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Pythagorean Theorem and More
MATHematics MAGnification™

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PYTHAGOREAN THEOREM and more MAGNIFICATION

This is one of a series of very short books on math, statistics, and physics called "Math Magnifications." The "magnification" refers to focusing on a particular topic that is pivotal in or emblematic of mathematics.

OUTLINE

Arguably the greatest successes in math have involved relating geometry (pictures) to algebra (calculations). Geometry has intuition while algebra provides precision.

This Magnification presents three examples of the foregoing, more specifically, examples of popular algebraic formulas that may be demonstrated with drawings of rectangles and triangles.

The first formula is the **Pythagorean theorem**

$$a^2 + b^2 = c^2,$$

where a and b are the lengths of the legs of a right triangle and c is the length of the hypotenuse (see Section I).

The second formula is the square of a sum

$$(a+b)^2 = a^2 + b^2 + 2ab,$$

for any numbers a and b (see Section II).

The third formula is the **distributive law**

$$a(b+c) = ab + ac,$$

for any numbers a, b, and c (see Section III).

Unlike our drawings in Section I, the drawings in Sections II and III do not qualify as proofs, unless a, b, and c are positive. They may still be considered illustrative; if nothing else, the drawings in Sections II and III could be used to derive the corresponding formulas, if one had forgotten said formulas.

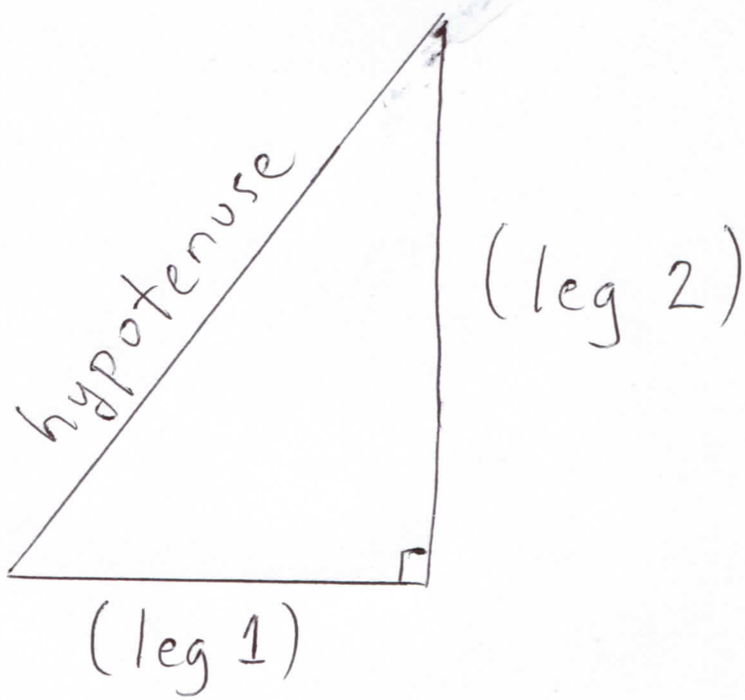
See [1], Chapter 10, for interesting consequences of the Pythagorean theorem, including the drawings in Sections II and III.

An example of a popular number that may be defined geometrically or algebraically is the *golden ratio*; see [1] and [2].

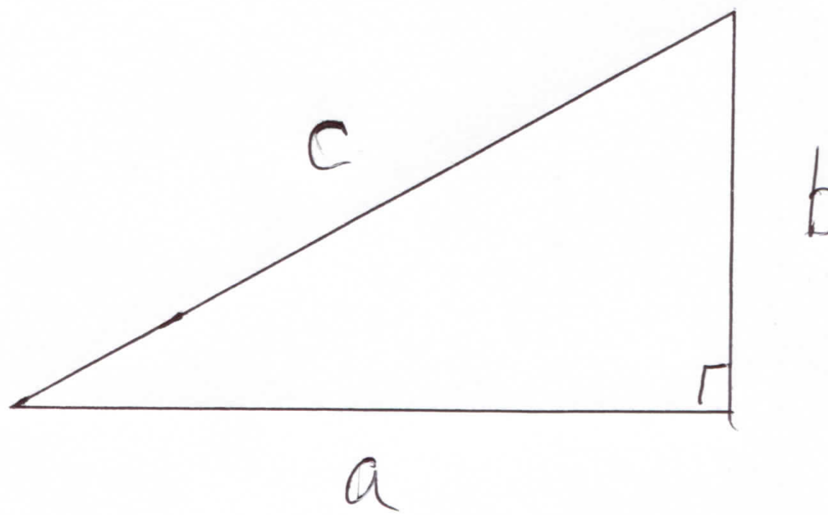
Prerequisites for this Magnification are square roots, the formula for area of a rectangle, the definition of a right angle and a right triangle, and the use of letters for general numbers, as in the formulas above; reference [3] is more than sufficient.

