TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI

# Does Ontogeny Recapitulate Phylogeny in Learning Freshman Calculus?

Dr. Ralph deLaubenfels

TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI

Teacher-Scholar Institute

Columbus, Ohio

2014

**DOES ONTOGENY RECAPITULATE PHYLOGENY**

**in learning freshman calculus?**

Ralph deLaubenfels

**ABSTRACT.** The answer is NO in what we teach; YES in students' absorption and interpretation of what we teach or could teach, that is, in natural conceptions and misconceptions that arise or could arise if exposed to calculus ideas.

**I. INTRODUCTION.** Ontogeny is the development of an individual; phylogeny is the development or evolution of species. The phrase "Ontogeny recapitulates phylogeny" is attributed to Ernst Haeckel in the mid nineteenth century (first published 1866), although the idea appears as early as the 1790s. "Recapitulate," at least in biology, means to repeat. Here is a description of the capitalized phrase in the title due to Dr. Spock ([6, p. 223]), famous alleged baby expert:

"Each child as he develops is retracing the whole history of mankind, physically and spiritually, step by step. A baby starts off in the womb as a single tiny cell, just the way the first living thing appeared in the ocean. Weeks later, as he lies in the amniotic fluid of the womb, he has gills like a fish..."

In biology, this idea seems to be no longer believed. Other fields that have proposed analogues include philosophy, art, anthropology, psychology, and education. Here's an online example of a discussion of math educational recapitulation:

http://unlearningmath.com/2009/02/13.

As with the outline I'm undertaking, the educational analogue of ontogeny recapitulating phylogeny would be that the development of ideas in an individual follows patterns similar to the development of those ideas over centuries.

My thesis is that this educational analogue is not true if the ontogeny is calculus as we teach it (Section II), but is true if the ontogeny refers to how students tend to naturally formulate the ideas of calculus; that is, student's conceptions and misconceptions of calculus mirror history in their development (Section III).

My data is based on fifty years of teaching including more than twenty years of homeschooling my own children and other children. In teaching math to middle school and high school homeschooled students, the absence of mandatory lock-step curricula enabled me to expose the students to calculus ideas much earlier than the moment prescribed by educationists. Middle school and high school students are capable of understanding many calculus ideas; it is mostly shaky algebra skills that might make enrolling in a college freshman calculus class difficult.

Because of my limited goals, the historical exposition throughout this paper will be sketchy. I have tried to include detailed references, for those who want more complete history.

**II. ONTOGENY in teaching freshman (single-variable) calculus.** Here is the traditional order of topics in which single-variable calculus is taught in college, especially to students beginning a science or engineering undergraduate degree.

**(1)** function (precalculus review)

**(2)** limit

**(3)** continuity

**(4)** derivative

**(5)** applications of derivative

**(6)** Riemann sums to definite integrals

**(7)** Fundamental Theorem of Calculus and integration techniques

**(8)** applications of integration

**(9)** limit of sequence of numbers

**(10)** series of numbers

**(11)** power or Taylor series.

**III. PHYLOGENY; that is, ordered history of single-variable calculus ideas.** Here are **(1)**–**(11)** from Section II, placed in historical chronological order, based on when the idea appeared.

My thesis is that this phylogeny comes close to the ontogeny of students' understanding (this includes the unfortunately usually hypothetical category of what they *could* understand at a given time) of the calculus concepts numbered in Section II. Throughout this section, I will be comparing the historical development of calculus to the development of calculus concepts that a student *could* learn.

**(9)** Although not formally defined as a limit of a sequence, the idea of making a sequence of approximations arbitrarily close to a desired quantity goes back at least as far as the classical Greeks. Emblematic of this is the Greeks' finding of the area of a disc by approximating with polygons of an increasing number of sides. This was proven by Archimedes (third century B.C.) although it was known before him.

This is an idea that students can learn at an early age (the author has worked with home-schooled children of middle-school age on this) by looking at a decimal expansion of a famous irrational, such as $\pi$ or $\sqrt{2}$, or by performing iterations with a calculator.

**(10)** Geometric series appear at least as early as Zeno and one of his paradoxes: in order to travel a certain distance, he must first travel half the distance, then half of that half, etc. The perceived impossibility of adding up the infinite sum $\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k \equiv \frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\ldots$ to travel that distance makes the trip seemingly impossible.

