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# Statistics: Confidence Intervals
# MATHematics MAGnification™

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## STATISTICS: CONFIDENCE INTERVALS MAGNIFICATION

This is one of a series of very short books on math, statistics, and physics called "Math Magnifications." The "magnification" refers to focusing on a particular topic that is pivotal in or emblematic of mathematics.

## OUTLINE

Statistical inference comes primarily in two forms: confidence intervals and hypothesis testing. This magnification, after some general results and motivation, will talk about a special case of confidence intervals; a future magnification will talk about the same special case for hypothesis testing. Another future magnification will show how the same constructions work for most other popular confidence intervals and hypothesis tests.

This magnification will construct confidence intervals and upper and lower confidence bounds for the mean of a normal population, with standard deviation known. Some of the desirabilities of large samples will be exhibited.

Prerequisites for this magnification are algebra ([4] is more than sufficient); and [2] or other references, such as [3], that contain the basic terminology of statistical inference. A class in probability, such as [1] or (for those who have had calculus) [3] would be helpful, but is not necessary, since we will, in Chapter 2, give a crash course in normal random variables.

## 1. INTRODUCTION.

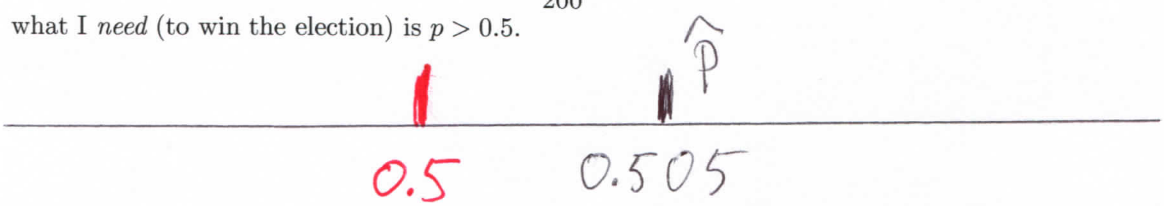
We introduced, in [2], the primary mission (with special terminology) of statistical inference: making an intelligent guess about an entire population by looking at a sample chosen from said population. More specifically, we want to estimate the value of a population parameter $\theta$ with a specified calculation from the data, denoted an *estimator* $\hat{\theta}$ of $\theta$.

For example, let's say I'm running for Dogcatcher of Columbus. I poll 200 residents of Columbus, and find that 101 of them will vote for me.

In the language of the penultimate paragraph, the last paragraph is described as follows. The population parameter is the proportion of the voting population of Columbus that will vote for me, denoted $p$, and the estimator is the proportion of a sample that will vote for me, denoted $\hat{p}$. For the sample I took,

$$\hat{p} = \frac{101}{200} = 0.505;$$

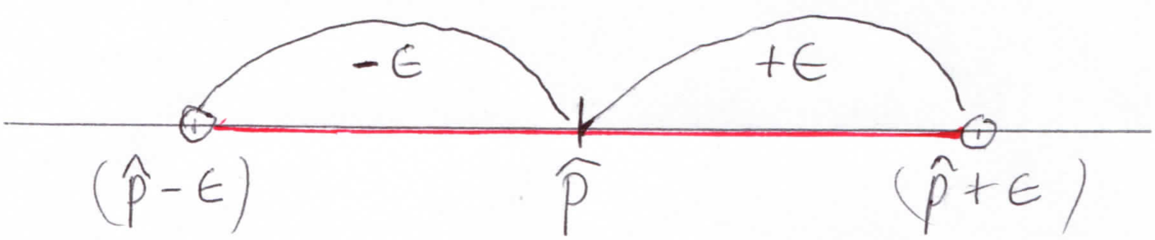
what I *need* (to win the election) is $p > 0.5$.



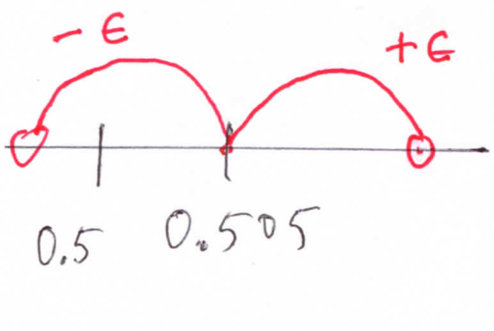

Will I win the election? If I think the answer is "yes," how confident am I? Should I spend money, based on an assumption of winning? For example, should I buy a special, expensive Dogcatcher Tuxedo prior to the election, so that it will arrive in time for post-election ceremonies?

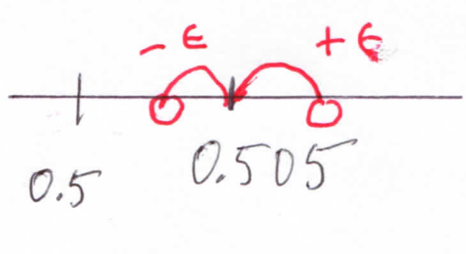
What's needed is a *margin of error*, meaning a positive number $\epsilon$ ("epsilon," standing for "error") that you add to and subtract from $\hat{p}$. The "error" in this case is the fact that the population is much bigger than the sample, thus the population proportion is not likely to be the same as the sample proportion.

The terminology here is $\hat{p} \pm \epsilon$, denoting all the real numbers between $(\hat{p} - \epsilon)$ and $(\hat{p} + \epsilon)$.

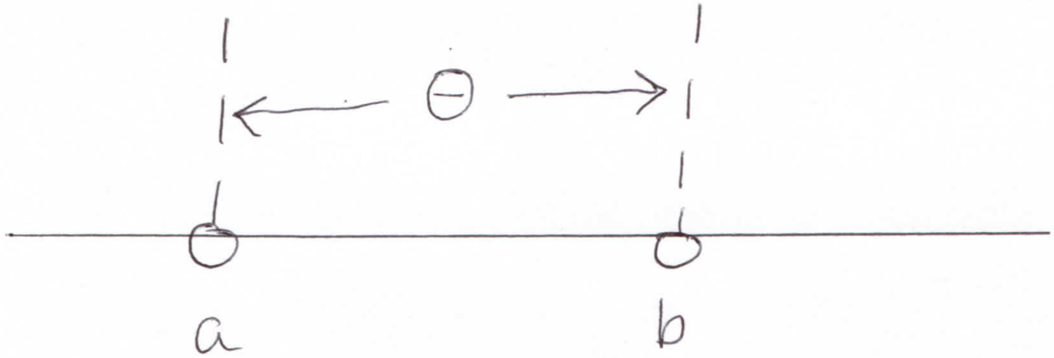



Notice that the size of the error $\epsilon$ can change the results of the election (assuming my interval is correct). In the first picture below, I might lose; in the second picture below, I'm guaranteed to win.



In general, rather than a single number (the *estimator*, denoted $\hat{\theta}$) to estimate a population parameter $\theta$, we prefer a *pair* of numbers $a, b$, along with some measure of confidence that $\theta$ is between them.




We have changed our goal from guessing "$\theta$ equals $\hat{\theta}$" to "$\theta$ is between $a$ and $b$." The set of numbers between $a$ and $b$ is a **confidence interval** for $\theta$, denoted

$$(a, b) \equiv \{\text{numbers } c \,|\, a < c < b\}.$$

This magnification will be restricted to the population parameter $\mu$, the population mean, for a normal population with population standard deviation $\sigma$ assumed to be known (see [2]). Messier constructions for other parameters use almost exactly the same strategy, thus we are focusing on this cleanest and simplest special case, to clarify the techniques without getting bogged down in algebra.

We should mention here the most popular estimator $\hat{\mu}$ for $\mu$, as discussed in [2]; see especially [2, Example 20].

**Definition 1.1.** The **sample mean** $\overline{X}$, from a random sample $X_1, X_2, \ldots, X_n$, is the random variable

$$\overline{X} \equiv \frac{1}{n}(X_1 + X_2 + \cdots + X_n);$$

$n$ is the **sample size.**

When $x_1, x_2, \ldots, x_n$ is a sequence of numbers,

$$\overline{x} \equiv \frac{1}{n}(x_1 + x_2 + \cdots + x_n)$$

is called the **mean** of the sequence. If the sequence arises from a random sample after measurements are made, $\overline{x}$ is (also) called the **sample mean.**

Terminology customs are important here: capital $X$ means random variables and lower-case $x$ means numbers.

## 2. NORMAL RANDOM VARIABLES.

See [1, Chapter VI] or [3, Section 4.3] for more about normal random variables.

The following definition, included for intellectual completeness (see [2, Definitions 9 and 12 and Examples 15(2)]), does not tell us how to calculate normal probabilities; said calculations are achieved with Theorem 2.7 and the $Z$ table at the end of this magnification, as in Examples 2.8.

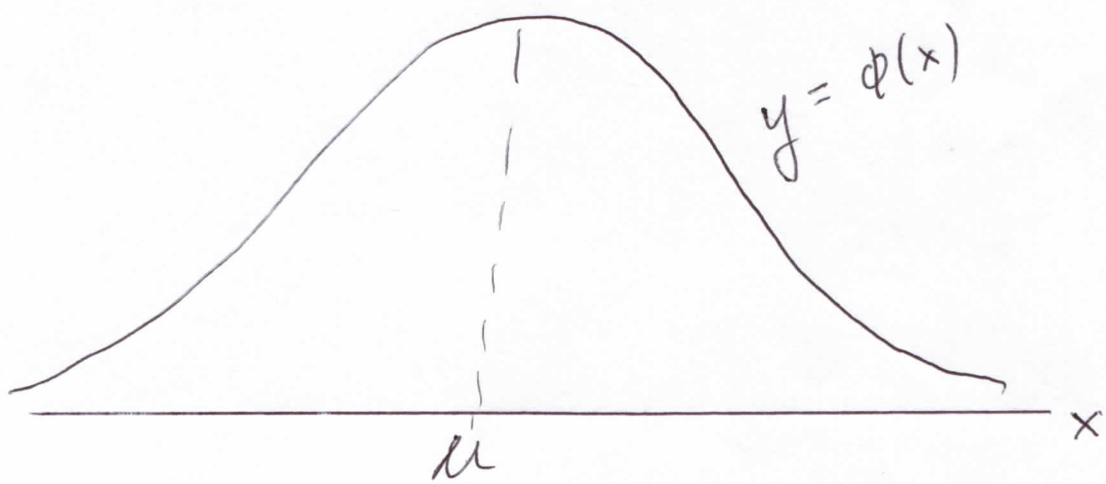
**Definition 2.1.** A **normal** random variable $X$ is a continuous random variable with probability density function

$$\phi(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

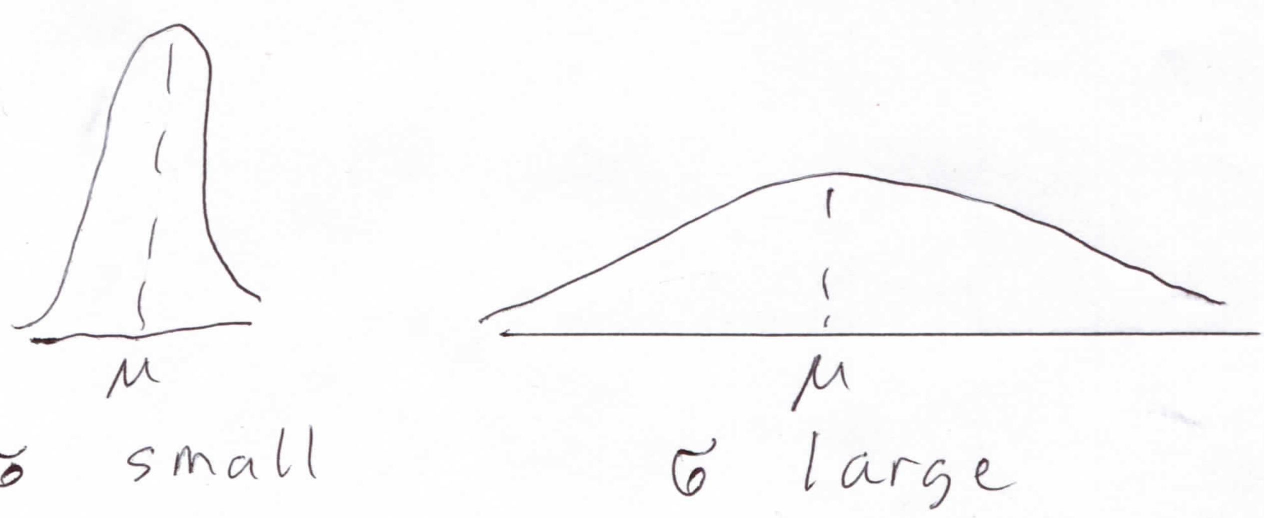
$\mu$ is the **mean** of the random variable, $\sigma$ is the **standard deviation**, and $e$ is a famous irrational number.

Equivalent terminology is "$X$ is normally distributed," or "$X$ has a normal distribution."