SECTION I: Pythagorean Theorem

In a right triangle, the side opposite the right angle is the hypotenuse; the other sides are legs:

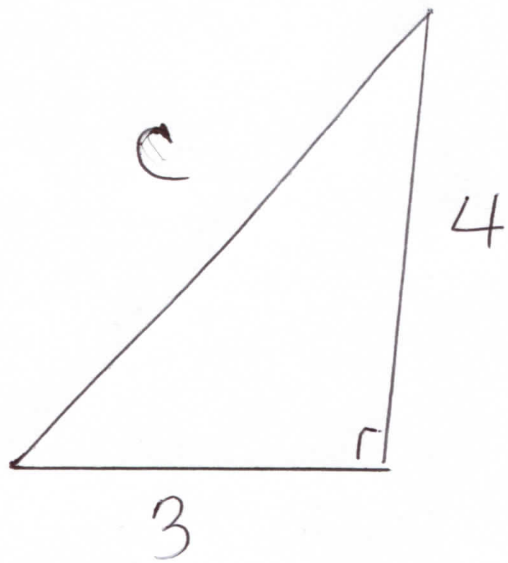


Denote by c the length of the hypotenuse, by a the length of (leg 1), and by b the length of (leg 2):

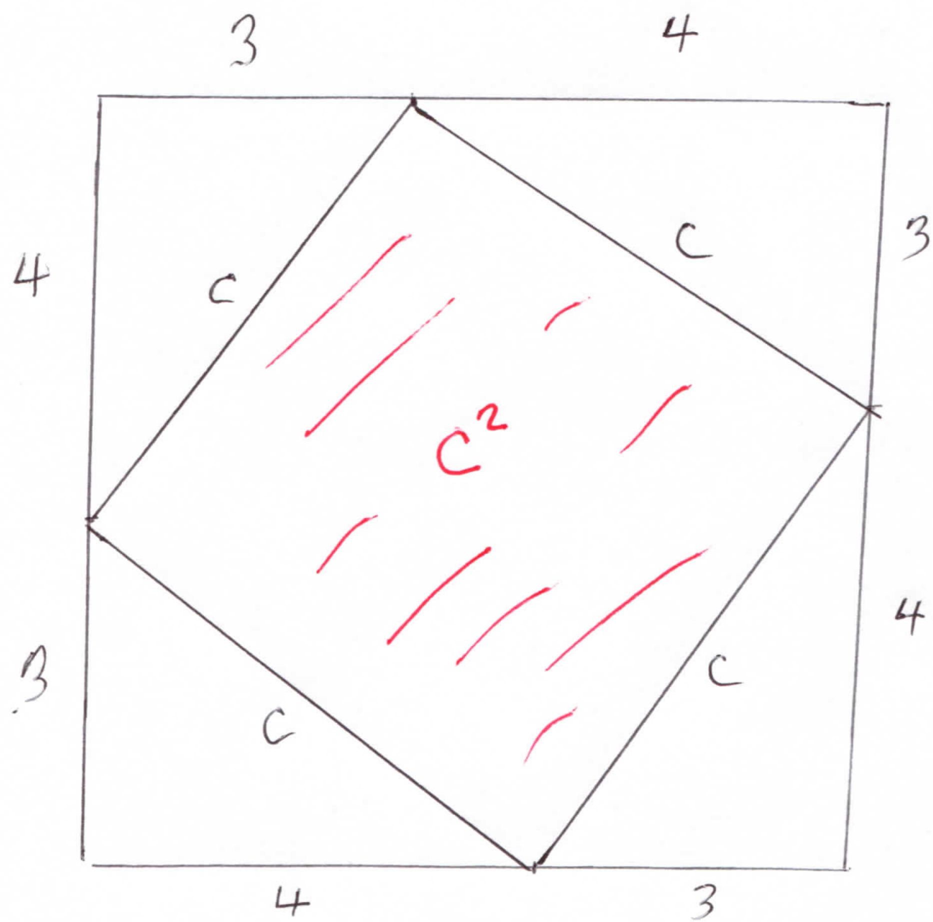


Our goal is to get c in terms of a and b.