Archimedes studied geometric series, partly to find areas related to parabolic segments; see **(6)** below.

English and French mathematicians in the first half of the 14th century worked on some infinite series, including many that were not geometric series (see [3, pp. 91–93]). This included showing that the harmonic series $\sum_{k=1}^{\infty}\frac{1}{k}$ diverges, by grouping the terms as follows:

$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\cdots+\frac{1}{16}\right)+\ldots;$$

each group in parentheses is greater than $\frac{1}{2}$.

Regarding students' ontogeny, the author and home-schooled students as early as middle school the author has worked with have calculated geometric series and applied them to repeating decimals.

**(6)** Archimedes used analogues of Riemann sums (triangles instead of rectangles) to get areas under line segments and parabolic segments (among many other areas and volumes; see [3, Chapter 2]);

what we now call $\int_0^a x\,dx$ and $\int_0^a x^2\,dx$; as students do today, he used formulas for $\sum_k k, \sum_k k^2$. See [5, pp. 111–116], [3, Chapter 2 and p. 109], and [7, pp. 44–47].

The "invention" of calculus, as credited to Newton and Leibniz independently in the second half of the 17th century, is defined as the discovery of the fundamental theorem of calculus. There were many calculus results, both in integration (that is, in calculating definite integrals, sometimes called *quadrature*) and differentiation (see **(4)** and **(5)** below), during the first half of the 17th century. See [3, Chapter 4] and [7, Sections 8.2 and 8.4]. Cavalieri got volumes by integrating cross sections (special cases taught in calculus classes usually include the "disc" method for solids of revolution, the "disc" referring to the shape of the cross sections) and calculated $\int_a^b x^n\,dx, n = 1, 2, ..., 9$, via Riemann sums $\sum_k k^n$; he speculated the general formula for $\int_a^b x^n\,dx$ for all nonnegative integral $n$, which was proved in the 1630s by Fermat, Descartes, and Roberval; Fermat also integrated fractional powers. Toricelli produced the solid with infinite surface area and finite volume $y = \frac{1}{x}$ revolved about the $x$ axis. Wallis did a more "direct" approach to integrating fractional powers than Fermat; see [7, Section 8.4].

Even Cauchy, in finally creating our modern calculus rigor in the early 19th century, did not quite have the full generality of our present Riemann sums. Cauchy's sums had the form

$$\sum_i f(x_{i-1})(x_i - x_{i-1}),$$

instead of the form (due to Riemann)

$$\sum_i f(\overline{x}_i)(x_i - x_{i-1}),$$

for arbitrary $\overline{x}_i$ in $[x_{i-1}, x_i]$. See [4].

Regarding students' ontogeny, I must again be partly hypothetical, since they are normally exposed to derivatives before integration. When I present integration via Riemann sums to (home-schooled) high school students, they find the idea simpler than differentiation. Area is more intuitive than rate of change and the idea of approximating a region with increasingly skinny rectangles, to approximate area, is a natural one.

**(4) and (5)** Many derivatives and applications thereof occurred prior to the "invention" of calculus by Newton and Leibniz, as described under **(6)** above.

In the early 14th century, a sort of prederivative, velocity and motion with uniform acceleration, was studied (see [3, pp. 86–93]) by many of the English and French mathematicians mentioned under **(10)** in the third paragraph. This was an adventurous improvement over the classical Greeks, who studied motion only with constant velocity. In terms of analogous ontogeny of a (future) student, the author recalls noticing, as a child, that a car in motion could still have something constant, namely the speedometer reading. Constant acceleration (felt as a constant pressure on the chest) along with a changing speedometer reading could have been introduced.

Work on both tangent lines, and maxima and minima via setting a derivative equal to zero, appeared in the early 17th century, in work by Fermat, Cavalieri, and Wallis; see [1, pp. 153–162], [3, pp. 122-123], and [7, pp. 103-104].

See [4], especially "The Debate over Foundations," pp. 31–36, where the function $f(x) \equiv x^2$ is focused on in particular, for very interesting descriptions of attempts to define the derivative rigorously, especially in the 18th century.

**(7) and (8)** Most of the calculus techniques taught in freshman calculus classes were developed by Newton and Leibniz in the second half of the 17th century.

