e. $P_N = 3P_{N-1}$

f. Population doubles every day

g. Population increases by 5 every hour

h. Add 7 to the population every minute

i. Multiply population by 1.5 every month.

5. A population follows a linear growth model, with P_0, the initial population today, equal to 8, and P_{10}, the population after 10 days, equal to 68.

a. What is the common difference d?

b. Find the population P_{54}, the population after 54 days.

c. Find the explicit description of $P_N \equiv$ the population after N days.

6. A population P_N, the population N years from now, grows linearly, with $P_{10} = 64, P_{15} = 84$.

a. By how much does the population increase each year?

b. What is the initial population P_0, the population now?

c. Find the explicit description $P_N \equiv$ the population N years from now.

7. If a population P_N grows linearly, with $P_{402} = 5,124$ and $P_{405} = 5,136$, what is P_{411}? (NOTE: you don't need to get P_0)

8. If a population P_N grows exponentially, with initial population $P_0 = 40, P_1 = 160$, find

a. the common ratio r;

b. P_2;

c. P_{10}; and

d. an explicit description P_N of the population.

9. Suppose a relative population p_N grows according to the logistic growth model, with growth parameter $r = 3$, and $p_0 = 0.4$. Find p_1, p_2, p_3, and p_4.

10. Suppose a relative population p_N grows logistically, with $p_0 = 0.00023$ and growth parameter $r = 1.5$. Describe the long-term behaviour of p_N.

11. Suppose a population P_N has $P_{97} = 100$ and $P_{98} = 110$. Find P_{99}, if

a. the population grows linearly;

b. the population grows exponentially.

(NOTE: you do not need to get P_0)

12. Suppose the sequence $p_N, N = 0, 1, 2, ...$, satisfies the recursive relation

$$p_N = 2.4p_{N-1}(1 - p_{N-1}), \quad N = 1, 2, 3, ...$$

What does p_N converge to, as N goes to infinity?

13. A geometric sequence has initial term $P_0 = 5$ and common ratio $r = 1.6$.

(a) Find P_1.

(b) Find P_2.

(c) Find P_{17}.

(d) Give an explicit description of P_N (N arbitrary).

14. A relative population p_N satisfies the logistic growth model, with initial relative population $p_0 = 0.053$ and growth parameter $r = 2.2$. What is the long-term behaviour of p_N?

15. Rewrite

$$P_0 = 5, P_N = 3 \times P_{N-1}, \quad N = 1, 2, 3, \ldots$$

as an explicit description.

16. Rewrite

$$P_N = 7 + 2N$$

as a recursive description.

LINEAR GROWTH
add 5 every day
(5 = d = common difference)
population of 12 today
($P_0 = 12$)

	RECURSIVE	EXPLICIT
N	$P_N = P_{N-1} + 5$: Add 5 to previous day	$P_N = 12 + 5N$: 12 plus 5x(number of days)
0	12	12
	(= P_0)	($P_0 = 12 + 5x0$)
1	17 = 12 + 5	17 = 12 + 5x1
	($P_1 = P_0 + 5$)	($P_1 = 12 + 5x1$)
2	22 = 17 + 5	22 = 12 + 5x2
	($P_2 = P_1 + 5$)	($P_2 = 12 + 5x2$)
3		
	($P_3 = P_2 + 5$)	($P_3 = 12 + 5x3$)
4		
	($P_4 = P_3 + 5$)	($P_4 = 12 + 5x4$)
5		
	($P_5 = P_4 + 5$)	($P_5 = 12 + 5x5$)
6		
	($P_6 = P_5 + 5$)	($P_6 = 12 + 5x6$)
.	....	....
.	....	....
10	....	
		($P_{10} = 12 + 5x10$)
.	....	
.	....	
50	....	