The graph of $\phi$ has the infamous **bell-shaped curve**, symmetric about the vertical line $x = \mu$; that is, the curve to the left of the vertical dotted line $x = \mu$ is the mirror image of the curve to the right of $x = \mu$.



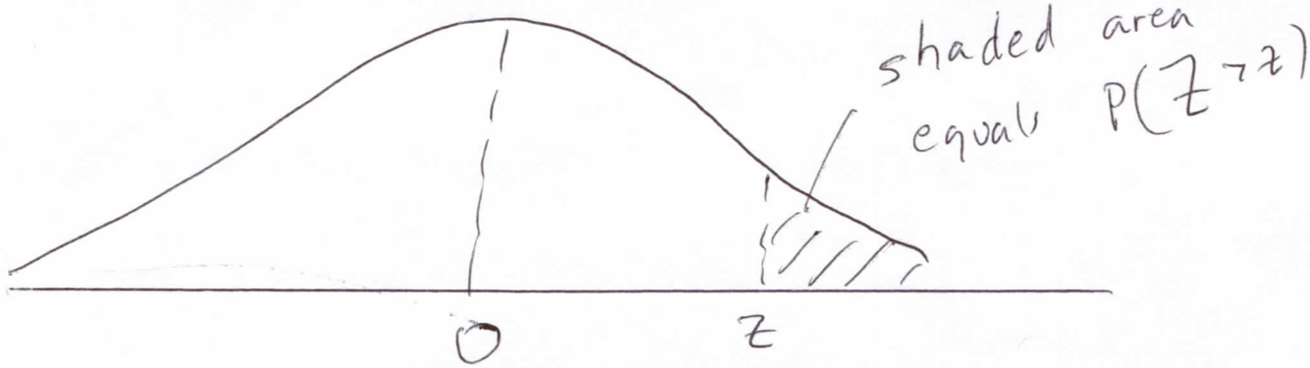

The standard deviation $\sigma$ measures how spread out $\phi$ (and hence the values of $X$) is:



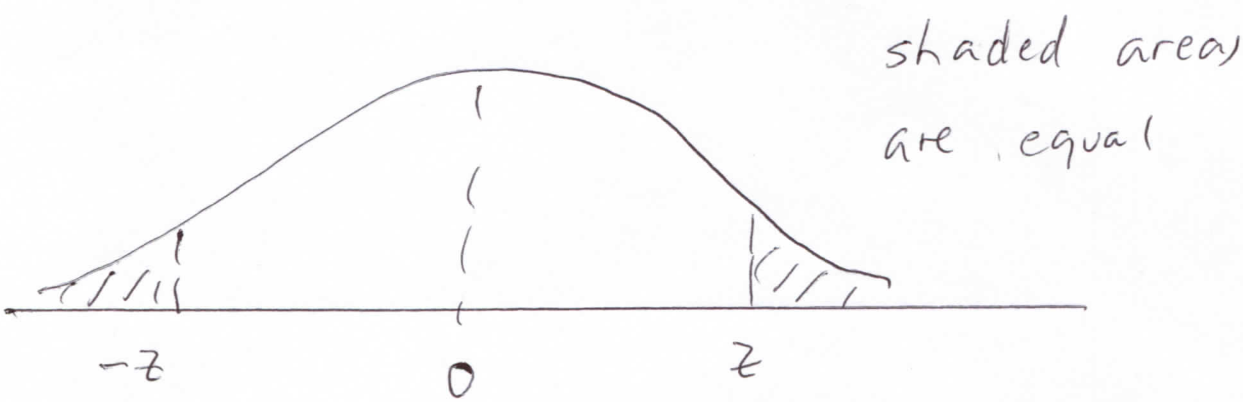
**Definition 2.2.** The **standard normal random variable**, denoted $Z$, is the normal random variable with $\mu = 0$ and $\sigma = 1$.

The good news about $Z$ is that tables of $Z$ probabilities have been compiled: For $a$ and $b$ a pair of numbers of absolute value less than or equal to three, rounded to two decimal places, tables (called *Z tables*) such as the one at the end of this magnification will give us (approximately) $P(a < Z < b)$, shorthand for the probability that $Z$ is between $a$ and $b$. See Examples 2.4.

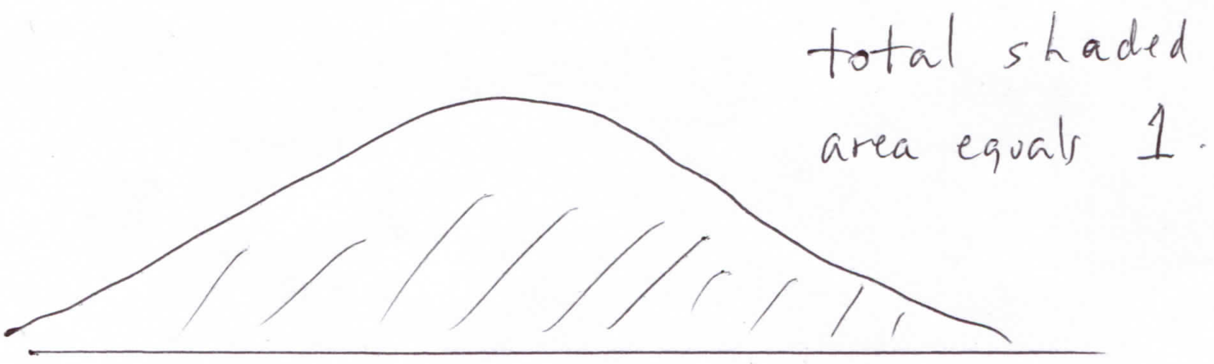
Here is the picture from our $Z$ table, where, for any real number $z$, $P(Z > z) = P(Z \geq z)$ denotes the probability that $Z$ is greater than the number $z$ and $P(Z < z) = P(Z \leq z)$ denotes the probability that $Z$ is less than the number $z$.




**Properties 2.3.** (1) (symmetry) For any real $z$, $P(Z > z) = P(Z < -z)$.




(2) (true for any random variable) $P(-\infty < Z < \infty) = 1$.



**Examples 2.4.** (a) Get $P(Z > 1.23)$.

(b) Get $P(Z < 1.23)$.

(c) Get $P(Z < -1.23)$.

(d) Get $P(Z > -1.23)$.

(e) Get $P(Z \geq -1.23)$.

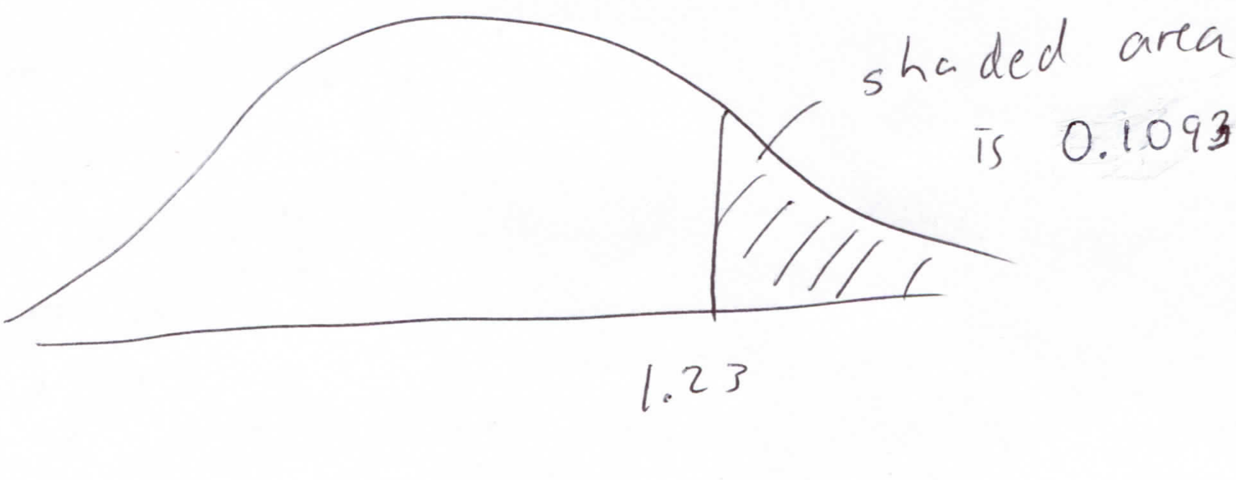
(f) Get $P(-1.23 < Z < 1.23)$.

(g) If $z_1 > z_2$, what (if anything) can be said about the relationship between $P(Z > z_1)$ and $P(Z > z_2)$?

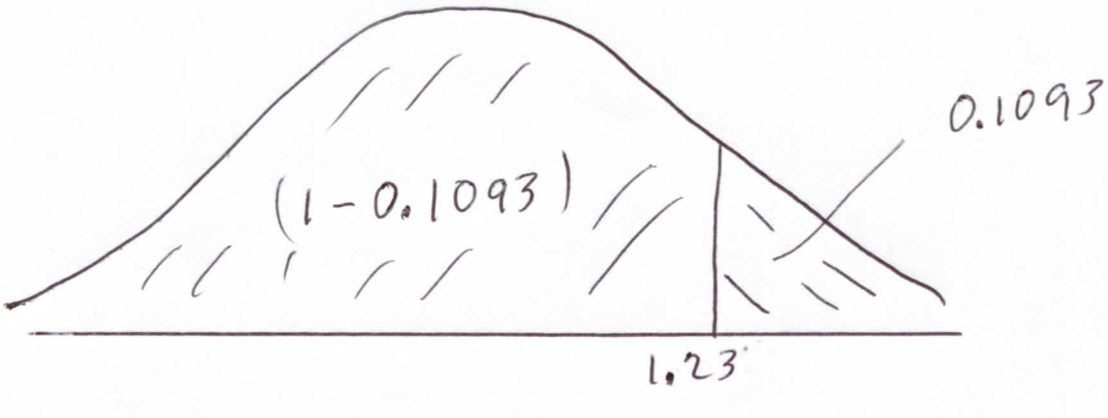
**Solutions.** (a) Writing 1.23 as $1.2 + 0.03$, we look, in the $Z$ table at the end of this magnification, for the entry in the 1.2 row and 0.03 column, as reproduced below:

| z | 0.03 |
|---|---|
| 1.2 | 0.1093 |

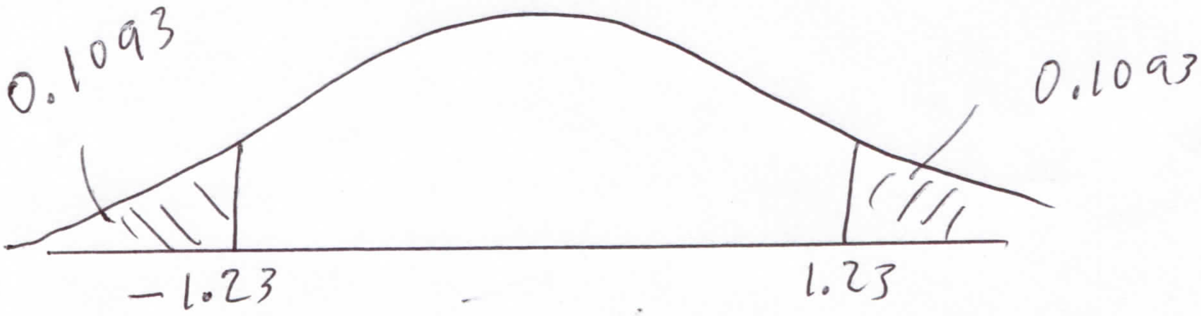
so that $P(Z > 1.23) = 0.1093$



(b) By Properties 2.3(2), $P(Z < 1.23) = 1 - P(Z \geq 1.23) = 1 - P(Z > 1.23) = 1 - 0.1093 = 0.8907$, from (a).




(c) By Properties 2.3(1), $P(Z < -1.23) = P(Z > 1.23) = 0.1093$, from (a).

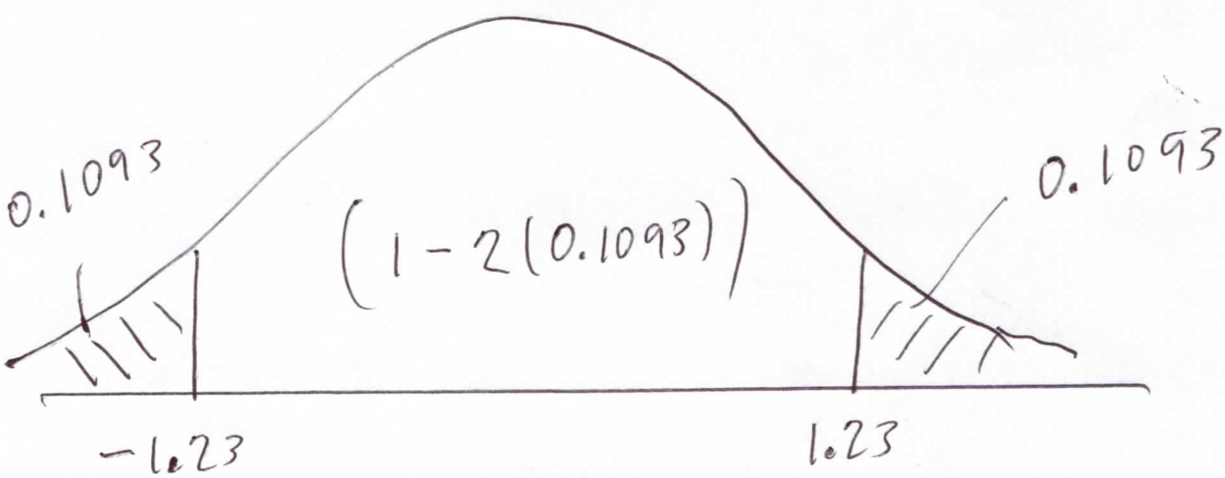



(d) By Properties 2.3(1), $P(Z > -1.23) = P(Z < 1.23) = 0.8907$, from (b).