Newton defined only derivatives with respect to time, denoted by a dot over the function being differentiated, e.g., $\dot{y}$ for the time derivative of $y$. For slope of the tangent line to the graph of a function $y = f(x)$, he used $\frac{\dot{y}}{\dot{x}}$. Leibniz developed the more general notion of the derivative of one variable with respect to another; he called the slope of the tangent line just mentioned $\frac{dy}{dx}$, the derivative of $y$ with respect to $x$. Note how Leibniz's notation makes Newton's formula for slope of a tangent line believable:

$$\frac{\dot{y}}{\dot{x}} = \frac{\frac{dy}{dt}}{\frac{dx}{dy}} = \frac{dy}{dx},$$

the chain rule appearing to be no more mysterious than cancelling fractions.

Newton's attempt to make calculus rigorous had velocities as a first principle; this is analogous to students thinking of derivative or rate of change only meaning velocity.

It is interesting that the greater emphasis on applications due to Newton gave his approach more popularity at the time, just as it appeals more to students learning calculus now. However Leibniz's generality and excellent suggestive terminology, both for derivatives and integrals, made his formulation the main vehicle for extending and applying calculus in the 18th century.

**(11)** If convergence is not a concern, infinite sums need not seem that much different than finite sums, a branch of algebra (see [4, Chapter 2]). This was the outlook in the pre-Cauchy days of calculus; for example, Newton used many functions represented as a (usually infinite) series.

This certainly coincides with students' first introduction to infinite sums and makes it pedagogically desirable, soon after introducing the formula for sum of a geometric series

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r},$$

to insert $r = 2$ into this formula to get the amazing result

$$1 + 2 + 4 + 8 + \cdots = \sum_{k=0}^{\infty} 2^k = -1.$$

Since algebra brings the potential for rigor, Lagrange, in the second half of the 18th century, used Taylor series to define derivatives (see [4, p. 39], [2, pp. 296-8]): If

$$f(x) = \sum_{k=0}^{\infty} a_k (x - c)^k,$$

then the $j^{th}$ derivative of $f$ at $c$ *by definition* is

$$f^{(j)}(c) \equiv j! a_j.$$

It is interesting, and, again, consistent with students' evolving ideas, that it was assumed that any continuous function has a Taylor series. See [2, p. 267].

**(2), (3)** The modern "$\epsilon$-$\delta$" definition of limit of a function, hence definition of a continuous function, did not appear until the early part of the 19th century, introduced independently by Cauchy and Bolzano.

Ideas of limits, as with approximations, appeared in the early days of calculus, e.g., d'Alembert spoke of a derivative as a "limit of a ratio" ([3, pp. 295-6]), but did not give a definition of limit. Newton essentially talks about limit (really a supremum), but doesn't define it.

The first example of a continuous but not differentiable function appeared in 1834, due to Bolzano; see [2, p. 269]. For students as with mathematicians historically, this is a subtlety.

In an incomplete but developing idea, Euler uses the word "continuous" to mean more "contiguous" (consistent with students' first impression).

**(1)** Our modern idea of function, as taught in a freshman calculus class, did not appear until the late 19th century, when Dirichlet defined it as a map that assigns to each point in the domain a unique point in the range.

In the 18th century, a function meant an explicit formula, sometimes an infinite series. This is consistent with students' first impression; a function defined in two pieces (e.g., $f(x) \equiv x^2$ for $x > 0$, $f(x) \equiv x$ for $x \leq 0$, is often perceived by students as being two functions).

One source of the need to generalize the idea of function is described in [4, pp. 89–90]: if a function was *defined* only as the solution of a differential equation, it might not have an explicit representation.

**IV. SOME CONCLUSIONS.** I don't see that it's automatically necessary or desirable for teachers to repeat history, in choosing the order or style of presentation of ideas. The sifting and reformulation of ideas that occurs over time does create more unified and simplified approaches that mean both greater and more efficiently acquired understanding.

But there is the same lesson from history of ideas that one should get from any history: that events are not inevitable. Initial expressions of ideas, in particular, need not be immutable. Comparing the ordering in Section II to that in Section III demonstrates that even the *order* of ideas can change considerably.

Something I object to in much math teaching is the implicit air of things being precisely preordained. Rather than ideas that anyone can and naturally would create, this style of teaching presents a totalitarian monotheism of ideas, and their order of presentation, that must be force fed to students. This misconception of mathematical creation is very similar to the phrase "settled science," recently made popular by authority figures attempting to use misrepresentations of science for political ends. Even as this short note is being written, the latest incarnation of the nationally centralized pseudo-scientific educational theocracy is imposing standardized *methods* of solutions of mathematical problems; this is an infinitely intolerant conformity of thought completely counter to the spirit of math.