EXPONENTIAL GROWTH
multiply by 2 every day
($2 = r =$ common ratio)
population of 3 today
($P_0 = 3$)

	RECURSIVE	EXPLICIT
N	$P_N = 2 \times P_{N-1}$:	$P_N = 3 \times (2^N)$:
	multiply previous day by 2	3 times (2 raised to Nth power)
0	3	$3 = 3 \times 1$
	($= P_0$)	($P_0 = 3 \times (2^0)$)
1	$6 = 2 \times 3$	$6 = 3 \times 2$
	($P_1 = 2 \times P_0$)	($P_1 = 3 \times (2^1)$)
2	$12 = 2 \times 6$	$12 = 3 \times 4$
	($P_2 = 2 \times P_1$)	($P_2 = 3 \times (2^2)$)
3		
	($P_3 = 2 \times P_2$)	($P_3 = 3 \times (2^3)$)
4		
	($P_4 = 2 \times P_3$)	($P_4 = 3 \times (2^4)$)
5		
	($P_5 = 2 \times P_4$)	($P_5 = 3 \times (2^5)$)
6		
	($P_6 = 2 \times P_5$)	($P_6 = 3 \times (2^6)$)
.	....	....
.	....	....
10	....	
		($P_{10} = 3 \times (2^{10})$)
.	....	
.	....	
20	....	

LOGISTIC GROWTH I:

$p_N = rp_{N-1}(1 - p_{N-1})$

rounded to four decimal places

N	p_N, r = 2	p_N, r = 2	p_N, r = 2	p_N, r = 2.5	p_N, r = 2.5
0	0.6000	0.0100	0.5000	0.6000	0.0100
	(= p_0)	(= p_0)	(= p_0)	(= p_0)	(= p_0)
1					
	($p_1 = 2p_0(1-p_0)$)			($p_1 = (2.5)p_0(1-p_0)$)	
2					
	($p_2=2p_1(1-p_1)$)			($p_2=(2.5)p_1(1-p_1)$)	
3					
	($p_3 = 2p_2(1-p_2)$)			($p_3=(2.5)p_2(1-p_2)$)	
4					
5					
6					
7					
8					
9					
10					
11					
12					
13					
14					
15					

NOTE: $(1 - (1/2)) = 0.5$ $(1 - (1/2.5)) = 0.6$

SOME HOMEWORK HINTS

0. See 2.4, 3.4, and 4.6.

1. To check for being arithmetic, look at the differences between consecutive terms; for being geometric, look at the ratios of consecutive terms.

2. "Increases by 5" means linear growth with common difference 5.

3. "Multiplied by four" means exponential growth with common ratio 4.

5. a. $68 = P_{10} = P_0 + d \times 10 = 8 + 10d$.

6. a. What's asked for is also known as the common difference d. To get from P_{10} to P_{15} requires adding d five times; thus

$$84 = P_{15} = P_{10} + d \times 5 = 64 + 5d.$$

b. We can go backwards from P_{10}, using d from a, but *subtracting*, since we *decrease* by d every time we go back a year:

$$P_0 = P_{10} - 10d.$$

Or, we could set up

$$64 = P_{10} = P_0 + d \times 10 = P_0 + 10d.$$

7. Moving ahead three generations, from P_{402} to P_{405}, increases the population by $(P_{405} - P_{402}) = (5,136 - 5,124) = 12$, thus we can figure the increase in moving ahead six generations, from P_{405} to P_{411}, must be $2 \times 12 = 24$.

8. a. r is the ratio of consecutive terms, in particular, the ratio of P_1 to P_0.
b. Use r from part a: $P_2 = rP_1$. Or use $P_2 = P_0(r^2)$.

9. We only have the recursive definition.

10. Don't calculate p_N, only use our $(1 - \frac{1}{r})$ formula for long-term behavior.

11. a. Add the common difference to P_{98}.
b. Multiply P_{98} by the common ratio.