(e) This is the same probability as (d), 0.8907.

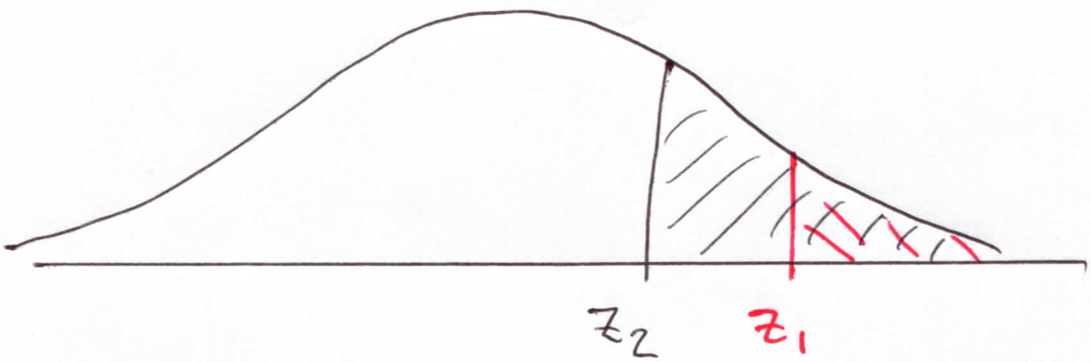
(f) By (a) and Properties 2.3(1)and 2.3(2), this is

$$1 - [P(Z \geq 1.23) + P(Z \leq -1.23)] = 1 - 2P(Z \geq 1.23) = 1 - 2(0.1093) = 0.7814.$$



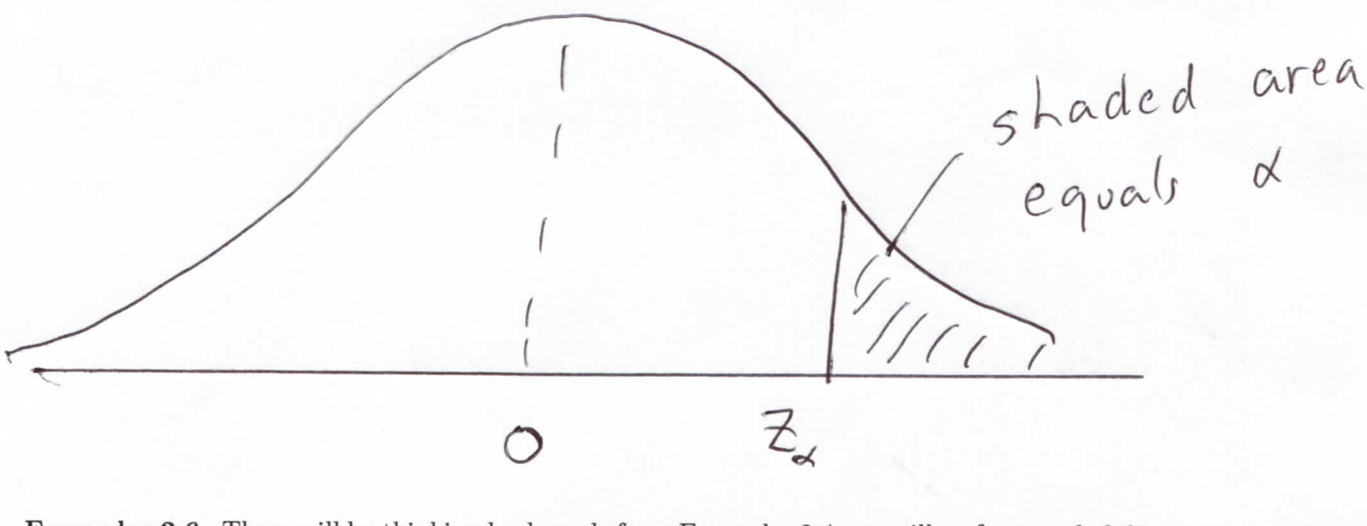
(g) In the drawing below, we have written and shaded $z_1$ and corresponding probabilities in red, $z_2$ in black. It is clear from the drawing that the black shading contains the red shading, hence has larger area; in other words,

$$P(Z > z_2) > P(Z > z_1).$$




**Definition 2.5.** For $\alpha$ a positive number less than one, the $Z$ **critical value for** $\alpha$ is a number, denoted $z_\alpha$, such that

$$P(Z > z_\alpha) = \alpha.$$




**Examples 2.6.** These will be thinking backwards from Examples 2.4: we will go from probabilities to $z$ values, whereas Examples 2.4 went from $z$ values to probabilities.

(a) Find the critical value for 0.2.

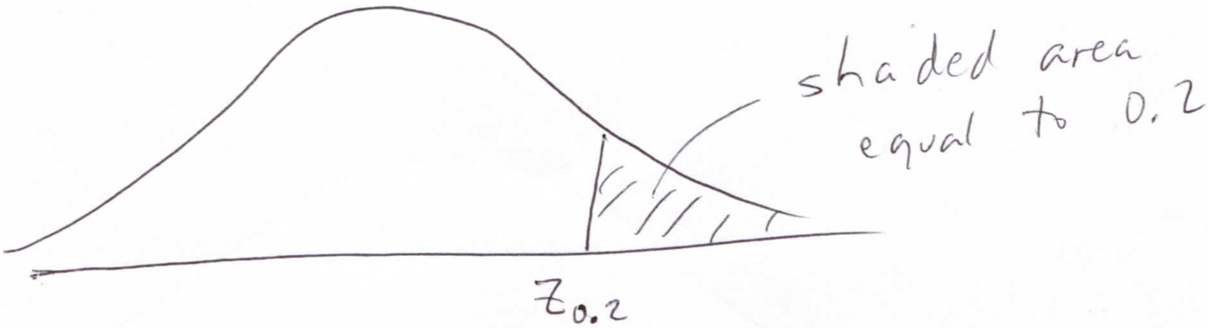
(b) Find a number $c$ so that $P(Z < c) = 0.75$; identify any critical values used.

(c) Find a positive number $c$ so that $P(-c < Z < c) = 0.9$; identify any critical values used.

(d) For arbitrary positive $\alpha$ less than one, what is $P(-z_{\frac{\alpha}{2}} < Z < z_{\frac{\alpha}{2}})$?

(e) If $\alpha_1 > \alpha_2$, what (if anything) can be said about the relationship between $z_{\alpha_1}$ and $z_{\alpha_2}$?

**Solutions.** (a) Here is the picture we'd like.



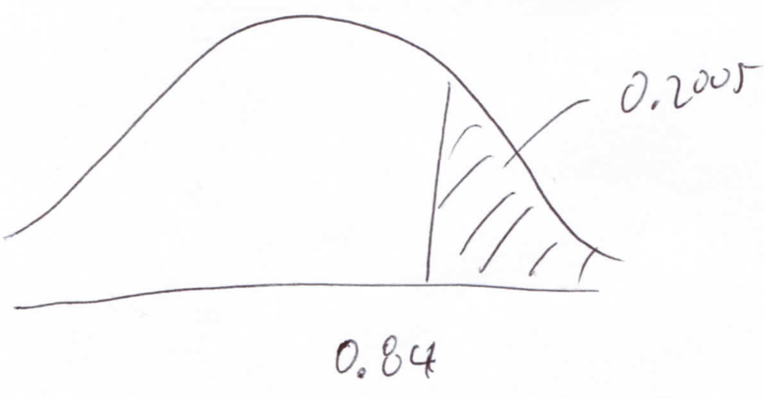

In the $Z$ tables at the end of this magnification, get as close as possible to the probability 0.2 in the inside: this occurs in the row 0.8, between the columns 0.04 and 0.05, as in the reproduction below.

| z | 0.04 | 0.05 |
|---|---|---|
| 0.8 | 0.2005 | 0.1977 |

Compare the two drawings we get from the $Z$ table with our desired picture at the beginning of these Solutions; we conclude that

$$0.84 < z_{0.2} < 0.85.$$

For this magnification, we will accept $z_{0.2} = 0.84$ or $0.85$.




(b) By Properties 2.3(2), $P(Z > c) = 1 - P(Z < c) = 1 - 0.75 = 0.25$; that is, we want the critical value $c = z_{0.25}$ for 0.25. Looking at the $Z$ table we have

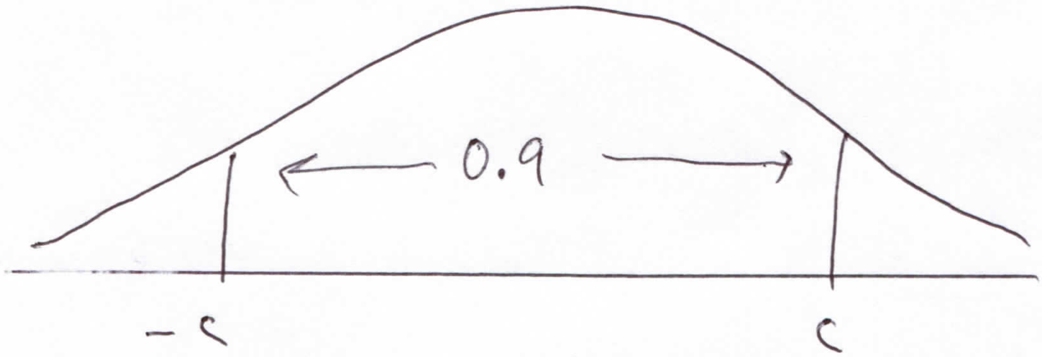
$$P(Z > 0.67) = 0.2514, P(Z > 0.68) = 0.2483,$$

so that, as in (a),

$$0.67 < c < 0.68;$$

choose $c = 0.67$ or $0.68$.

(c) Here is the picture we want.




By Properties 2.3,

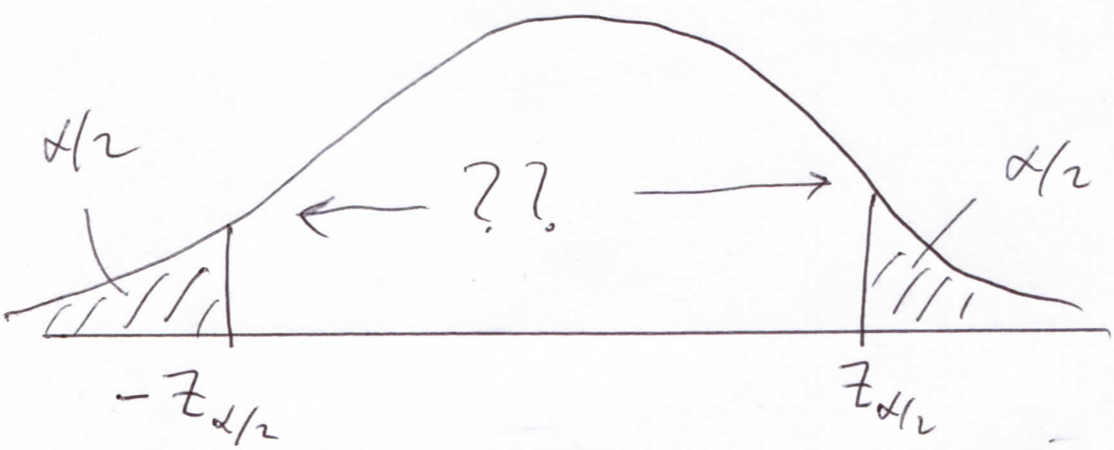
$$P(Z > c) = P(Z < -c) = \frac{1}{2}[P(Z < -c) + P(Z > c)] = \frac{1}{2}[1 - (P(-c < Z < c))] = \frac{1}{2}(1-0.9) = 0.05;$$

that is, we want $c = z_{0.05}$, the critical value for 0.05.

We leave it to the reader to use the methods of (a) to get $c = 1.64$ or $1.65$.