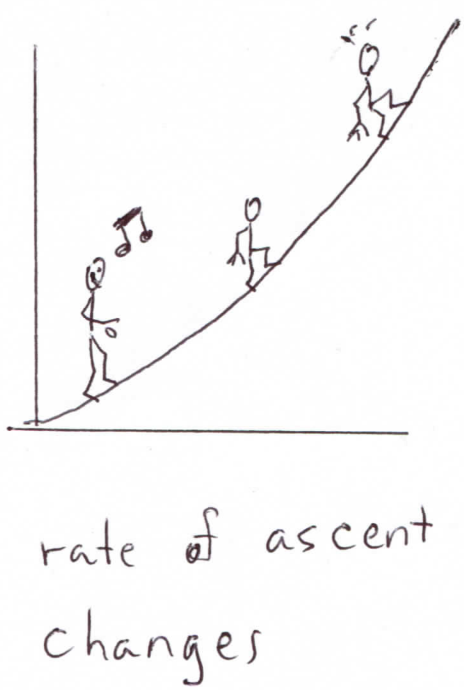
A small infusion of the history from Section III into calculus classes would have psychological and pedagogical benefits. The teacher benefits by getting clues about what their students are thinking. Students would find it quite reassuring to hear that the greatest mathematicians had the same confusion and misunderstanding in their initial attempts to understand that they do. Even a superficial knowledge of the history of ideas would help students to see they should be forming ideas themselves rather than blindly obeying or absorbing someone else's ideas. At least as short asides, historical notes about the history of the ideas would remove much of the intimidating, and mostly unnecessary, alienness of the subject.

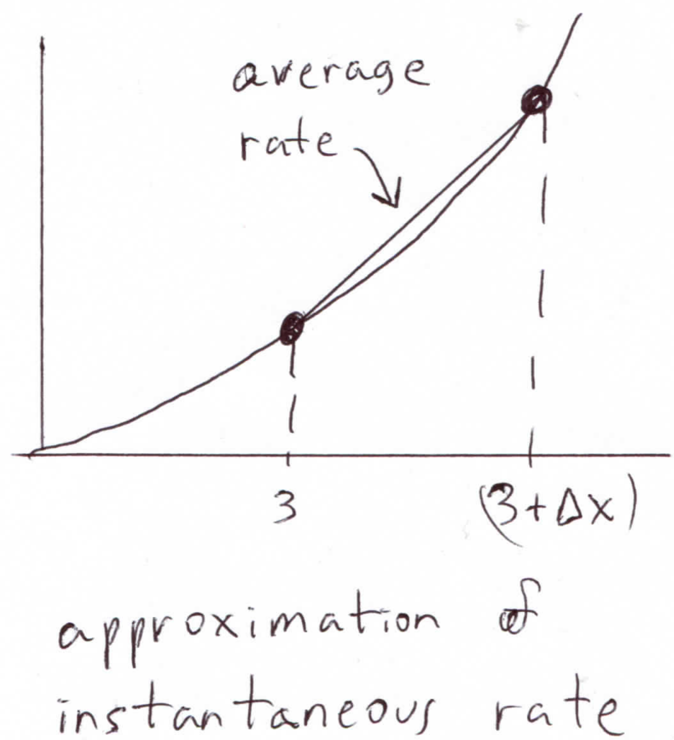
A more radical suggestion is to devote the first lecture of a calculus class to two calculations, in a pre-Newton/Leibniz style, as in Section III **(4)**–**(6)**:

**Calc. I.** The instantaneous rate of ascent, at a point, when walking up a parabolic hillside; and

**Calc. II.** The area between $y = x^2, y = 0$, and $x = 1$.

For **Calc. I,** say our hillside has the shape $y = f(x) \equiv x^2$, and we want the (instantaneous) rate of ascent at $x = 3$ (see drawing below.)