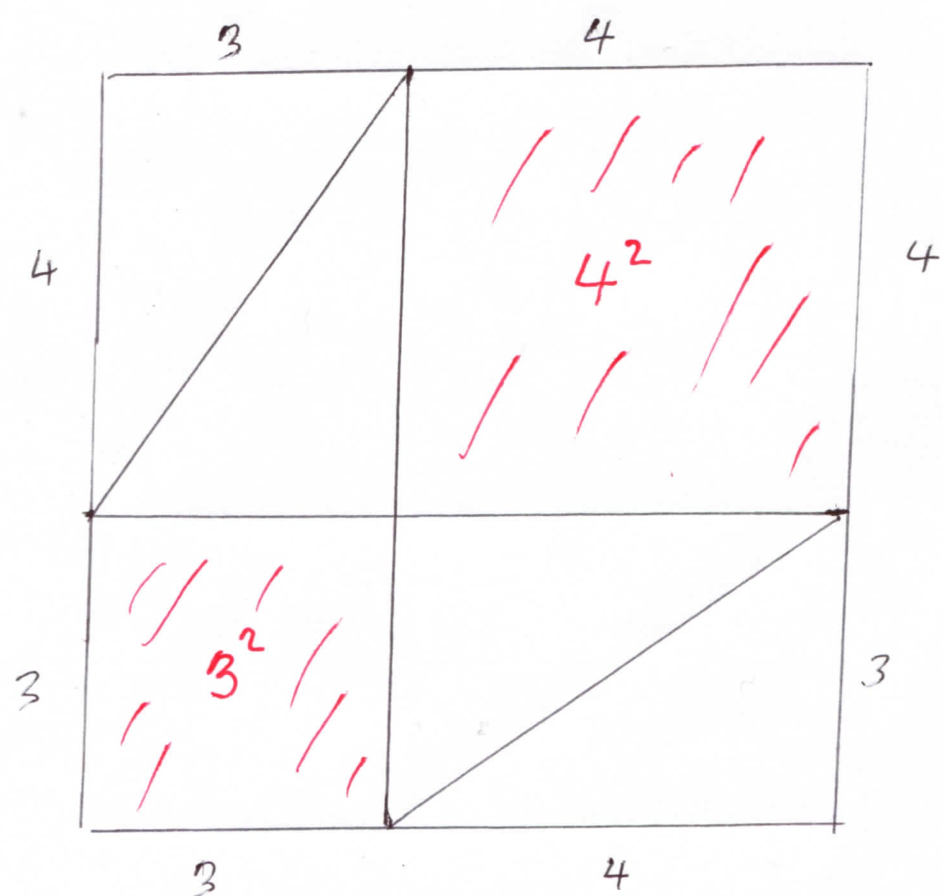
Let's begin with a and b known; we'd like c in the following:



Draw (not necessarily to scale) two squares of side 7 labelled (i) and (ii) respectively:



(i)



(ii)

From (i), adding up areas,

$$7^2 = c^2 + 4 \times (\text{area of triangle});$$

similarly, from (ii),

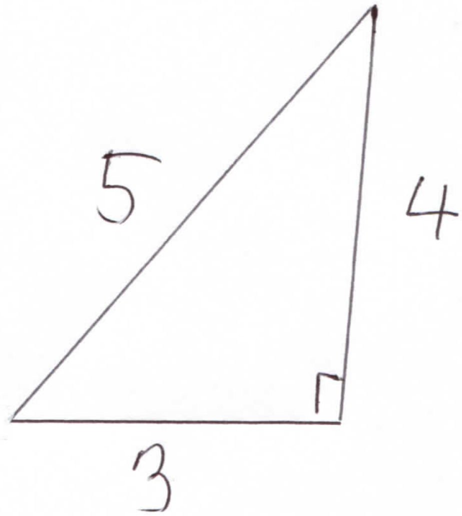
$$7^2 = 3^2 + 4^2 + 4 \times (\text{area of triangle}).$$