(d) By Properties 2.3 and the definition of critical value,

$$P(-z_{\frac{\alpha}{2}} < Z < z_{\frac{\alpha}{2}}) = 1 - \left[P(Z \geq z_{\frac{\alpha}{2}}) + P(Z \leq -z_{\frac{\alpha}{2}})\right] = 1 - 2P(Z > z_{\frac{\alpha}{2}}) = 1 - 2(\frac{\alpha}{2}) = (1 - \alpha).$$

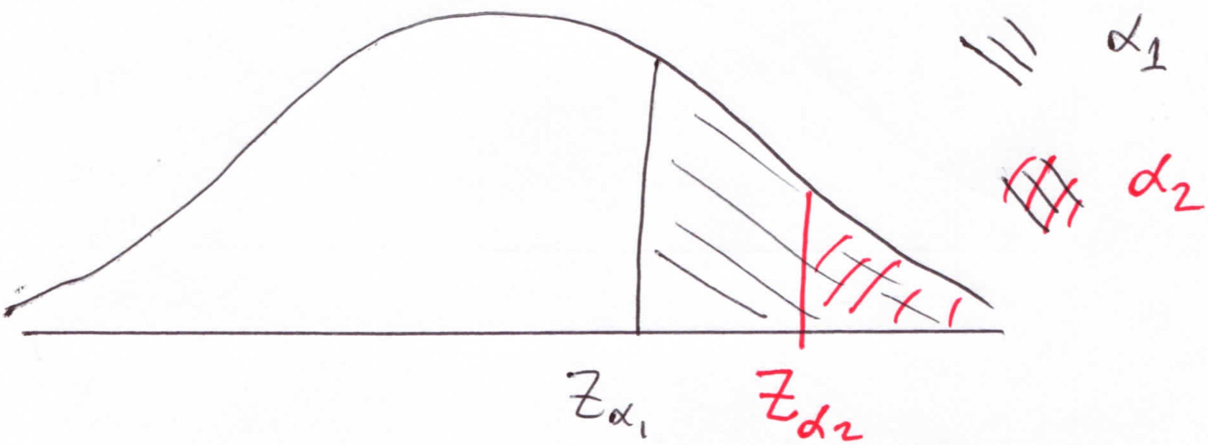



$$?? = 1 - \left(\frac{\alpha}{2} + \frac{\alpha}{2}\right) = (1 - \alpha)$$

(e) The converse of Examples 2.4(g) is also true, by the same argument as the solution of Examples 2.4(g); that is, if $P(Z > z_2) > P(Z > z_1)$, then $z_1 > z_2$. All we need now is to translate into the language of critical values:

$$P(Z > z_{\alpha_1}) = \alpha_1 > \alpha_2 = P(Z > z_{\alpha_2}),$$

thus $z_{\alpha_2} > z_{\alpha_1}$. See the drawing below, where $z_{\alpha_1}$ and $\alpha_1$ are shaded in black, $z_{\alpha_2}$ and $\alpha_2$ are shaded in red.




**Theorem 2.7.** (a) If $X$ is normal, with mean $\mu$ and standard deviation $\sigma$, then

$$Z = \frac{(X - \mu)}{\sigma}.$$

(b) If $X$ is normal, with mean $\mu_X$ and standard deviation $\sigma_X$, then the sample mean $\overline{X}$ (see Definitions 1.1) is also normal, with mean $\mu_{\overline{X}} = \mu_X$ and standard deviation $\sigma_{\overline{X}} = \frac{\sigma_X}{\sqrt{n}}$, where $n$ is the sample size.

Theorem 2.7(b) explains why we like large committees. The larger the sample size, the smaller $\sigma_{\overline{X}}$ is. This implies that extreme, or weird, behavior (by definition, being far from the mean $\mu$) is less likely (see the picture at the end of Definition 2.1, comparing small standard deviation to large standard deviation).

**Examples 2.8.** Suppose wolverine weight is normally distributed, with a mean of 12 pounds and a standard deviation of 2 pounds; that is, the random variable $X$ defined to be the weight, in pounds, of a randomly chosen wolverine, is normally distributed, with mean $\mu = 12$, standard deviation $\sigma = 2$.

(a) What is the probability that a randomly chosen wolverine weighs more than 12.5 pounds?

(b) What is the probability that twenty-five randomly chosen wolverines weigh, on average, more than 12.5 pounds?

(c) What is the probability that one hundred randomly chosen wolverines weigh, on average, more than 12.5 pounds?

**Solutions.** (a) We want

$$P(X > 12.5) = P(\frac{X - \mu}{\sigma} > \frac{12.5 - \mu}{\sigma}) = P(Z > \frac{12.5 - 12}{2}) = P(Z > 0.25) = 0.4013,$$

from the $Z$ tables.

(b) Now we want, with sample size $n = 25$,

$$P(\overline{X} > 12.5) = P(\frac{\overline{X} - \mu_{\overline{X}}}{\sigma_{\overline{X}}} > \frac{12.5 - \mu_{\overline{X}}}{\sigma_{\overline{X}}}) = P(Z > \frac{12.5 - 12}{\frac{2}{\sqrt{25}}}) = P(Z > 1.25) = 0.1056,$$

from the $Z$ tables.

(c) This is the same as (b), with $n = 100$,

$$P(\overline{X} > 12.5) = P(\frac{\overline{X} - \mu_{\overline{X}}}{\sigma_{\overline{X}}} > \frac{12.5 - \mu_{\overline{X}}}{\sigma_{\overline{X}}}) = P(Z > \frac{12.5 - 12}{\frac{2}{\sqrt{100}}}) = P(Z > 2.5) = 0.0062,$$

from the $Z$ tables.

Notice that unusual behavior, in this case defined as weighing more than 12 pounds, of the average wolverine, gets less likely as the sample size increases. Another way of saying this is that we have more precise information about the sample average, in the sense that we have a higher probability of being near the mean, when the sample size is larger.

Here is the probability statement that will lead to our confidence intervals and bounds in the next chapter.

**Corollary 2.9.** If $X$ is normal with mean $\mu$ and standard deviation $\sigma$, then for any positive $\alpha$ less than one,

$$(1 - \alpha) = P\left(-z_{\frac{\alpha}{2}} < \frac{(\overline{X} - \mu)}{\frac{\sigma}{\sqrt{n}}} < z_{\frac{\alpha}{2}}\right).$$

**Proof:** In Examples 2.6(d), we hope you have shown that

$$(1 - \alpha) = P(-z_{\frac{\alpha}{2}} < Z < z_{\frac{\alpha}{2}}).$$

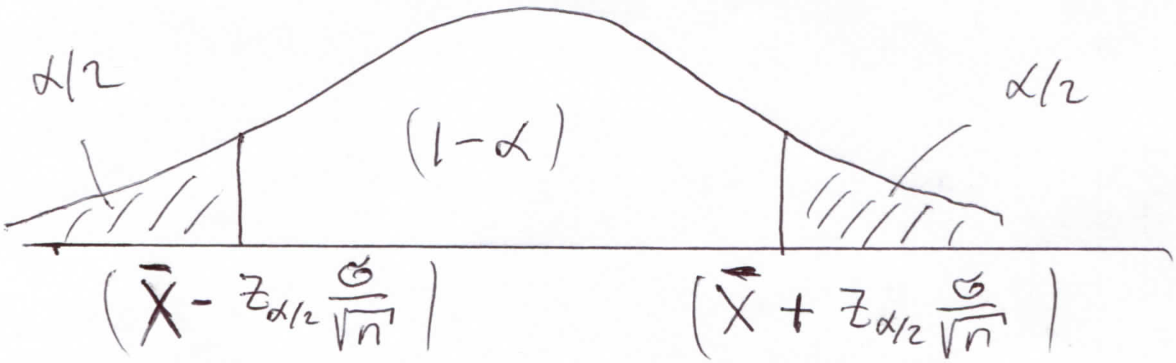
Our result then follows from Theorem 2.7. □

## 3. DERIVATION of CONFIDENCE INTERVALS and CONFIDENCE BOUNDS.

Throughout this section, $X$ is normal, with (unknown) mean $\mu$ and (known) standard deviation $\sigma$, $\overline{X}$ is the sample mean, from random samples of size $n$ (see Definitions 1.1), and $\alpha$ is a positive number less than one.

**Derivation of confidence intervals 3.1.** Recall the definition of *critical value* Definition 2.5. Corollary 2.9 implies

$$(1-\alpha) = P(-z_{\frac{\alpha}{2}} < \frac{(\overline{X}-\mu)}{\frac{\sigma}{\sqrt{n}}} < z_{\frac{\alpha}{2}}) = P(-z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} < (\overline{X}-\mu) < z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}})$$
$$= P(-z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} < (\mu-\overline{X}) < z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}) = P(\overline{X} - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} < \mu < \overline{X} + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}).$$




**Definitions 3.2.** When a sample $x_1, x_2, \ldots, x_n$ from $X$ is taken, denote as usual (see, for example, Definitions 1.1 and [2, Definitions 6]) the sample mean of numbers

$$\overline{x} \equiv \frac{1}{n}(x_1 + x_2 + \cdots + x_n).$$

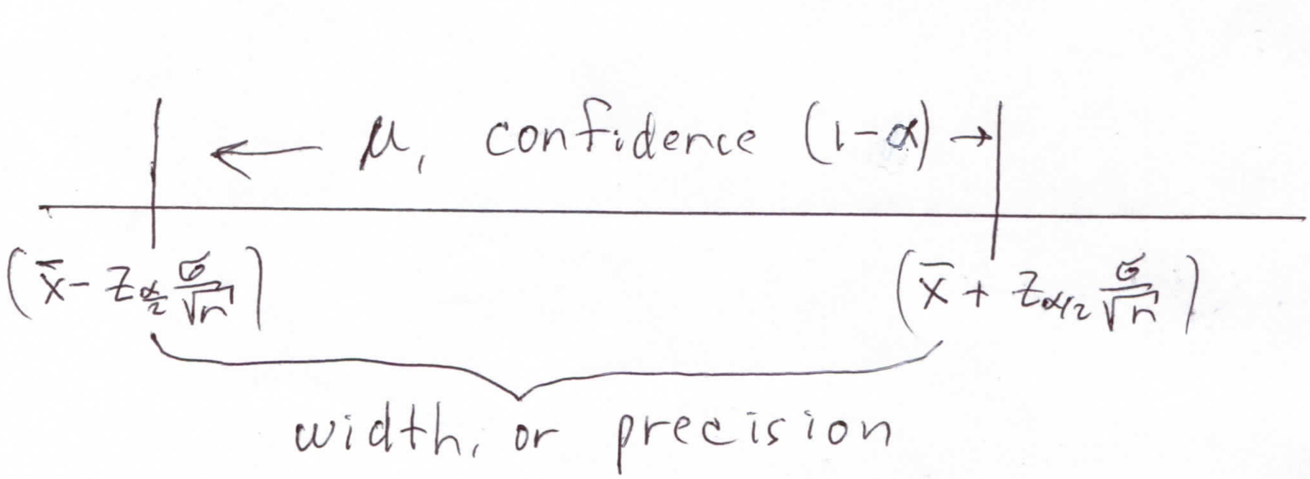
Derivation 3.1 leads to the following definition.

The $100(1-\alpha)\%$ **confidence interval** for $\mu$ is

$$\left(\left(\overline{x} - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right), \left(\overline{x} + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right)\right) \equiv \text{ the set of all real numbers } c \text{ satisfying } \left(\overline{x} - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right) < c < \left(\overline{x} + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right);$$

this is also denoted

$$\overline{x} \pm \left(z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right).$$



$(1-\alpha)$ is the **confidence level** or **reliability** of the confidence interval; $2z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}$ is the **width** or **precision** of the confidence interval. We have **estimated** $\mu$ **to within** $\left(z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right)$.

Very informally, we are $100(1-\alpha)\%$ confident that $\mu$ is between $\left(\overline{x} - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right)$ and $\left(\overline{x} + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right)$.