All the student needs is the definition of slope of a straight line: "rise over run" $\equiv \frac{\Delta y}{\Delta x}$, where "$\Delta$" means change. One approximates the rate of ascent at $x = 3$ by calculating the average rate of ascent over the interval $[3, 3 + \Delta x]$

$$\frac{\Delta y}{\Delta x} = \frac{f(3+\Delta x) - f(3)}{\Delta x}$$

with the approximation improving as $\Delta x$ gets smaller:

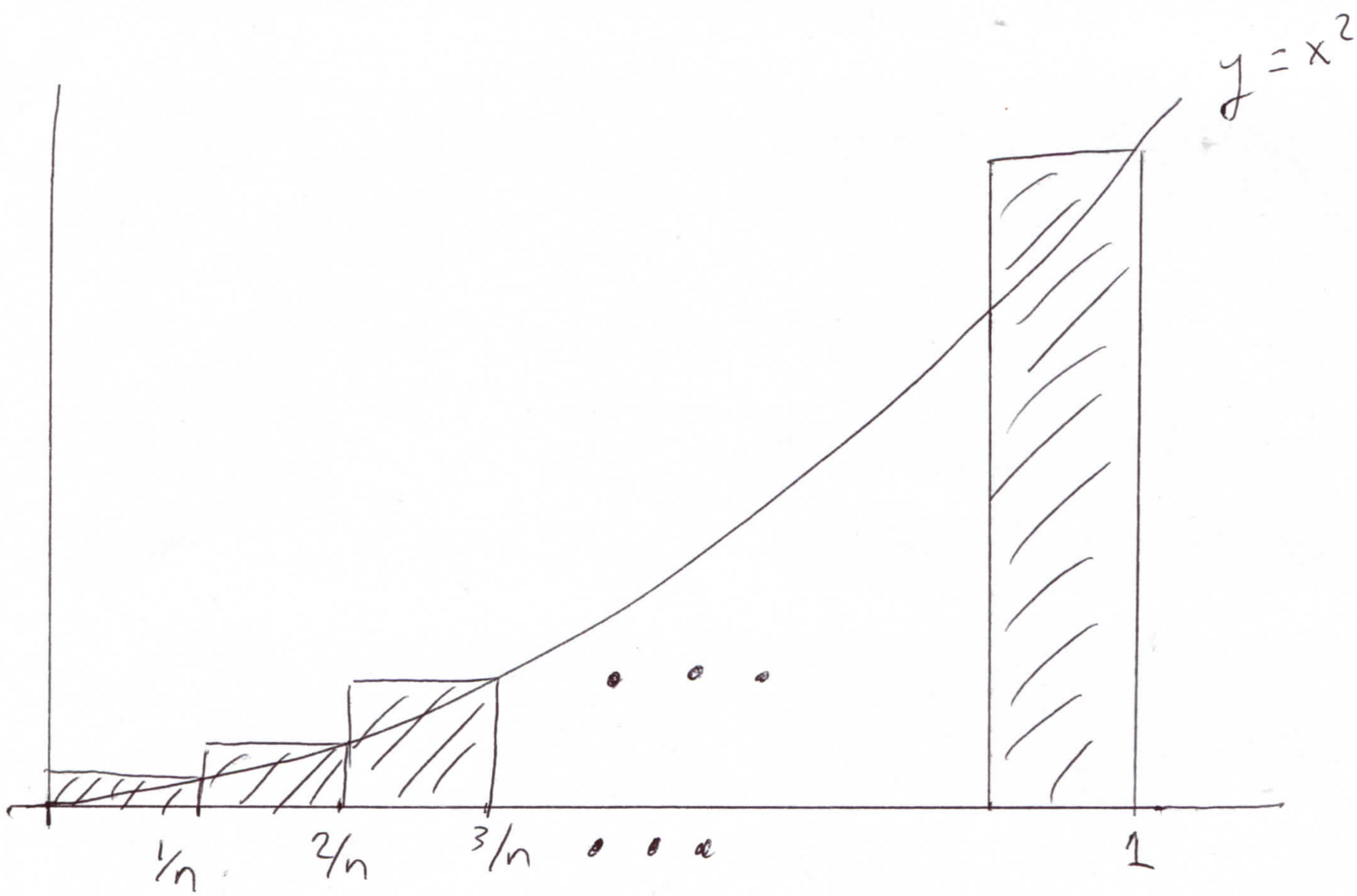
| $\Delta x$ | $\Delta y = (3+\Delta x)^2 - 3^2$ | $\frac{\Delta y}{\Delta x}$ |
|---|---|---|
| 1 | $4^2 - 3^2 = 7$ | 7 |
| 0.1 | $(3.1)^2 - 3^2 = 0.61$ | 6.1 |
| 0.01 | $(3.01)^2 - 3^2 = 0.0601$ | 6.01 |
| 0.001 | $(3.001)^2 - 3^2 = 0.006001$ | 6.001 |
| | $(9 + 6\Delta x + (\Delta x)^2) - 9 = 6\Delta x + (\Delta x)^2$ | $6 + \Delta x$ |