12. This is asking about long-term behavior only.

14. See the hint for no. 10.

15. You need the common ratio 3.

16. You need the common difference 2.

HOMEWORK ANSWERS

0. See the filled-in spreadsheets after the other homework answers.

1. a. ARITHMETIC, $d = 3$.
b. NEITHER
c. GEOMETRIC, $r = 2$.
d. NEITHER
e. NEITHER

2. $12 + 5 \times 30 = 162$.

3. $12(4^{10}) = 12,582,912$.

4. a. logistic, recursive.

b. exponential, explicit; $r = 3$.

c. linear, recursive; $d = 3$.

d. linear, explicit; $d = 3$.

e. exponential, recursive; $r = 3$.

f. exponential, recursive; $r = 2$.

g. linear, recursive; $d = 5$.

h. linear, recursive; $d = 7$.

i. exponential, recursive; $r = 1.5$.

5. a. $\frac{68-8}{10} = 6$.

b. $8 + 6 \times 54 = 332$.

c. $P_N = 8 + 6N$.

6. a. $\frac{84-64}{15-10} = 4$.

b. $64 - 4 \times 10 = 24$.

c. $P_N = 24 + 4N$.

7. $5,136 + 2 \times (5,136 - 5,124) = 5,160$.

8. a. $\frac{160}{40} = 4$.

b. $160 \times 4 = 640$.

c. $40(4^{10}) = 41,943,040$.

d. $P_N = 40(4^N)$.

9. $p_1 = 3 \times 0.4 \times (1 - 0.4) = 0.72; p_2 = 3 \times 0.72 \times (1 - 0.72) = 0.6048; p_3 = 3 \times 0.6048 \times (1 - 0.6048) \sim 0.7171; p_4 = 3 \times p_3 \times (1 - p_3) \sim 0.6807$.

10. converges (that is, gets arbitrarily close to) $(1 - \frac{1}{1.5}) = \frac{1}{3}$ as N gets large.

11. a. $d = 10$, so $P_{99} = 110 + 10 = 120$.

b. $r = 1.1$, so $P_{99} = 110 \times 1.1 = 121$.

12. $(1 - \frac{1}{2.4}) = \frac{7}{12}$.

13. (a) $5 \times 1.6 = 8$.

(b) $8 \times 1.6 = 12.8$.

(c) $5(1.6^{17}) \sim 14,757$.

(d) $P_N = 5(1.6^N)$.

14. converges (gets arbitrarily close to) $(1 - \frac{1}{2.2}) = \frac{6}{11}$ as N gets large.

15. $P_N = 5 \times 3^N, \;\; N = 0, 1, 2, \ldots.$

16. $P_N = P_{N-1} + 2, \;\; N = 1, 2, 3, \ldots.$

p. 34

LINEAR GROWTH
add 5 every day
(5 = d = common difference)
population of 12 today
($P_0 = 12$)

	RECURSIVE	EXPLICIT
N	$P_N = P_{N-1} + 5$: Add 5 to previous day	$P_N = 12 + 5N$: 12 plus 5x(number of days)
0	12	12
	(= P_0)	($P_0 = 12 + 5x0$)
1	17 = 12 + 5	17 = 12 + 5x1
	($P_1 = P_0 + 5$)	($P_1 = 12 + 5x1$)
2	22 = 17 + 5	22 = 12 + 5x2
	($P_2 = P_1 + 5$)	($P_2 = 12 + 5x2$)
3	27 = 22 + 5	27 = 12 + 15
	($P_3 = P_2 + 5$)	($P_3 = 12 + 5x3$)
4	32 = 27 + 5	32 = 12 + 20
	($P_4 = P_3 + 5$)	($P_4 = 12 + 5x4$)
5	37 = 32 + 5	37 = 12 + 25
	($P_5 = P_4 + 5$)	($P_5 = 12 + 5x5$)
6	42 = 37 + 5	42 = 12 + 30
	($P_6 = P_5 + 5$)	($P_6 = 12 + 5x6$)
.	……………………	………………………………
.	……………………	………………………………………
10	………………………	62 = 12 + 50
		($P_{10} = 12 + 5x10$)
.	………………………	
.	………………………	
50	………………………	262 = 12 + 5x50