Identically,

$$(1-\alpha) = P(\frac{(\overline{X}-\mu)}{\frac{\sigma}{\sqrt{n}}} > -z_\alpha)$$

leads to the definition of a $100(1-\alpha)\%$ **upper confidence bound** for $\mu$:

$$\left(\overline{x} + z_\alpha\frac{\sigma}{\sqrt{n}}\right)$$

and

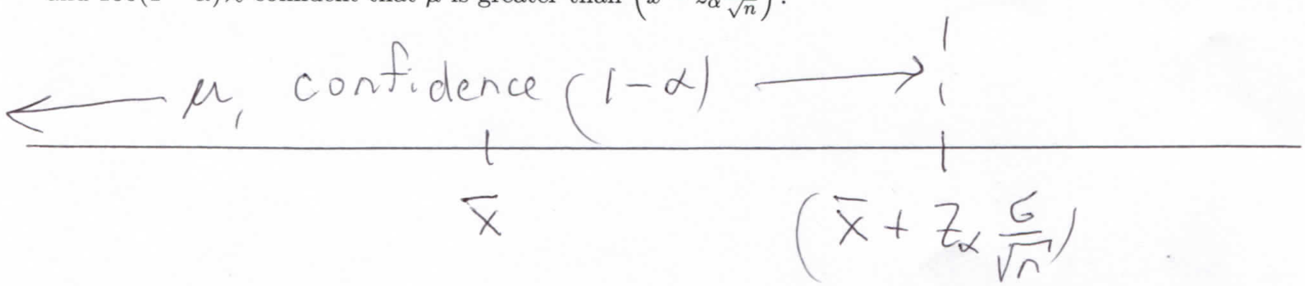
$$(1-\alpha) = P(\frac{(\overline{X}-\mu)}{\frac{\sigma}{\sqrt{n}}} < z_\alpha)$$

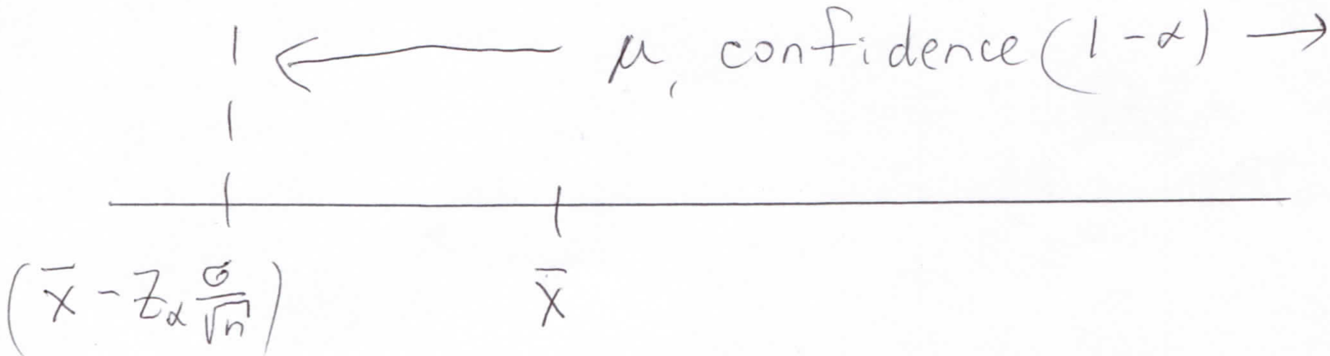
leads to the definition of a $100(1-\alpha)\%$ **lower confidence bound** for $\mu$:

$$\left(\overline{x} - z_\alpha\frac{\sigma}{\sqrt{n}}\right).$$

$(1-\alpha)$ is the **confidence level** or **reliability** of each bound.

Again very informally, we are $100(1-\alpha)\%$ confident that $\mu$ is less than $\left(\overline{x} + z_\alpha\frac{\sigma}{\sqrt{n}}\right)$ and $100(1-\alpha)\%$ confident that $\mu$ is greater than $\left(\overline{x} - z_\alpha\frac{\sigma}{\sqrt{n}}\right)$.



**Remark 3.3.** The critical reader might wonder why the confidence level is written in the awkward and backwards form $(1-\alpha)$ or $100(1-\alpha)\%$, so that a high confidence level corresponds to a small, positive $\alpha$.

The $(1-\alpha)$ terminology for confidence level has become standard because it anticipates *hypothesis testing* (to appear in future magnifications), where $\alpha$ is the *significance* of a hypothesis test.

## 4. EXAMPLES and INTERPRETATIONS.

See Definitions 3.2 for relevant formulas and definitions.

**Example 4.1.** Get a 99.5% confidence interval for the average height of all human beings, assuming human height is normally distributed with a standard deviation of three inches, if we measured four people and got the following heights, in inches

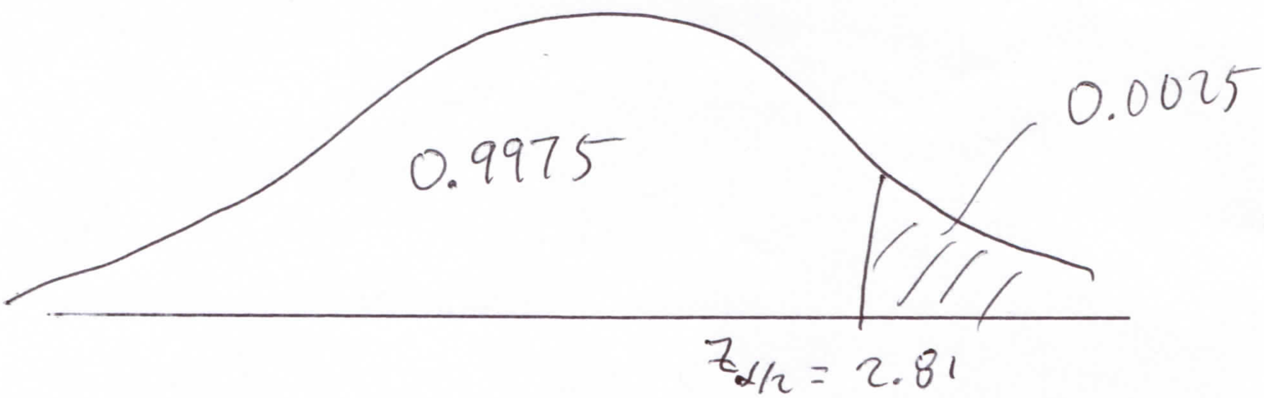
$$70, 68, 72, 74.$$

**Solution.** Let's fill in the pieces of our $100(1-\alpha)\%$ confidence interval formula in Definitions 3.2:

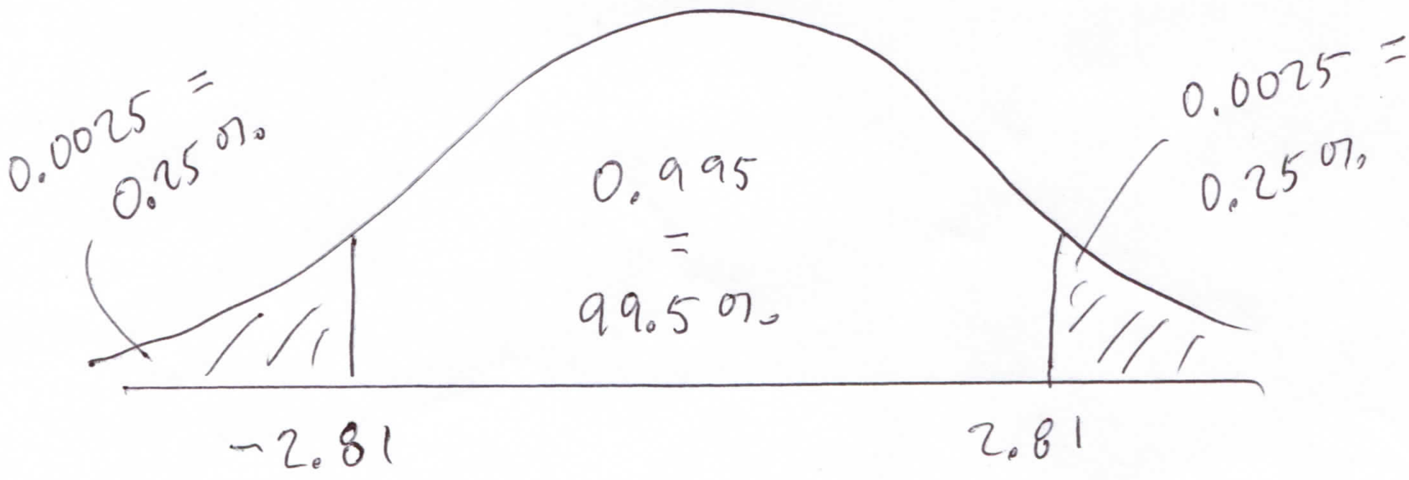
$$\overline{x} \pm \left( z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \right).$$

$$100(1-\alpha) = 99.5 \rightarrow (1-\alpha) = 0.995 \rightarrow \alpha = 0.005 \rightarrow \frac{\alpha}{2} = 0.0025.$$

From our $Z$ tables we get the critical value $z_{\frac{\alpha}{2}} = z_{0.0025} = 2.81$.




Although not required for solving this example, our strategy is illuminated by the following modification of the $Z$ table picture we just drew.



Our sample size $n = 4$, population standard deviation $\sigma = 3$, and we may quickly calculate the sample mean

$$\overline{x} = \frac{1}{4}(70 + 68 + 72 + 74) = 71.$$

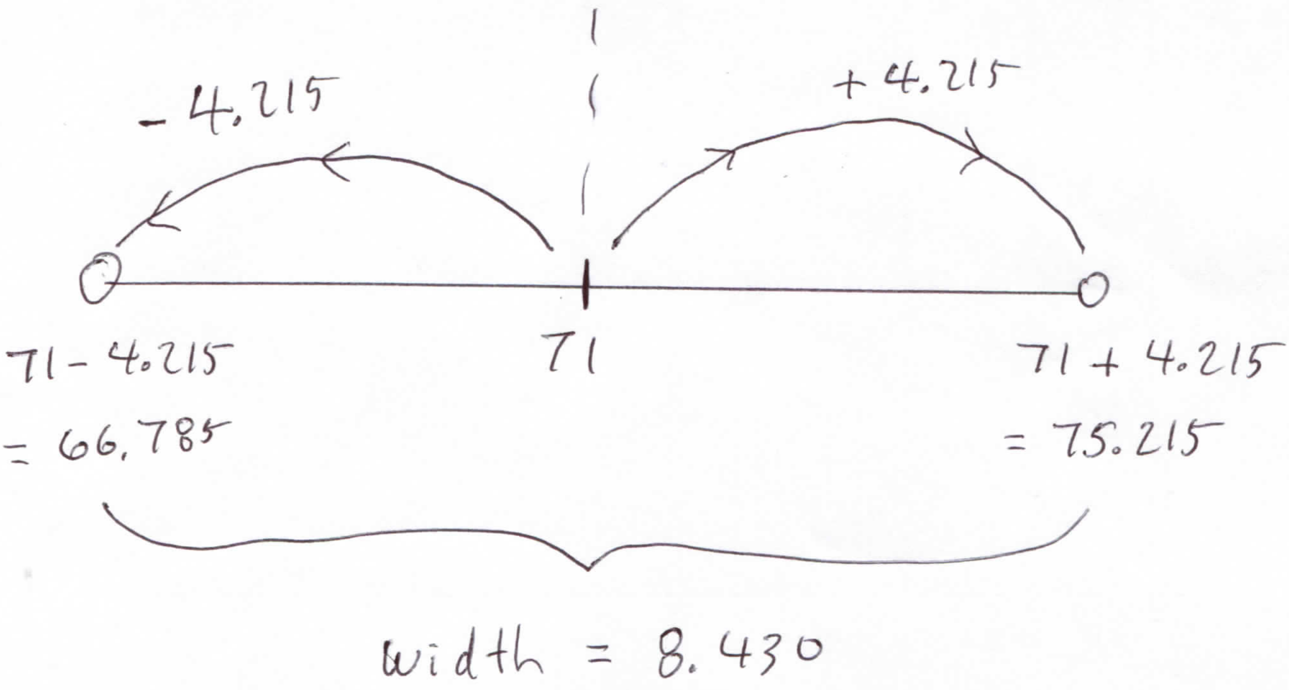
We appear to have all the ingredients for our confidence interval:

$$\overline{x} \pm \left( z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \right) = 71 \pm \left( 2.81 \frac{3}{\sqrt{4}} \right) = 71 \pm 4.215.$$

This can be written as an interval of numbers

$$(71 - 4.215, 71 + 4.215) = (66.785, 75.215),$$

meaning the set of all real numbers $c$ such that $66.785 < c < 75.215$.



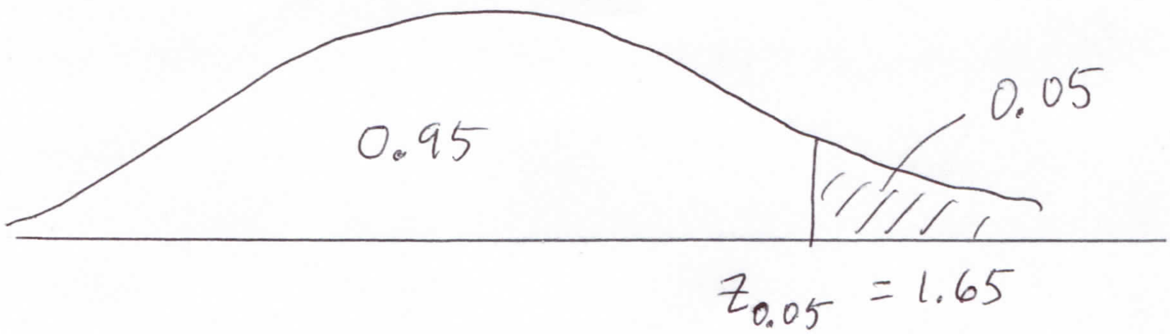

99.5% is the *confidence level* or *reliability* of our confidence interval. $8.430 = 75.215 - 66.785 = 2(4.215)$ is the *width* or *precision* of said interval. We have *estimated* $\mu$ *to within* 4.215; with 99.5% confidence,

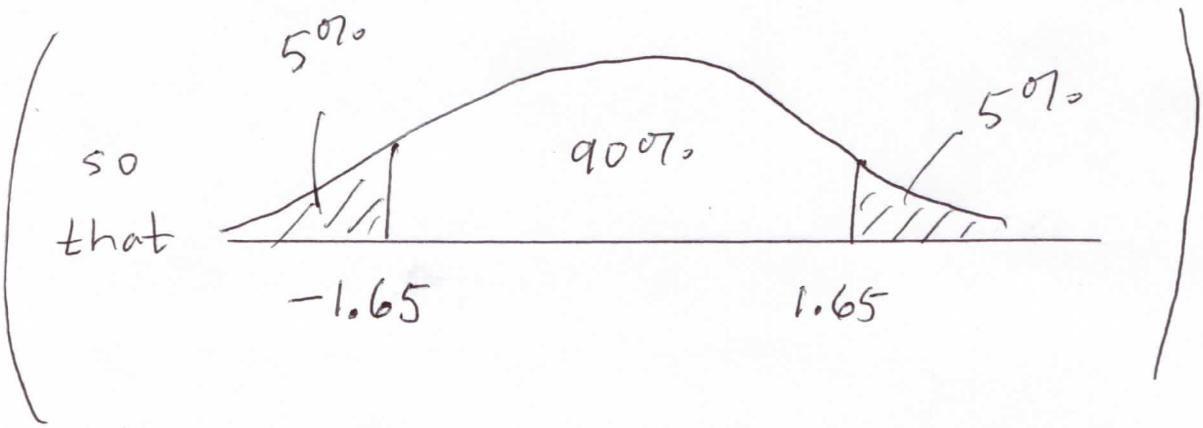
$$|\mu - 71| < 4.215.$$

**Examples 4.2.** Suppose wolverine mass is normally distributed, with standard deviation 100.

(a) Get a 90% confidence interval for the average mass of all wolverines, if a sample of 25 wolverines has a mean of 8,439 grams.