Subtracting one equation from the other:

$$\begin{array}{rl} & 7^2 = c^2 + 4 \times (\text{area of triangle}) \\ - & \left(7^2 = 3^2 + 4^2 + 4 \times (\text{area of triangle})\right) \\ \hline \end{array}$$

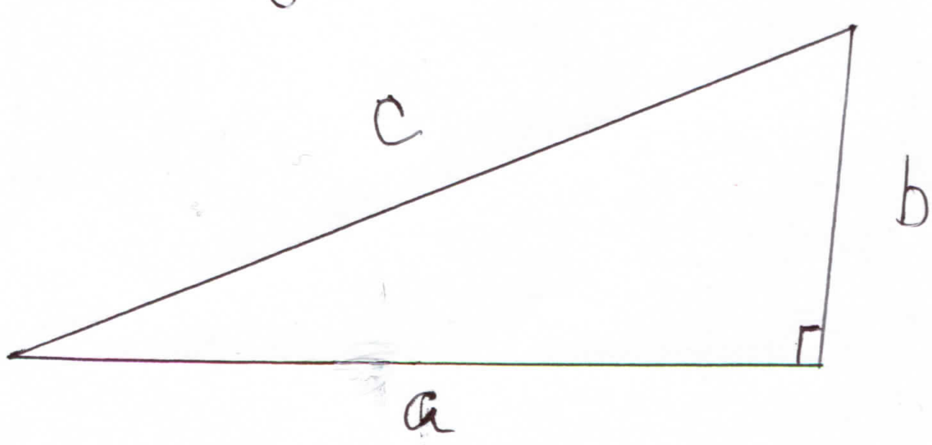
$$\rightarrow 0 = c^2 - (3^2 + 4^2) \rightarrow$$

$$c^2 = (3^2 + 4^2) = 25 \rightarrow \boxed{c = 5}$$

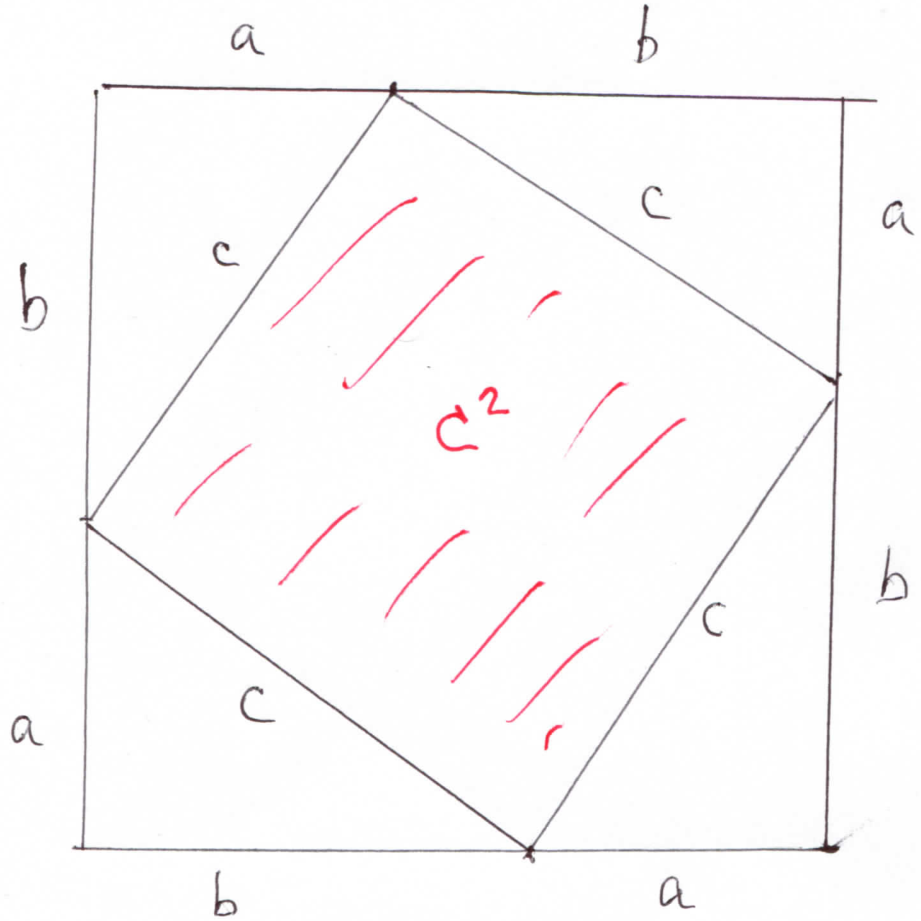


(3, 4, 5) is an example of a <u>Pythagorean triple</u>, meaning integers that form the sides of a right triangle.