EXPONENTIAL GROWTH
multiply by 2 every day
(2 = r = common ratio)
population of 3 today
(P_0 = 3)

	RECURSIVE	EXPLICIT
N	P_N = 2xP_{N-1}:	P_N = 3x(2^N):
	multiply previous day by 2	3 times (2 raised to Nth power)
0	3	3 = 3x1
	(= P_0)	(P_0 = 3x (2^0))
1	6 = 2x3	6 = 3x2
	(P_1 = 2x P_0)	(P_1 = 3x (2^1))
2	12 = 2x6	12 = 3x4
	(P_2 = 2 x P_1)	(P_2 = 3 x (2^2))
3	24 = 2x12	24 = 3x8
	(P_3 = 2 x P_2)	(P_3 = 3 x (2^3))
4	48 = 2x24	48 = 3x16
	(P_4 = 2 x P_3)	(P_4 = 3 x (2^4))
5	96 = 2x48	96 = 3x32
	(P_5 = 2x P_4)	(P_5 = 3 x (2^5))
6	192 = 2x96	192 = 3x64
	(P_6 = 2 x P_5)	(P_6 = 3 x (2^6))
.	....	....
.	....	....
10	....	3,072 = 3x1,024
		(P_{10} = 3 x (2^{10}))
.	....	
.	....	
20	....	3,145,728 = 3x(2^{20})

LOGISTIC GROWTH I:

p_N = rp_{N-1}(1 - p_{N-1})

round to four decimal places

N	p_N, r = 2	p_N, r = 2	p_N, r = 2	p_N, r = 2.5	p_N, r = 2.5
0	0.6000	0.0100	0.5000	0.6000	0.0100
	(= p_0)	(= p_0)	(= p_0)	(= p_0)	(= p_0)
1	0.4800	0.0198	0.5000	0.6000	0.0248
	(p_1 = 2p_0(1-p_0))			(p_1 = (2.5)p_0(1-p_0))	
2	0.4992	0.0388	0.5000	0.6000	0.0603
	(p_2=2p_1(1-p_1))			(p_2=(2.5)p_1(1-p_1))	
3	0.5000	0.0746	0.5000	0.6000	0.1418
	(p_3 = 2p_2(1-p_2))			(p_3=(2.5)p_2(1-p_2))	
4	0.5000	0.1381	0.5000	0.6000	0.3042
5	0.5000	0.2381	0.5000	0.6000	0.5291
6	0.5000	0.3628	0.5000	0.6000	0.6229
7	0.5000	0,4623	0.5000	0.6000	0.5873
8	0.5000	0.4972	0.5000	0.6000	0.6060
9	0.5000	0.5000	0.5000	0.6000	0.5969
10	0.5000	0.5000	0.5000	0.6000	0.6015
11	0.5000	0.5000	0.5000	0.6000	0.5992
12	0.5000	0.5000	0.5000	0.6000	0.6004
13	0.5000	0.5000	0.5000	0.6000	0.5998
14	0.5000	0.5000	0.5000	0.6000	0.6001
15	0.5000	0.5000	0.5000	0.6000	0.6000

p_N converges to 0.5, as $N \to \infty$, since

(1 - (1/2)) = 0.5

p_N converges to 0.6, as $N \to \infty$, since

(1 - (1/2.5)) = 0.6

REFERENCES

1. R. deLaubenfels, "Fibonacci Numbers and the Golden Ratio Magnification," https://teacherscholarinstitute.com/MathMagnificationsReadyToUse.html.

2. J. Saxon, "Algebra 1. An Incremental Development," Second Edition, Saxon Publishers, Inc., 1990.