**Solution.** See Definitions 3.2. We set $100(1-\alpha) = 90$, and get $\alpha = 0.1$, so that $\frac{\alpha}{2} = 0.05$, and, from the $Z$ tables, our critical value is $z_{\frac{\alpha}{2}} = z_{0.05} = 1.65$.



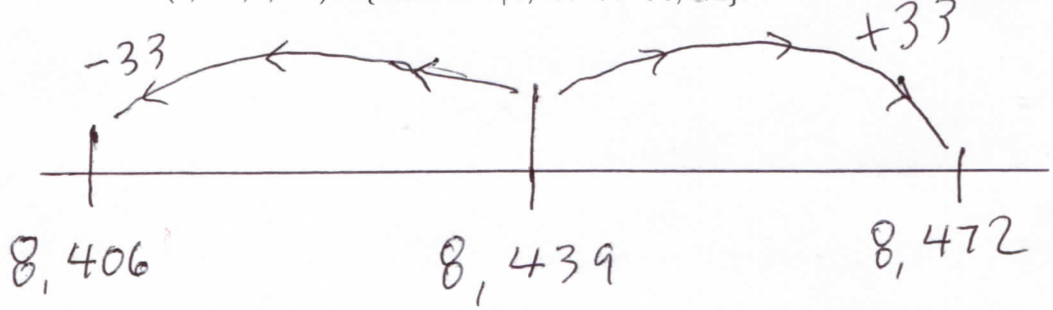




We have been told that $\sigma = 100, n = 25$, and $\overline{x} = 8,439$, thus our confidence interval is

$$\overline{x} \pm \left( z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \right) = 8,439 \pm \left( 1.65 \frac{100}{\sqrt{25}} \right) = 8,439 \pm 33,$$

or

$$(8,406, 8,472) \equiv \{\text{numbers } c \,|\, 8,406 < c < 8,472\}.$$




That's a precision of $2 \times 33 = 66$ and a confidence level of 90%.

(b) Same as (a), except a 99% confidence level.

**Solution.** All that is changing is $\alpha$: $100(1-\alpha) = 99 \to \frac{\alpha}{2} = 0.005$, so that, from the $Z$ tables, $z_{\frac{\alpha}{2}} = z_{0.005} = 2.58$.


As with (a), our confidence interval is

$$\overline{x} \pm \left( z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \right) = 8,439 \pm \left( 2.58 \frac{100}{\sqrt{25}} \right) = 8,439 \pm 51.6,$$

or

$$(8,387.4, 8,490.6).$$

That's a precision of $2 \times 51.6 = 103.2$ and a confidence level of 99%.

**NOTE:** More confidence implies (and requires) less precision, that is, a wider confidence interval. Since (b) replaced 90% confidence with 99% percent confidence, our interval went from a precision of 66 to a precision of 103.2.

Think of a confidence interval as a basket, placed outdoors to catch hailstones falling out of the sky. To catch more hailstones (equivalent to more confidence in the interval), we need a bigger basket (equivalent to a wider interval).

(c) Same as (b), except the sample size is 2500.

**Solution.** All that changes from (b) is $n = 2500$, instead of 25:

$$\overline{x} \pm \left( z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \right) = 8,439 \pm \left( 2.58 \frac{100}{\sqrt{2500}} \right) = 8,439 \pm 5.16,$$

or

$$(8,433.84, 8,444.16).$$

That's a precision of $2 \times 5.16 = 10.32$ (ten times as precise as (b)) with the same confidence as in (b)

**NOTE:** A larger sample size creates more precision, that is, a skinnier confidence interval. Since the sample size in (c) is 100 times as large as in (b), and $\frac{1}{\sqrt{100}} = \frac{1}{10}$, the width of the interval in (c) is $\frac{1}{10}$ the width of the interval in (b).

The intuition here is that a larger sample means more information, which means more precise, hence more informative, statements about the population mean $\mu$.

For a more extreme illustration, stating that $\mu$ is between $-1$ and 1 is more informative than stating that $\mu$ is between $-1,000,000$ and $1,000,000$.

(d) How large a sample is required to get a width less than 6 in a 99% confidence interval for the average mass of all wolverines?

**Solution.** We know from our general formula in Definitions 3.2 that the width (precision) of our confidence interval is

$$2z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} = \text{(see (b) and (c))}\ 2(2.58)(\frac{100}{\sqrt{n}}).$$

Set $2(2.58)(\frac{100}{\sqrt{n}}) < 6$, and solve for $n$:

$$n > (86)^2 = 7,396.$$

We need a sample of at least 7,397 wolverines.

(e) Get a 95% upper confidence bound for $\mu$ and a 95% lower confidence bound for $\mu$, where $\mu$ is the average mass of all wolverines, if, as in (a), a sample of 25 wolverines has a mean of 8,439 grams.

**Solution.** See Definitions 3.2 for formulas for upper and lower confidence bounds. Setting $100(1-\alpha) = 95$ implies that $\alpha = 0.05$, thus the critical value we need is $z_\alpha = z_{0.05} = 1.65$ (see the Solution to (a)), and thus

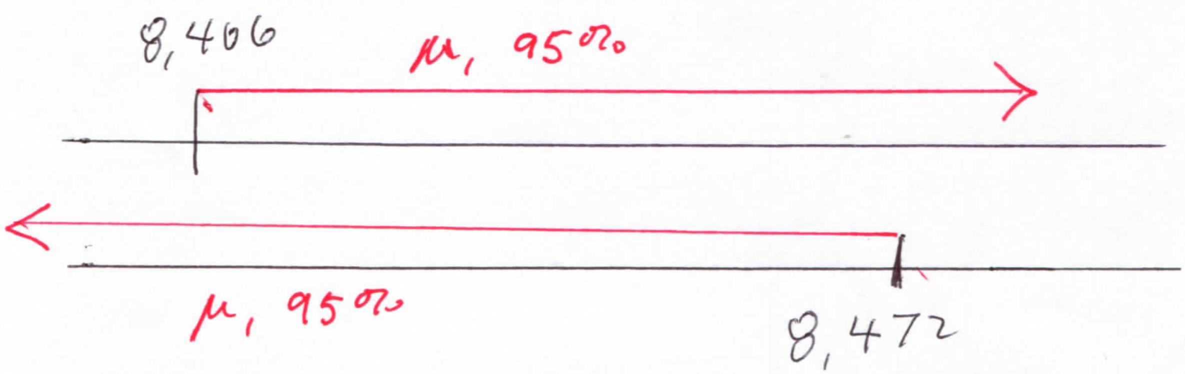
$$\overline{x} + \left(z_\alpha\frac{\sigma}{\sqrt{n}}\right) = 8,439 + \left(1.65\frac{100}{\sqrt{25}}\right) = 8,439 + 33 = 8,472$$

is the desired upper confidence bound for $\mu$, and

$$\overline{x} - \left(z_\alpha\frac{\sigma}{\sqrt{n}}\right) = 8,439 - \left(1.65\frac{100}{\sqrt{25}}\right) = 8,439 - 33 = 8,406$$

is the desired lower confidence bound for $\mu$.

Informally, we are 95% confident that $\mu$ is less than 8,472 and we are 95% confident that $\mu$ is greater than 8,406.



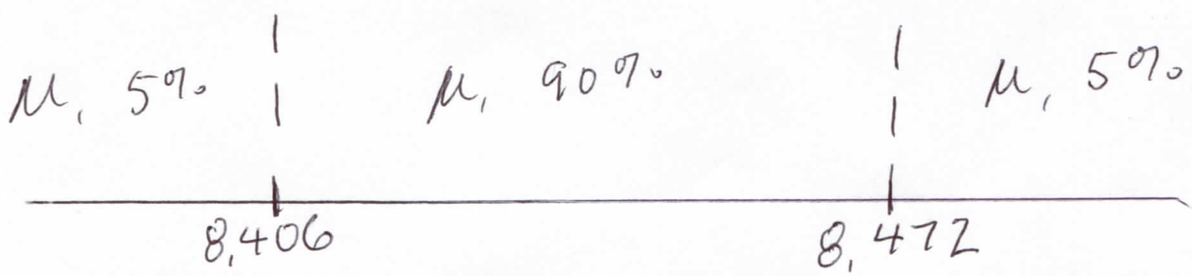

Look back at the solution of part (a). The confidence interval constructed there tells us that we are 90% confident that $\mu$ is between 8,406 and 8,472. This may be stated in language that sounds like upper and lower confidence bounds:

we are 90% confident that $\mu$ is both greater than 8,406 and less than 8,472.

Since 8,406 and 8,472 are 95% lower and upper (respectively) confidence bounds, why doesn't the confidence interval of part (a) have 95% confidence rather than merely 90% confidence?

One way to answer that last question is that the sets producing 95% confidence, for upper and lower confidence bounds for $\mu$, are different. To get the confidence of the confidence interval, we need to be in *both* those sets; that is, we are taking an intersection of two sets, each of which has 95% confidence. Intersections usually shrink sets, so we shouldn't expect to hold onto that 95%.

A better answer to that question in the penultimate paragraph comes from a negative outlook. That 95% upper confidence bound of 8,472 is saying that we are as much as 5% confident that $\mu \geq 8,472$; identically, the 95% lower confidence bound of 8,406 is saying that we are as much as 5% confident that $\mu \leq 8,406$. That is, we threw out as much as 5% confidence to get $\mu$ less than 8,472 and we threw out as much as 5% confidence to get $\mu$ greater than 8,406. To get *both* $\mu$ less than 8,472 and $\mu$ greater than 8,406, as is needed for our confidence interval, we have to throw out as much as both those 5% (unwanted) confidences. Thus we are throwing out as much as 10%, to get inside our confidence interval, making it a $(100 - 10) = 90\%$ confidence interval.




**Examples 4.3.** "Confidence interval" here will mean the confidence interval for the population mean $\mu$ constructed in Definitions 3.2, under the hypotheses of Chapter 3.

(a) Suppose the sample size is multiplied by 36. What happens to the precision of a confidence interval, assuming the confidence stays the same?

(b) How much should we enlarge the sample size, to make the precision quadruple, that is, make the width be multiplied by one quarter, assuming the confidence stays the same?

(c) Suppose, in our confidence interval, we change from 95 percent to 80 percent confidence; what happens to the precision? Assume the sample size stays the same.

(d) Suppose, in a 90% confidence interval, we quintuple the precision; that is, the width of the interval is divided by five. What is the new confidence level? Assume the sample size stays the same.

(e) Same as (d), except we begin with a 99% confidence interval.

(f) If we increase the precision of a confidence interval, that is, make the interval skinnier, what happens to the confidence? Assume the sample size stays the same.

(g) If $(3, 7)$ is a 95% confidence interval, what is, using the same data, an 80% confidence interval?

**Solutions.** (a) The sample size in the precision or width $2z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}$ appears only in the $\frac{1}{\sqrt{n}}$ term. Since we are replacing $n$ with $36n$, our new precision is

$$2z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{36n}} = 2z_{\frac{\alpha}{2}}\frac{\sigma}{6\sqrt{n}} = \frac{1}{6}\left(2z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right),$$

that is, the new width is one-sixth of the old width.

(b) As in the solution to (a), if $n'$ is the new sample size, we need

$$2z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n'}} = \frac{1}{4}\left(2z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right), \quad \text{so that } \frac{1}{\sqrt{n'}} = \frac{1}{4}\left(\frac{1}{\sqrt{n}}\right),$$

thus

$$\frac{1}{n'} = \frac{1}{16}\left(\frac{1}{n}\right), \quad \text{implying } n' = 16n;$$

we must multiply the sample size by 16.