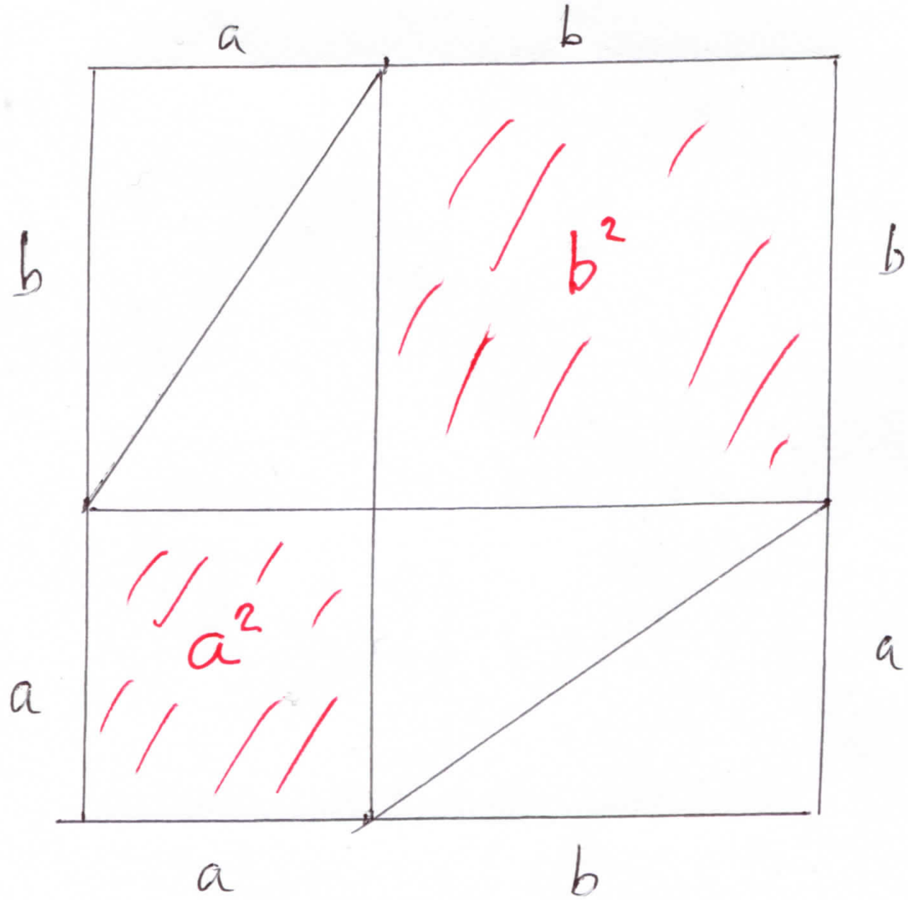
Now let's look at an arbitrary right triangle



DRAW SQUARES!



(i)



(ii)

As before, add up areas inside each square of side $(a+b)$:

(i) $(a+b)^2 = c^2 + 4 \times \left(\begin{array}{c}\text{area of}\\ \text{triangle}\end{array}\right)$

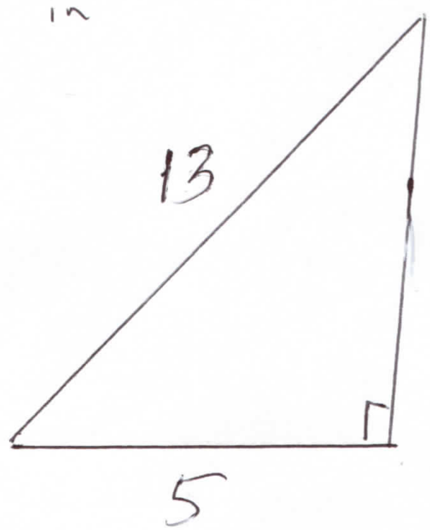
(ii) $(a+b)^2 = a^2 + b^2 + 4 \times \left(\begin{array}{c}\text{area of}\\ \text{triangle}\end{array}\right)$,

and again subtracting (ii) from (i) gives us

$c^2 = a^2 + b^2$ $\left(\begin{array}{c}\text{Pythagorean}\\ \text{theorem}\end{array}\right)$