At this point, one may gesture at the sequence $7, 6.1, 6.01, 6.001$, and say it sure looks like it's approaching 6. Or one may infuriate the early critics of calculus (see [4], especially Berkeley), by letting $\Delta x$ equal zero in the expression $\frac{\Delta y}{\Delta x} = 6 + \Delta x$. The word "limit" (as $\Delta x$ goes to zero) could be ostentatiously spoken at this point.

For **Calc. II,** all students need is area of a rectangle (base times height) and the formula (known, e.g., to Aristotle)

$$\sum_{k=1}^{n} k^2 \equiv (1^2 + 2^2 + 3^2 + 4^2 + 5^2 + \cdots + n^2) = (1 + 4 + 9 + 16 + 25 + \cdots + n^2) = \frac{n(n+1)(2n+1)}{6}.$$

One approximates with $n$ rectangles of equal width, upper right vertex on the curve $y = x^2$, with the approximation improving as $n$ gets larger (see drawing below).



The sum of the areas of the $n$ rectangles described and drawn may be simplified into a single expression; we calculate height times base, going from left to right:

$$\text{area} = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \frac{1}{n} \equiv \left(\left(\frac{1}{n}\right)^2 \frac{1}{n} + \left(\frac{2}{n}\right)^2 \frac{1}{n} + \left(\frac{3}{n}\right)^2 \frac{1}{n} + \left(\frac{4}{n}\right)^2 \frac{1}{n} + \cdots + \left(\frac{n}{n}\right)^2 \frac{1}{n}\right) = \frac{1}{n^3}\left(1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2\right)$$

$$\equiv \frac{1}{n^3} \sum_{k=1}^{n} k^2 = \frac{(n+1)(2n+1)}{6n^2}.$$

| $n$ = number of rectangles | area $= \frac{(n+1)(2n+1)}{6n^2}$ |
|---|---|
| 1 | $\frac{2\times 3}{6} = 1$ |
| 2 | $\frac{3\times 5}{24} = \frac{5}{8}$ |
| 10 | $\frac{11\times 21}{600} = \frac{231}{600}$ |
| 100 | $\frac{101\times 201}{60{,}000} = \frac{20{,}301}{60{,}000}$ |
| $1,000$ | $\frac{1{,}001\times 2{,}001}{6{,}000{,}000} = \frac{2{,}003{,}001}{6{,}000{,}000}$ |

Staring at the areas as $n$ gets larger might convince the student that they get arbitrarily close to $\frac{1}{3}$. Alternatively, one could expand the numerator in the fractional expression for the area:

$$\frac{(n+1)(2n+1)}{6n^2} = \frac{2n^2 + 3n + 1}{6n^2} = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2},$$

and use a calculator to believe that the last two fractions get arbitrarily small as $n$ get large.

Exaggerated proclamations of the word "limit" can also be made here, with the observation that this is a limit as n goes to infinity, whereas the prior limit was as a quantity ($\Delta x$) goes to zero. The ability to divide by zero or infinity should be assigned significant mystical respect; students should be warned that it is a super power, that brings with it super responsibility.

## REFERENCES


**1.** W.S. Anglin and J. Lambek, "The Heritage of Thales," Springer, 1995.

**2.** Carl B. Boyer, "The History of the Calculus and its Conceptual Development," Dover, 1949.

**3.** C.H. Edwards, Jr., "The Historical Development of the Calculus," C.H. Edwards, Jr., Springer-Verlag, 1979.

**4.** J. V. Grabiner, "Origins of Cauchy's Rigorous Calculus," The MIT Press, 1981.

**5.** H.L. Resnikoff and R.O. Wells, Jr., "Mathematics in Civilization," Dover, 1984.

**6.** "Baby and Child Care," B. Spock, Pocket Books, New York, 1957.

**7.** J. Stillwell, "Mathematics and Its History," Springer-Verlag, 1989.