Note that $\sqrt{16} = 4$.

(c) Let $\alpha \equiv 0.05$ and $\alpha' \equiv 0.2$, so that $(1-\alpha)$ is the original 95% confidence and $(1-\alpha')$ is the new 80% confidence.

We are changing from a precision of $2z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}$ to a precision of $2z_{\frac{\alpha'}{2}}\frac{\sigma}{\sqrt{n}}$.

We need those critical values (Definition 2.5) in the formula above for precision: $z_{\frac{\alpha}{2}}$ and $z_{\frac{\alpha'}{2}}$. Staring at the $Z$ table at the end of the magnification, as in Chapter 2, we get

$$z_{\frac{\alpha}{2}} = z_{0.025} = 1.96 \quad \text{and} \quad z_{\frac{\alpha'}{2}} = z_{0.1} = 1.28,$$

so the new precision divided by the old precision is $\frac{1.28}{1.96} \sim 0.653$; that is, the width of the 80% confidence interval is 0.653 times the width of the 95% confidence interval.

Compare this to the precisions in Examples 4.2(a) and (b), especially the "NOTE" after the solution of Examples 4.2(b).

(d) Let $\alpha \equiv 0.1$, so that $(1-\alpha)$ is the confidence level of the original confidence interval, and let $\alpha'$ be such that $(1-\alpha')$ is the confidence level of the new confidence interval, modified from the original by having its width divided by 5.

The width of the original confidence interval is $2z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} = 2(1.65)\frac{\sigma}{\sqrt{n}}$, while the width of the new confidence interval is

$$2z_{\frac{\alpha'}{2}}\frac{\sigma}{\sqrt{n}},$$

thus

$$2z_{\frac{\alpha'}{2}}\frac{\sigma}{\sqrt{n}} = \frac{1}{5}\left(2(1.65)\frac{\sigma}{\sqrt{n}}\right),$$

so that

$$z_{\frac{\alpha'}{2}} = \frac{1}{5}(1.65) = 0.33.$$

By definition of critical value,

$$\frac{\alpha'}{2} = P(Z > z_{\frac{\alpha'}{2}}) = P(Z > 0.33) = 0.3707,$$

from the $Z$ tables, thus $\alpha' = 0.7414$, so that the new confidence level is $(1-\alpha') = (1-0.7414) = 0.2586$, or 25.86%.

(e) With $\alpha = 0.01$ and $\alpha'$ as in (d), the same reasoning as in (d) shows that

$$z_{\frac{\alpha'}{2}} = \frac{1}{5}(2.58) = 0.516,$$

so that

$$\frac{\alpha'}{2} = P(Z > z_{\frac{\alpha'}{2}}) = P(Z > 0.516) \sim P(Z > 0.52) = 0.3015,$$

thus the new confidence level is $(1-\alpha') = (1-0.603) = 0.397$, or 39.7%.

(f) The different answers to parts (d) and (e) show that, even if we stated precisely what was happening to the precision, a precise quantitative answer to (f) would be impossible. Let's see how much we can say.

Let $\alpha$ and $\alpha'$ be as in (d) and (e). Since the width is decreased, we have

$$\frac{z_{\frac{\alpha'}{2}}}{z_{\frac{\alpha}{2}}} = \frac{2z_{\frac{\alpha'}{2}}\frac{\sigma}{\sqrt{n}}}{2z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}} < 1,$$

or

$$z_{\frac{\alpha'}{2}} < z_{\frac{\alpha}{2}},$$

which implies, by Examples 2.4(g), that $\frac{\alpha'}{2} > \frac{\alpha}{2}$, which implies that $(1-\alpha') < (1-\alpha)$; that is, the confidence has decreased.

(g) Let $\alpha \equiv 0.05$ and $\alpha' \equiv 0.2$, so that $(1-\alpha)$ is the original confidence level and $(1-\alpha')$ is the confidence level of the confidence interval to be constructed. It is clear from Definitions 3.2 of a confidence interval that

$$\overline{x} = \frac{1}{2}(3+7) = 5,$$

the *midpoint* of the interval $(3, 7)$. Thus

$$7 = \overline{x} + z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} = 5 + 1.96\frac{\sigma}{\sqrt{n}},$$

thus

$$\frac{\sigma}{\sqrt{n}} = \frac{2}{1.96},$$

so that our desired confidence interval is

$$\overline{x} \pm z_{\frac{\alpha'}{2}} \frac{\sigma}{\sqrt{n}} = 5 \pm 1.28\frac{\sigma}{\sqrt{n}} = 5 \pm 1.28(\frac{2}{1.96}) \sim 5 \pm 1.306 = (3.694, 6.306).$$

Notice that the width of the 80% confidence interval is $(6.306 - 3.694) = 2.612$, the width of the 95% confidence interval is 4, and

$$\frac{z_{\frac{\alpha'}{2}}}{z_{\frac{\alpha}{2}}} = \frac{z_{0.1}}{z_{0.025}} = \frac{1.28}{1.96} \sim \frac{2.612}{4} \text{ (approximation due to rounding).}$$

In fact, we could have done this more quickly, after getting the midpoint 5: the 80% confidence interval is

$$5 \pm \frac{1.28}{1.96}(2),$$

since 2 is half the width of the 95% confidence interval.

**Interpretation 4.4.** The suspicious reader might have noticed that we seem to be avoiding the word "probability," at least after Derivation 3.1. We've instead spoken glibly about "confidence," which, if you compare Derivation 3.1 and the definition of a confidence interval in Definitions 3.2, seems related to probability.

But here we must save the reader embarrassment and social disapproval: if $(a, b)$, the set of all numbers between $a$ and $b$, is a confidence interval for a population parameter $\theta$, say of confidence level 99% (replace with whatever confidence you'd like), it is *not* correct to say

"The probability that $\theta$ is between $a$ and $b$ is 99%."

For emphasis, we hit the offending quote with a math-busters symbol

Semantically the quoted statement makes no sense: $\theta$ is a fixed, unvarying (although unknown) number, like the average length of all cats, and probabilities are applied to uncertain events, like the length of an unspecified cat chosen at random.

There are probabilities in Derivation 3.1, but those probabilities apply to the random variable $\overline{X}$, rather than the parameter $\mu$.

Here is the interpretation of the confidence level $100(1-\alpha)\%$ for a $100(1-\alpha)\%$ confidence interval for a population parameter $\theta$. If you construct many $100(1-\alpha)\%$ confidence intervals for $\theta$, following the same strategy and taking samples of the same sample size, approximately $100(1-\alpha)\%$ of those (different) confidence intervals will contain $\theta$.

For example, in Examples 4.2(a), we constructed, from a sample of 25 measurements, the 90% confidence interval for $\mu$

$$8,439 \pm 33 \equiv (8,406, 8,472).$$

It is not correct to say that

$\mu$ is between 8,406 and 8,472 with probability 90%.

The 90% confidence means that, if we construct many 90% confidence intervals for $\mu$ of the form

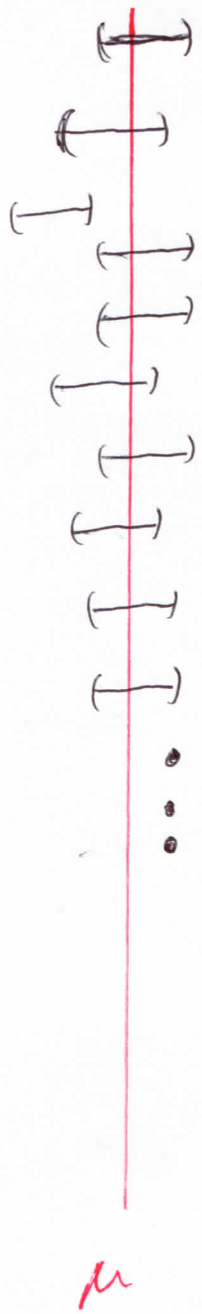
$$\overline{x} \pm 33 \quad (\text{equals } \overline{x} \pm z_{0.05}\frac{\sigma}{\sqrt{25}}),$$

the same way as we constructed $(8,406, 8,472)$, from samples $x_1, x_2, \ldots x_{25}$ of size 25, where

$$\overline{x} \equiv \frac{1}{25}(x_1 + x_2 + \cdots + x_{25}),$$

then approximately 90% of those (*different*) intervals will contain $\mu$.

In the drawing below, the confidence intervals are drawn in black, the vertical red line is $\mu$, and the red line crosses approximately 90% of the confidence intervals.



**Remark 4.5.** For a fixed sample size, there is a trade-off between precision and confidence (see Definitions 3.2): increasing confidence decreases precision (meaning making a confidence interval wider) while increasing precision (meaning making a confidence interval less wide) decreases confidence. This is sad, since we would *like* to be both confident and precise.

It is when we increase sample size that we may increase confidence without losing precision or increase precision without losing confidence.

## HOMEWORK

All confidence intervals are for $\mu$, the population mean of a random variable $X$, as in Chapter 3.

**1.** Suppose that, when sampling from a normal population with standard deviation 30, we get a sample mean of 10.

(a) Find a 90% confidence interval, if the sample size is 9.

(b) Find a 90% confidence interval, if the sample size is 900.

(c) Find a 99% confidence interval, if the sample size is 9.

(d) Find a 99% confidence interval, if the sample size is 900.

**2.** In each of the confidence intervals in no. **1**, find the precision.

**3.** Suppose I make the following measurements from a normal population with standard deviation 8:

$$1, 0, 0, -1, 2, 3, 3, 2, -5, 0, 1, 2, -3, 0, -2, -35.$$

(a) Find an 80% confidence interval.

(b) Find a 95% confidence interval.

(c) Find a 99.5% confidence interval.

(d) Find a confidence interval whose width is 6. What is the confidence level of this interval?

**4.** Suppose a normal population has a standard deviation of 0.5. How large a sample must be taken, to make a 99% confidence interval have width less than 1?

**5.** Same as no. **4**, except make the width less than 0.01.

**6.** Same as no. **5**, except make the confidence level 70%.

**7.** A sample of 25 people gave a sample mean of 12 stresses per day. Assuming stresses per day is normally distributed with a standard deviation of 4 stresses, calculate a 90% confidence interval for the average number of stresses per day among all people.

**8.** I measure ten frogs and get a sample mean of 6 grams. Assuming frog mass is normally distributed with a standard deviation of 1.2 grams, get

(a) an 80% confidence interval for the average mass of all frogs;

(b) an 80% upper confidence bound for the average mass of all frogs;

(c) an 80% lower confidence bound for the average mass of all frogs.

(d) a 90% upper confidence bound for the average mass of all frogs; and

(e) a 90% lower confidence bound for the average mass of all frogs.

**9.** Suppose $X$ is normal, with known standard deviation.

(a) How much must you increase the sample size, to decrease the width of a confidence interval by a factor of three, if the confidence level remains the same?

(b) If a 95% confidence interval is $(-1, 5)$, find a 90% confidence interval, using the same data.

(c) Suppose $\sigma = 25$. Get a 90% confidence interval, if $\overline{x} = 10$ and $n = 400$.

(d) What happens to the precision of a confidence interval, if the confidence level changes from 75% to 99%?

**10.** For the same data, taken from a normal population with known standard deviation, if the confidence level changes from 90% to 60%, will the width of the confidence interval increase or decrease?

**11.** Suppose $X$ is a normal population with known standard deviation $\sigma = 100$. How large must $n$, the sample size, be, so that the width of a 99.8% confidence interval is less than 0.2?

**12.** If the sample size changes from 50 to 400, while maintaining the same confidence level, will the width of the confidence interval increase or decrease?

**13.** Suppose the sugar content of a randomly chosen "Big Drink"(BD) is normal. Your sidekick samples 9 BDs and gets a sample average of 3 grams. Get a 95% upper confidence bound for the average sugar content of all BDs, under the assumption that the standard deviation of the sugar content of all BDs is 0.6 grams.

**14.** Suppose a population is normal, with known standard deviation. If we increase the sample size, while maintaining the same precision, will the confidence level increase or decrease?

**15.** If $z > 1$, what can be said about $P(Z > z)$? Use the $Z$ tables at the end of this magnification.

**16.** Which is larger, $z_{0.1}$ or $z_{0.01}$?

**17.** If $P(Z > c) < 0.02$, what can be said about $c$? Use the $Z$ tables at the end of this magnification.

## HOMEWORK ANSWERS

**1.** (a) $10 \pm z_{0.05}\frac{30}{\sqrt{9}} = 10 \pm 1.65(\frac{30}{\sqrt{9}}) = 10 \pm 16.5 = (-6.5, 26.5)$.

(b) $10 \pm z_{0.05}\frac{30}{\sqrt{900}} = 10 \pm 1.65(\frac{30}{\sqrt{900}}) = 10 \pm 1.65 = (8.35, 11.65)$.