Examples

(1) Find the missing side length in



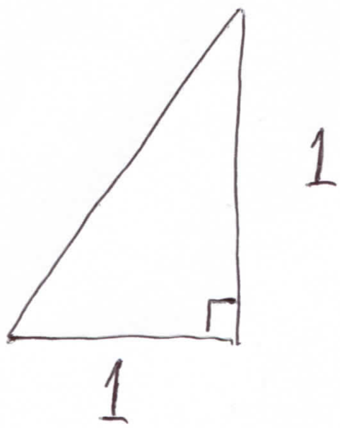
ANSWER Call the missing length b. Then

$5^2 + b^2 = 13^2 \rightarrow b^2 = 13^2 - 5^2$

$= 144 \rightarrow b = \boxed{12}$

((5, 12, 13) is another Pythagorean triple)

(2) Find the length of the hypotenuse in



ANSWER Denote by c the desired length. By the Pythagorean theorem,

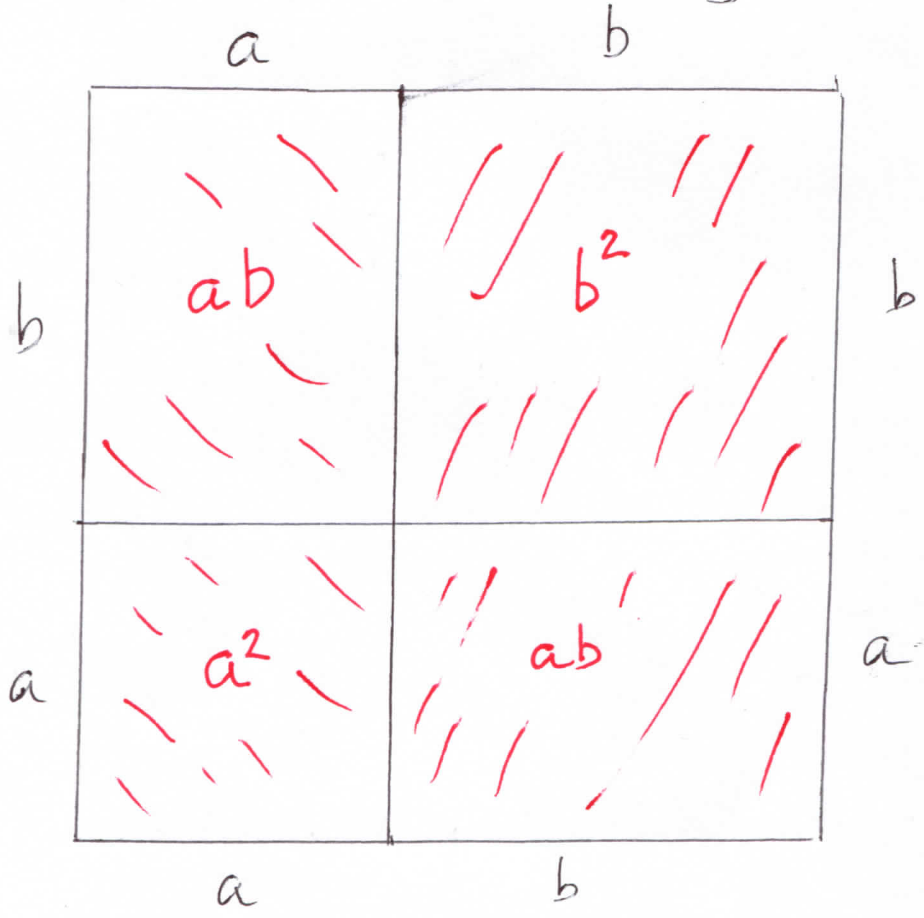
$$c^2 = 1^2 + 1^2 = 2,$$

so that $c = \boxed{\sqrt{2}}$

It can be shown that $\sqrt{2}$ is irrational; that is, it cannot be written as a ratio of integers. This was very upsetting to the classical Greeks, since rational numbers are not hard to define and understand, while arbitrary real numbers are much more mysterious. Real numbers were not defined until the 19th century.