(c) $10 \pm z_{0.005}\frac{30}{\sqrt{9}} = 10 \pm 2.58(\frac{30}{\sqrt{9}}) = 10 \pm 25.8 = (-15.8, 35.8)$.

(d) $10 \pm z_{0.005}\frac{30}{\sqrt{900}} = 10 \pm 2.58(\frac{30}{\sqrt{900}}) = 10 \pm 2.58 = (7.42, 12.58)$.

**2.** (a) $2 \times 16.5 = 33$; (b) $2 \times 1.65 = 3.3$; (c) $2 \times 25.8 = 51.6$; (d) $2 \times 2.58 = 5.16$.

**3.** We calculate $\overline{x} = -2$.

(a) $-2 \pm z_{0.1}(\frac{8}{\sqrt{16}}) = -2 \pm 1.28(\frac{8}{\sqrt{16}}) = -2 \pm 2.56 = (-4.56, 0.56)$.

(b) $-2 \pm z_{0.025}(\frac{8}{\sqrt{16}}) = -2 \pm 1.96(\frac{8}{\sqrt{16}}) = -2 \pm 3.92 = (-5.92, 1.92)$.

(c) $-2 \pm z_{0.0025}(\frac{8}{\sqrt{16}}) = -2 \pm 2.81(\frac{8}{\sqrt{16}}) = -2 \pm 5.62 = (-7.62, 3.62)$.

(d) Since 3 is half of six, our confidence interval is $-2 \pm 3 = (-5, 1)$.

To get the confidence level $(1 - \alpha)$, we set $2z_{\frac{\alpha}{2}}\frac{8}{\sqrt{16}} = 6$, so that $z_{\frac{\alpha}{2}} = 1.5$, meaning (using the $Z$ tables at the end of this magnification)

$$0.0668 = P(Z > 1.5) = P(Z > z_{\frac{\alpha}{2}}) = \frac{\alpha}{2},$$

so that $\alpha = 0.1336$, and the confidence level is $(100 - 13.36)\% = 86.64\%$.

**4.** Since the width is $2z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} = 2z_{0.005}\frac{0.5}{\sqrt{n}} = 2(2.58)\frac{0.5}{\sqrt{n}}$, set

$$2(2.58)\frac{0.5}{\sqrt{n}} < 1,$$

and solve for $n$: $n > (2(2.58)(0.5))^2 = 6.6564$; we want a sample of more than six.

**5.** As with no. **4**, set

$$2(2.58)\frac{0.5}{\sqrt{n}} < 0.01,$$

and solve for $n$: $n > \left(\frac{2(2.58)(0.5)}{0.01}\right)^2 = 66,564$; we want a sample of more than $66,564$.

**6.** As with no. **5**, set

$$2(1.04)\frac{0.5}{\sqrt{n}} = 2z_{\frac{0.3}{2}}\frac{0.5}{\sqrt{n}} < 0.01,$$

and solve for $n$: $n > \left(\frac{2(1.04)(0.5)}{0.01}\right)^2 = 10,816$; we want a sample of more than $10,816$.

**7.** $12 \pm z_{0.05}\frac{4}{\sqrt{25}} = 12 \pm 1.65(\frac{4}{\sqrt{25}}) = 12 \pm 1.32 = (10.68, 13.32)$.

**8.** (a) $6 \pm z_{0.1}\frac{1.2}{\sqrt{10}} = 6 \pm 1.29(\frac{1.2}{\sqrt{10}}) = 6 \pm \frac{1.548}{\sqrt{10}} \sim 6 \pm 0.490 = (5.510, 6.490)$.

(b) $6 + z_{0.2}\frac{1.2}{\sqrt{10}} = 6 + 0.85(\frac{1.2}{\sqrt{10}}) = 6 + \frac{1.02}{\sqrt{10}} \sim 6.323$.

(c) $6 - z_{0.2}(\frac{1.2}{\sqrt{10}}) = 6 - 0.85(\frac{1.2}{\sqrt{10}}) = 6 - \frac{1.02}{\sqrt{10}} \sim 5.677$.

(d) $6 + z_{0.1}\frac{1.2}{\sqrt{10}} = 6 + 1.29(\frac{1.2}{\sqrt{10}}) = 6 + \frac{1.548}{\sqrt{10}} \sim 6.490$. Compare to (a).

(e) $6 - z_{0.1}\frac{1.2}{\sqrt{10}} = 6 - 1.29(\frac{1.2}{\sqrt{10}}) = 6 - \frac{1.548}{\sqrt{10}} \sim 5.510$. Compare to (a).

**9.** (a) Multiply the sample size by nine.

(b) $2 \pm 3(\frac{z_{0.05}}{z_{0.025}}) = 2 \pm 3(\frac{1.65}{1.96}) \sim 2 \pm 2.526 = (-0.526, 4.526)$.

Notice that we have more precision with the 90% confidence interval compared to the 95% confidence interval; a loss of confidence corresponds to a gain in precision.

(c) $10 \pm z_{0.05}\frac{25}{\sqrt{400}} = 10 \pm 1.65(\frac{25}{\sqrt{400}}) = 10 \pm 2.0625 = (7.9375, 12.0625)$.

(d) The precision diminishes; that is, the interval gets wider.

**10.** The width will decrease (more precision, when we permit a loss in the confidence level).

**11.** The width is $2z_{0.001}\frac{\sigma}{\sqrt{n}} = 2(3.08)\frac{100}{\sqrt{n}}$, so set

$$2(3.08)\frac{100}{\sqrt{n}} < 0.2$$

and solve for $n > \left(2(3.08)\frac{100}{0.2}\right)^2 = 9,486,400$; we need a sample of more than $9,486,400$.

**12.** Decrease (new width equals old width times $\sqrt{\frac{50}{400}}$)

**13.** $3 + z_{0.05}\frac{0.6}{\sqrt{9}} = 3 + 1.65(\frac{0.6}{\sqrt{9}}) = 3 + 0.33 = 3.33$.

**14.** The confidence level will increase.

**15.** $P(Z > z) < P(Z > 1) = 0.1587$.

**16.** $z_{0.01}$ is larger than $z_{0.1}$, since $0.01 < 0.1$. See Examples 2.6(e).

**17.** Since, from the $Z$ tables, $P(Z > 2.06) < 0.02$ and $P(Z > 2.05) > 0.02$, $c > 2.05$ is the most we can say about $c$.

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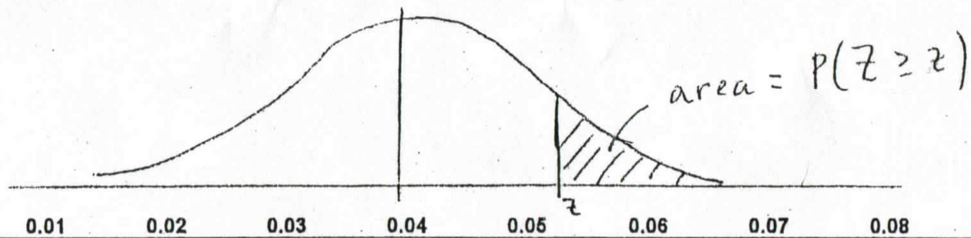
## The Standard Normal Distribution (Areas in the Right Tail)




| z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.5000 | 0.4960 | 0.4920 | 0.4880 | 0.4840 | 0.4801 | 0.4761 | 0.4721 | 0.4681 | 0.4641 |
| 0.1 | 0.4602 | 0.4562 | 0.4522 | 0.4483 | 0.4443 | 0.4404 | 0.4364 | 0.4325 | 0.4286 | 0.4247 |
| 0.2 | 0.4207 | 0.4168 | 0.4129 | 0.4090 | 0.4052 | 0.4013 | 0.3974 | 0.3936 | 0.3897 | 0.3859 |
| 0.3 | 0.3821 | 0.3783 | 0.3745 | 0.3707 | 0.3669 | 0.3632 | 0.3594 | 0.3557 | 0.3520 | 0.3483 |
| 0.4 | 0.3446 | 0.3409 | 0.3372 | 0.3336 | 0.3300 | 0.3264 | 0.3228 | 0.3192 | 0.3156 | 0.3121 |
| 0.5 | 0.3085 | 0.3050 | 0.3015 | 0.2981 | 0.2946 | 0.2912 | 0.2877 | 0.2843 | 0.2810 | 0.2776 |
| 0.6 | 0.2743 | 0.2709 | 0.2676 | 0.2643 | 0.2611 | 0.2578 | 0.2546 | 0.2514 | 0.2483 | 0.2451 |
| 0.7 | 0.2420 | 0.2389 | 0.2358 | 0.2327 | 0.2296 | 0.2266 | 0.2236 | 0.2206 | 0.2177 | 0.2148 |
| 0.8 | 0.2119 | 0.2090 | 0.2061 | 0.2033 | 0.2005 | 0.1977 | 0.1949 | 0.1922 | 0.1894 | 0.1867 |
| 0.9 | 0.1841 | 0.1814 | 0.1788 | 0.1762 | 0.1736 | 0.1711 | 0.1685 | 0.1660 | 0.1635 | 0.1611 |
| 1.0 | 0.1587 | 0.1562 | 0.1539 | 0.1515 | 0.1492 | 0.1469 | 0.1446 | 0.1423 | 0.1401 | 0.1379 |
| 1.1 | 0.1357 | 0.1335 | 0.1314 | 0.1292 | 0.1271 | 0.1251 | 0.1230 | 0.1210 | 0.1190 | 0.1170 |
| 1.2 | 0.1151 | 0.1131 | 0.1112 | 0.1093 | 0.1075 | 0.1056 | 0.1038 | 0.1020 | 0.1003 | 0.0985 |
| 1.3 | 0.0968 | 0.0951 | 0.0934 | 0.0918 | 0.0901 | 0.0885 | 0.0869 | 0.0853 | 0.0838 | 0.0823 |
| 1.4 | 0.0808 | 0.0793 | 0.0778 | 0.0764 | 0.0749 | 0.0735 | 0.0721 | 0.0708 | 0.0694 | 0.0681 |
| 1.5 | 0.0668 | 0.0655 | 0.0643 | 0.0630 | 0.0618 | 0.0606 | 0.0594 | 0.0582 | 0.0571 | 0.0559 |
| 1.6 | 0.0548 | 0.0537 | 0.0526 | 0.0516 | 0.0505 | 0.0495 | 0.0485 | 0.0475 | 0.0465 | 0.0455 |
| 1.7 | 0.0446 | 0.0436 | 0.0427 | 0.0418 | 0.0409 | 0.0401 | 0.0392 | 0.0384 | 0.0375 | 0.0367 |
| 1.8 | 0.0359 | 0.0351 | 0.0344 | 0.0336 | 0.0329 | 0.0322 | 0.0314 | 0.0307 | 0.0301 | 0.0294 |
| 1.9 | 0.0287 | 0.0281 | 0.0274 | 0.0268 | 0.0262 | 0.0256 | 0.0250 | 0.0244 | 0.0239 | 0.0233 |
| 2.0 | 0.0228 | 0.0222 | 0.0217 | 0.0212 | 0.0207 | 0.0202 | 0.0197 | 0.0192 | 0.0188 | 0.0183 |
| 2.1 | 0.0179 | 0.0174 | 0.0170 | 0.0166 | 0.0162 | 0.0158 | 0.0154 | 0.0150 | 0.0146 | 0.0143 |
| 2.2 | 0.0139 | 0.0136 | 0.0132 | 0.0129 | 0.0125 | 0.0122 | 0.0119 | 0.0116 | 0.0113 | 0.0110 |
| 2.3 | 0.0107 | 0.0104 | 0.0102 | 0.0099 | 0.0096 | 0.0094 | 0.0091 | 0.0089 | 0.0087 | 0.0084 |
| 2.4 | 0.0082 | 0.0080 | 0.0078 | 0.0075 | 0.0073 | 0.0071 | 0.0069 | 0.0068 | 0.0066 | 0.0064 |
| 2.5 | 0.0062 | 0.0060 | 0.0059 | 0.0057 | 0.0055 | 0.0054 | 0.0052 | 0.0051 | 0.0049 | 0.0048 |
| 2.6 | 0.0047 | 0.0045 | 0.0044 | 0.0043 | 0.0041 | 0.0040 | 0.0039 | 0.0038 | 0.0037 | 0.0036 |
| 2.7 | 0.0035 | 0.0034 | 0.0033 | 0.0032 | 0.0031 | 0.0030 | 0.0029 | 0.0028 | 0.0027 | 0.0026 |
| 2.8 | 0.0026 | 0.0025 | 0.0024 | 0.0023 | 0.0023 | 0.0022 | 0.0021 | 0.0021 | 0.0020 | 0.0019 |
| 2.9 | 0.0019 | 0.0018 | 0.0018 | 0.0017 | 0.0016 | 0.0016 | 0.0015 | 0.0015 | 0.0014 | 0.0014 |
| 3.0 | 0.0013 | 0.0013 | 0.0013 | 0.0012 | 0.0012 | 0.0011 | 0.0011 | 0.0011 | 0.0010 | 0.0010 |