SECTION II: Square of a Sum

Given positive numbers a, b, to deal with $(a+b)^2$, draw squares and rectangles very similar to the last drawing (ii) on p. 8:

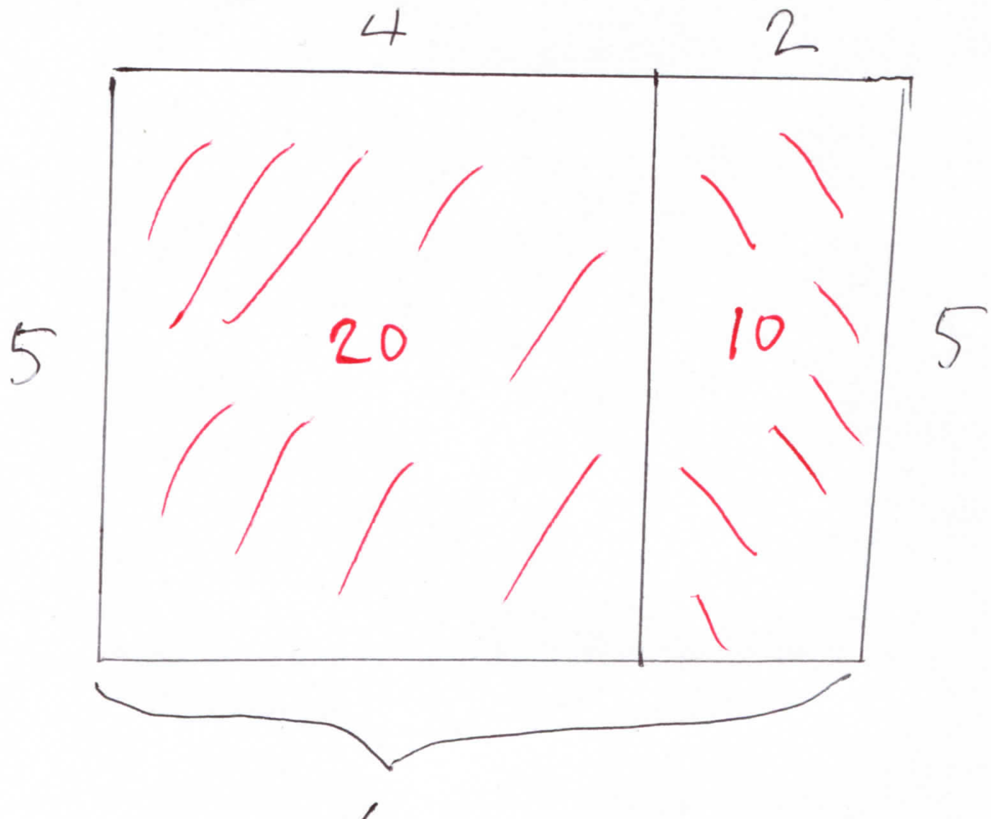


Write the area of the square of side $(a+b)$ as the sum of the rectangle areas inside:

$$(a+b)^2 = a^2 + b^2 + 2ab.$$

This formula also follows from the Distributive Law, in the next section.

First, let's use areas of rectangles to demonstrate the Distributive Law for specific numbers 5, 4, and 2:

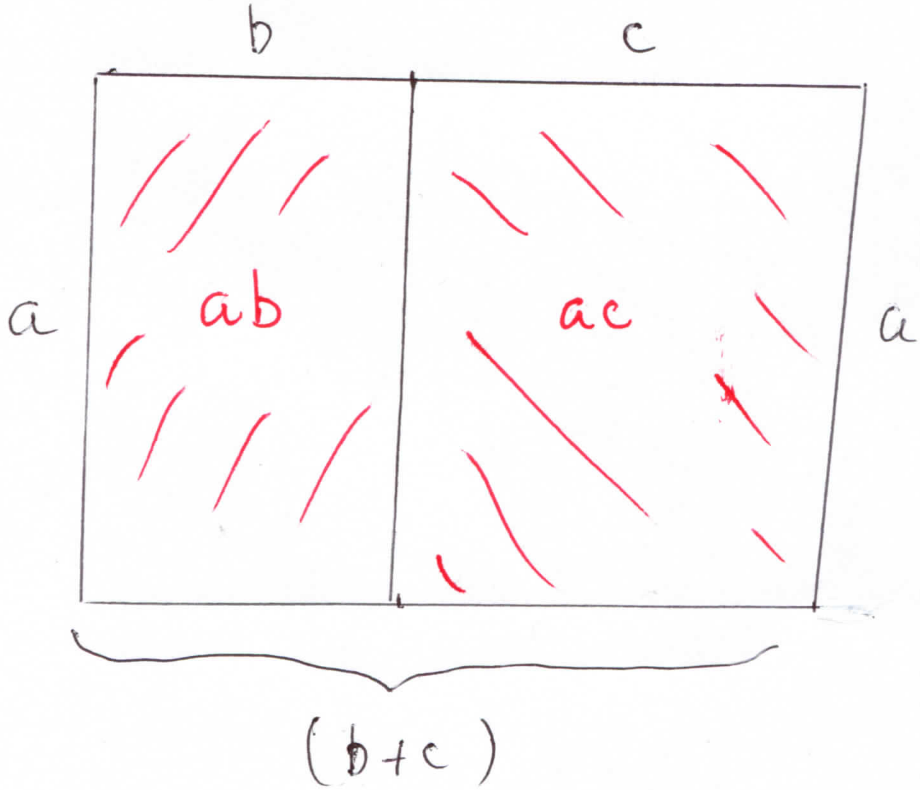


$5 \times (4+2) = 5 \times 6 = 30$

= total area = sum of areas of smaller rectangles

$= 20 + 10 = (5 \times 4) + (5 \times 2)$.

For arbitrary positive numbers a, b, c, draw rectangles and add areas analogously:



$a \times (b+c) =$

total area = sum of areas

of smaller rectangles

$= (ab + ac)$;

this is the Distributive Law.

<u>Example</u> We should mention

that the formula from Section II

follows from the Distributive

Law:

$$(a+b)^2 = (a+b) \times (a+b)$$

$$= (a+b)\,a + (a+b)\,b$$

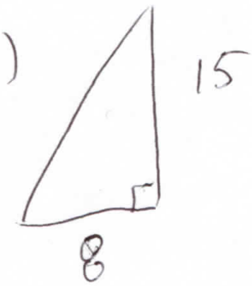
$$= a(a+b) + b(a+b)$$

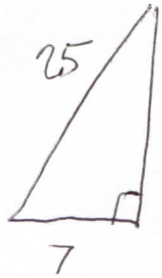
$$= (a^2 + ab) + (ba + b^2)$$

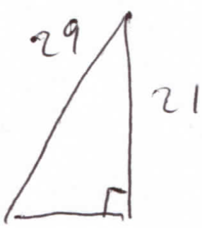
$$= a^2 + b^2 + 2ab.$$

HOMEWORK

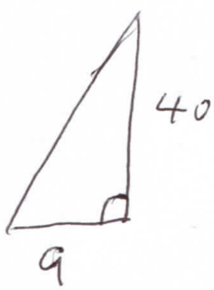
#1 In each of the right triangles drawn below, find the missing side length.

(a) 

(b) 

(c) 

(d) 

(e) 

(f) 

(g) 

p. 20

2. Use the results of Sections II and III to get an expression for

$$(a+b)^3$$

analogous to the result of Section II.

p. 21

HOMEWORK ANSWERS

1. Use the Pythagorean theorem.

(a) 17

(b) 24

(c) 20

(d) 35

(e) 41

(f) $\sqrt{51}$

(g) $\sqrt{45} = 3\sqrt{5}$

2. $(a+b)^3 = (a+b)(a+b)^2 = (a+b)(a^2+b^2+2ab) = a(a^2+b^2+2ab) + b(a^2+b^2+2ab) = (a^3+ab^2+2a^2b) + (ba^2+b^3+2ab^2) = a^3+b^3+3ab^2+3a^2b$.

p. 22

REFERENCES

1. W. S. Anglin and J. Lambek, "The Heritage of Thales," Springer, 1995.

2. R. deLaubenfels, "Fibonacci Numbers and the Golden Ratio," https://teacherscholarinstitute.com/MathMagnificationsReadyToUse.html.

3. J. Saxon, "Algebra 1. An Incremental Development," Second Edition, Saxon Publishers, Inc., 1990.